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GRADIENT ESTIMATES FOR A NONLINEAR ELLIPTIC EQUATION ON COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. In this short paper, we consider gradient estimates for positive solutions to the following nonlinear elliptic equation on a complete Riemannian manifold:

$$\Delta u + cu^{\alpha} = 0,$$

where c, α are two real constants and $c \neq 0$.

1. Introduction

It is well known that for complete noncompact Riemannian manifolds with nonnegative Ricci curvature, Yau [11] has proved that any positive or bounded solution to the equation

$$\Delta u = 0 \tag{1.1}$$

must be constant. In [1], Brighton studied f-harmonic functions on a mooth metric measure space. That is, he considered positive solutions to the equation

$$\Delta_f u = 0 \tag{1.2}$$

and obtained some similar results to Yau's under the Bakry–Émery Ricci curvature condition.

It is easy to see that equation (1.1) can be seen as a special case of

$$\Delta u + cu^{\alpha} = 0 \tag{1.3}$$

with c, α being two real constants. In particular, if $c = 0$ in (1.3), then the equation (1.3) becomes (1.1). If $c < 0$ and $\alpha < 0$, equation (1.3) on a bounded smooth domain in $\mathbb{R}^n$ is known as the thin film equation, which describes a steady state of the thin film (see [3]). For c a function, equation (1.3) is studied by Gidas and Spruck in [2] with $1 \leq \alpha \leq \frac{n+2}{n-2}$ when $n > 2$ and later it is studied by Li in [7] to achieve gradient estimates and Liouville type results with $1 < \alpha < \frac{n}{n-2}$ when $n > 3$. In particular, Li achieved a gradient estimate for positive solutions of (1.3) when c is a positive constant and $1 < \alpha < \frac{n}{n-2}$.

Therefore, it is natural to try to achieve gradient estimates for positive solutions to the nonlinear elliptic equation (1.3) with other $c \neq 0$ and α. In this direction Yang in [10] proved the following result.

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Theorem 1.1 (Yang). *Let M be a complete noncompact Riemannian manifold of dimension n without boundary. Let $B_p(2R)$ be a geodesic ball of radius $2R$ around $p \in M$. We denote $-K(2R)$ with $K(2R) \geq 0$ such that $Ric_{ij}(B_p(2R)) \geq -Kg_{ij}$. Suppose that $u(x)$ is a positive smooth solution to equation* (1.3) *with $\alpha < 0$. Then we have*

(i) *If $c > 0$, then $u(x)$ satisfies the estimate*

$$\frac{|\nabla u|^2}{u^2} + cu^{\alpha-1} \leq C(n,\alpha)\big(K + \frac{1}{R^2}(1 + \sqrt{K}R\coth(\sqrt{K}R))\big)$$

on $B_p(R)$ and $C(n,\alpha)$ is a positive constant which depends on n, α.

(ii) *If $c < 0$, then $u(x)$ satisfies the estimate*

$$\frac{|\nabla u|^2}{u^2} + cu^{\alpha-1} \leq C(n,\alpha)\big(|c|(\inf_{B_p(2R)} u)^{\alpha-1} + K + \frac{1}{R^2}(1 + \sqrt{K}R\coth(\sqrt{K}R))\big)$$

on $B_p(R)$ and $C(n,\alpha)$ is a positive constant which depends on n, α.

After studying Yang's argument carefully, we find in the case of $c > 0$ that the gradient estimate in (i) actually holds whenever $\alpha \leq 1$, that is, we have the following.

Theorem 1.2. *Let M be a complete noncompact Riemannian manifold of dimension n without boundary. Let $B_p(2R)$ be a geodesic ball of radius $2R$ around $p \in M$. We denote $-K(2R)$ with $K(2R) \geq 0$ such that $Ric_{ij}(B_p(2R)) \geq -Kg_{ij}$. Suppose that $u(x)$ is a positive smooth solution to equation* (1.3) *with $\alpha \leq 1$ and $c > 0$. Then we have*

$$\frac{|\nabla u|^2}{u^2} + cu^{\alpha-1} \leq C(n,\alpha)\big(K + \frac{1}{R^2}(1 + \sqrt{K}R\coth(\sqrt{K}R))\big)$$

on $B_p(R)$ and $C(n,\alpha)$ is a positive constant which depends on n, α.

The proof of the above theorem is the same as Yang's proof of Theorem 1.1, and we will only give a sketch of it in the appendix. As a corollary of the above theorem we have the following Liouville-type result.

Corollary 1.3. *Let M be a complete noncompact Riemannian manifold of dimension n without boundary. Suppose that the Ricci curvature of M is nonnegative. Then there does not exist a positive solution to equation* (1.3) *with $\alpha \leq 1$ and $c > 0$.*

Suppose that $u(x)$ is a positive solution to equation (1.3). Following Brighton's argument in [1] by choosing a test function $u^\epsilon (\epsilon \neq 0)$, we can also get the following gradient estimate to $u(x)$.

Theorem 1.4. *Let (M, g) be an n-dimensional complete Riemannian manifold with $R_{ij}(B_p(2R)) \geq -Kg_{ij}$, where $K \geq 0$ is a constant. If u is a positive solution to* (1.3) *on $B_p(2R)$ with c and α satisfying one of the following two cases:*

(1) *$c < 0$ and $\alpha > 0$,*

(2) *$c > 0$ and $\frac{n+2}{2(n-1)} < \alpha < \frac{2n^2+9n+6}{2n(n+2)}$ with $n \geq 3$,*

then we have for any $x \in B_p(R)$

$$|\nabla u(x)| \leq C(n,\alpha)M\sqrt{K + \frac{1}{R^2}\Big(1 + \sqrt{K}R\coth(\sqrt{K}R)\Big)}, \tag{1.4}$$

where $M = \sup_{x\in B_p(2R)} u(x)$ and the positive constant $C(n,\alpha)$ depends only on n, α.

Remark 1.1. In case (2), compared with Li's gradient estimate in [7] our right range for α is bigger than $\frac{n}{n-2}$ when $n \geq 13$.

Letting $R \to \infty$ in (1.4), we obtain the following gradient estimates on complete noncompact Riemannian manifolds.

Corollary 1.5. *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $R_{ij} \geq -Kg_{ij}$, where $K \geq 0$ is a constant. Suppose that u is a positive solution to* (1.3) *such that c, α satisfy one of the two cases given in Theorem* 1.4. *Then we have*

$$|\nabla u| \leq C(n, \alpha) M \sqrt{K}, \tag{1.5}$$

where $M = \sup_{x \in M} u(x)$.

Remark 1.2. Recently, using the ideas of Brighton in [1], some Liouville type results have been achieved to positive solutions of the nonlinear elliptic equation

$$\Delta u + au \log u = 0$$

in [4] (for more developments, see [6, 8]), and for porous medium and fast diffusion equations in [5].

2. Proof of Theorem 1.4

Let $h = u^\epsilon$, where $\epsilon \neq 0$ is a constant to be determined. Then we have

$$\begin{aligned} \Delta h =& \epsilon(\epsilon - 1) u^{\epsilon-2} |\nabla u|^2 + \epsilon u^{\epsilon-1} \Delta u \\ =& \epsilon(\epsilon - 1) u^{\epsilon-2} |\nabla u|^2 - c\epsilon u^{\alpha+\epsilon-1} \\ =& \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - c\epsilon h^{\frac{\alpha+\epsilon-1}{\epsilon}}, \end{aligned} \tag{2.1}$$

where in the second equality of (2.1), we used (1.3). Hence, we have

$$\begin{aligned} \nabla h \nabla \Delta h =& \nabla h \nabla \Big(\frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - c\epsilon h^{\frac{\alpha+\epsilon-1}{\epsilon}} \Big) \\ =& \frac{\epsilon - 1}{\epsilon} \nabla h \nabla \frac{|\nabla h|^2}{h} - c(\alpha + \epsilon - 1) h^{\frac{\alpha+\epsilon-1}{\epsilon}} \frac{|\nabla h|^2}{h} \\ =& \frac{\epsilon - 1}{\epsilon h} \nabla h \nabla (|\nabla h|^2) - \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^4}{h^2} - c(\alpha + \epsilon - 1) h^{\frac{\alpha+\epsilon-1}{\epsilon}} \frac{|\nabla h|^2}{h}. \end{aligned} \tag{2.2}$$

Applying (2.1) and (2.2) into the well-known Bochner formula for h, we have

$$\begin{aligned} \frac{1}{2} \Delta |\nabla h|^2 =& |\nabla^2 h|^2 + \nabla h \nabla \Delta h + \mathrm{Ric}(\nabla h, \nabla h) \\ \geq& \frac{1}{n} (\Delta h)^2 + \nabla h \nabla \Delta h - K |\nabla h|^2 \\ =& \frac{1}{n} \Big(\frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - c\epsilon h^{\frac{\alpha+\epsilon-1}{\epsilon}} \Big)^2 + \frac{\epsilon - 1}{\epsilon} \frac{\nabla h}{h} \nabla (|\nabla h|^2) \\ & - \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^4}{h^2} - c(\alpha + \epsilon - 1) h^{\frac{\alpha+\epsilon-1}{\epsilon}} \frac{|\nabla h|^2}{h} - K |\nabla h|^2 \\ =& \Big(\frac{(\epsilon - 1)^2}{n\epsilon^2} - \frac{\epsilon - 1}{\epsilon} \Big) \frac{|\nabla h|^4}{h^2} - c \left[\frac{n+2}{n} (\epsilon - 1) + \alpha \right] h^{\frac{\alpha+\epsilon-1}{\epsilon}} \frac{|\nabla h|^2}{h} \\ & + \frac{c^2 \epsilon^2}{n} h^{\frac{2(\alpha+\epsilon-1)}{\epsilon}} + \frac{\epsilon - 1}{\epsilon} \frac{\nabla h}{h} \nabla (|\nabla h|^2) - K |\nabla h|^2. \end{aligned} \tag{2.3}$$

By analyzing (2.3) we have the following lemmas.

Lemma 2.1. *Let u be a positive solution to* (1.3) *and let $R_{ij} \geq -Kg_{ij}$ for some nonnegative constant K. Denote $h = u^\epsilon$ with $\epsilon \neq 0$. If $c < 0$ and $\alpha > 0$, then there exists $\epsilon \in (0, 1)$ such that*

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 \geq&\Big(\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}\Big)\frac{|\nabla h|^4}{h^2}\\&+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2.\end{aligned}\tag{2.4}$$

Proof. In (2.3), if $c < 0$ and $\alpha > 0$, we can choose $\epsilon \in (0, 1)$ close enough to 1 such that

$$-c\left[\frac{n+2}{n}(\epsilon-1)+\alpha\right]\geq 0,$$

and then (2.4) follows directly. □

Lemma 2.2. *Let u be a positive solution to* (1.3) *and let $R_{ij} \geq -Kg_{ij}$ for some nonnegative constant K. Denote $h = u^\epsilon$ with $\epsilon \neq 0$. If $c > 0$ and for a fixed α, there exist two positive constants ϵ, δ such that*

$$c\left[\frac{n+2}{n}(\epsilon-1)+\alpha\right]>0\tag{2.5}$$

and

$$\frac{c^2\epsilon^2}{n}-\frac{c}{\delta}\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)>0.\tag{2.6}$$

Then we have

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 \geq&\Big[\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}-\delta c\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)\Big]\frac{|\nabla h|^4}{h^2}\\&+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2.\end{aligned}\tag{2.7}$$

Proof. For a fixed point p, if there exists a positive constant δ such that $h^{\frac{\alpha+\epsilon-1}{\epsilon}} \leq \delta\frac{|\nabla h|^2}{h}$, according to (2.5), then (2.3) becomes

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 \geq&\Big[\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}-\delta c\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)\Big]\frac{|\nabla h|^4}{h^2}\\&+\frac{c^2\epsilon^2}{n}h^{\frac{2(\alpha+\epsilon-1)}{\epsilon}}+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2\\ \geq&\Big[\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}-\delta c\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)\Big]\frac{|\nabla h|^4}{h^2}\\&+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2.\end{aligned}\tag{2.8}$$

On the contrary, at the point p, if $h^{\frac{\alpha+\epsilon-1}{\epsilon}} \geq \delta\frac{|\nabla h|^2}{h}$, then (2.3) becomes

(2.9)
$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 \geq&\Big(\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}\Big)\frac{|\nabla h|^4}{h^2}+\Big[\frac{c^2\epsilon^2}{n}-\frac{c}{\delta}\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)\Big]h^{\frac{2(\alpha+\epsilon-1)}{\epsilon}}\\&+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2\\ \geq&\Big\{\Big(\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}\Big)+\delta^2\Big[\frac{c^2\epsilon^2}{n}-\frac{c}{\delta}\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)\Big]\Big\}\frac{|\nabla h|^4}{h^2}\\&+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2\\ \geq&\Big[\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}-\delta c\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)\Big]\frac{|\nabla h|^4}{h^2}\\&+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2\end{aligned}$$

as long as

$$\frac{c^2\epsilon^2}{n}-\frac{c}{\delta}\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)>0. \tag{2.10}$$

In both cases, (2.7) always holds. We complete the proof of Lemma 2.2. □

In order to obtain the upper bound of $|\nabla h|$ by using the maximum principle, it is sufficient to choose the coefficients of $\frac{|\nabla h|^4}{h^2}$ in (2.4) and (2.7) such that they are positive. In case of Lemma 2.2, we need to choose appropriate ϵ, δ such that

$$\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}-\delta c\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)>0. \tag{2.11}$$

Under the assumption of (2.5), the inequality (2.6) becomes

$$\delta>\frac{nc}{c^2\epsilon^2}\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big) \tag{2.12}$$

and (2.11) becomes

$$\delta<\frac{\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}}{c\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)}. \tag{2.13}$$

In order to ensure we can choose a positive δ, from (2.12) and (2.13), we need to choose an ϵ satisfying

$$\frac{nc}{c^2\epsilon^2}\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)<\frac{\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}}{c\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)}. \tag{2.14}$$

In particular, (2.14) can be written as

$$\begin{aligned}n^2\Big(\frac{n+2}{n}(\epsilon-1)+\alpha\Big)^2<&n\epsilon^2\Big(\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}\Big)\\=&(\epsilon-1)^2-n\epsilon(\epsilon-1),\end{aligned} \tag{2.15}$$

which is equivalent to

$$\begin{aligned}&[n^2+5n+3]\epsilon^2+[2(\alpha-1)(n^2+2n)-(5n+6)]\epsilon\\&\qquad+(\alpha-1)^2n^2-4(\alpha-1)n+3<0.\end{aligned} \tag{2.16}$$

By a direct calculation, under the condition

$$\frac{-(n-4)-\sqrt{n^2+5n+3}}{2(n-1)} < \alpha - 1 < \frac{-(n-4)+\sqrt{n^2+5n+3}}{2(n-1)}, \tag{2.17}$$

we have

(2.18)

$$\begin{aligned}
&[2(\alpha-1)(n^2+2n)-(5n+6)]^2-4[n^2+5n+3][(\alpha-1)^2n^2-4(\alpha-1)n+3]\\
=&4(\alpha-1)^2[(n^2+2n)^2-n^2(n^2+5n+3)]+4(\alpha-1)[4n(n^2+5n+3)\\
&-(n^2+2n)(5n+6)]+(5n+6)^2-12(n^2+5n+3)\\
=&4(\alpha-1)^2[-n^3+n^2]+4(\alpha-1)[-n^3+4n^2]+13n^2\\
=&n^2\Big\{-4(n-1)(\alpha-1)^2-4(n-4)(\alpha-1)+13\Big\}\\
>&0,
\end{aligned}$$

which shows that the quadratic inequality (2.16) with respect to ϵ has two real roots.

Now we are ready to prove the following proposition.

Proposition 2.3. *Let u be a positive solution to (1.3) and let $R_{ij} \geq -Kg_{ij}$ for some nonnegative constant K. If we choose c and α satisfying one of the following two cases:*

(1) $c < 0$ and $\alpha > 0$,

(2) $c > 0$ and $\frac{n+2}{2(n-1)} < \alpha < \frac{2n^2+9n+6}{2n(n+2)}$ with $n \geq 3$,

then we have

$$\frac{1}{2}\Delta|\nabla h|^2 \geq C_1(n,\alpha)\frac{|\nabla h|^4}{h^2} - C_2(n,\alpha)\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2, \tag{2.19}$$

where $C_1(n,\alpha)$ and $C_2(n,\alpha)$ are positive constants.

Proof. We prove this proposition case by case.

(i) The case of $c < 0$ and $\alpha > 0$. In the proof of Lemma 2.1 we see that by choosing an $\epsilon = \epsilon(n,\alpha) \in (0,1)$ such that $\frac{n+2}{n}(\epsilon-1)+\alpha \geq 0$, we get

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 \geq&\Big(\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}\Big)\frac{|\nabla h|^4}{h^2}\\
&+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2.\end{aligned} \tag{2.20}$$

Then we see that $C_1(n,\alpha) = \frac{(\epsilon-1)^2}{n\epsilon^2} - \frac{\epsilon-1}{\epsilon} > 0$ and $C_2(n,\alpha) = \frac{1-\epsilon}{\epsilon} > 0$.

(ii) The case of $c > 0$ and $\frac{n+2}{2(n-1)} < \alpha < \frac{2n^2+9n+6}{2n(n+2)}$ when $n \geq 3$. In this case, (2.5) is equivalent to

$$\epsilon > 1 - \frac{n\alpha}{n+2}. \tag{2.21}$$

We can check

$$\frac{5n+6}{2(n^2+2n)} < \frac{-(n-4)+\sqrt{n^2+5n+3}}{2(n-1)}. \tag{2.22}$$

Hence, when $n \geq 3$, for any α satisfies

$$-\frac{n-4}{2(n-1)} < \alpha - 1 < \frac{5n+6}{2(n^2+2n)}, \tag{2.23}$$

which is equivalent to

$$\frac{n+2}{2(n-1)} < \alpha < \frac{2n^2+9n+6}{2n(n+2)}, \tag{2.24}$$

then (2.21) is satisfied by choosing

$$\epsilon := \tilde{\epsilon} = \frac{(5n+6)-2(\alpha-1)(n^2+2n)}{2(n^2+5n+3)}, \tag{2.25}$$

and it is easy to check that $\epsilon \in (0,1)$.

In particular, we let

$$\delta = \tilde{\delta} := \frac{1}{2}\left[\frac{nc}{c^2\tilde{\epsilon}^2}\Big(\frac{n+2}{n}(\tilde{\epsilon}-1)+\alpha\Big) + \frac{\frac{(\tilde{\epsilon}-1)^2}{n\tilde{\epsilon}^2} - \frac{\tilde{\epsilon}-1}{\tilde{\epsilon}}}{c\Big(\frac{n+2}{n}(\tilde{\epsilon}-1)+\alpha\Big)}\right]. \tag{2.26}$$

Then (2.10) and (2.11) are satisfied and (2.7) becomes

$$\frac{1}{2}\Delta|\nabla h|^2 \geq \tilde{C_1}(n,\alpha)\frac{|\nabla h|^4}{h^2} - \tilde{C_2}(n,\alpha)\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2, \tag{2.27}$$

where positive constants $\tilde{C_1}(n,\alpha)$ and $\tilde{C_2}(n,\alpha)$ are given by

$$\tilde{C_1}(n,\alpha) = \frac{1}{2}\left[\Big(\frac{(\tilde{\epsilon}-1)^2}{n\tilde{\epsilon}^2} - \frac{\tilde{\epsilon}-1}{\tilde{\epsilon}}\Big) - \frac{n}{\tilde{\epsilon}^2}\Big(\frac{n+2}{n}(\tilde{\epsilon}-1)+\alpha\Big)^2\right],$$

$$\tilde{C_2}(n,\alpha) = \frac{4(\alpha-1)n(n+2)+n(2n+5)}{(5n+6)-4(\alpha-1)n(n+2)},$$

respectively.

We conclude the proof of Proposition 2.3. □

Now we are in a position to prove our Theorem 1.4. Denote by $B_p(R)$ the geodesic ball centered at p with radius R. Let ϕ be a cut-off function (see [9]) satisfying $\mathrm{supp}(\phi) \subset B_p(2R)$, $\phi|_{B_p(R)} = 1$, and

$$\frac{|\nabla\phi|^2}{\phi} \leq \frac{C}{R^2}, \tag{2.28}$$

$$-\Delta\phi \leq \frac{C}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right), \tag{2.29}$$

where C is a constant depending only on n. We define $G = \phi|\nabla h|^2$ and will apply the maximum principle to G on $B_p(2R)$. Moreover, we assume G attains its maximum at the point $x_0 \in B_p(2R)$ and assume $G(x_0) > 0$ (otherwise the proof is trivial). Then at the point x_0, it holds that

$$\Delta G \leq 0, \quad \nabla(|\nabla h|^2) = -\frac{|\nabla h|^2}{\phi}\nabla\phi$$

and

$$\begin{aligned}0 \geq&\Delta G\\ =&\phi\Delta(|\nabla h|^2)+|\nabla h|^2\Delta\phi+2\nabla\phi\nabla|\nabla h|^2\\ =&\phi\Delta(|\nabla h|^2)+\frac{\Delta\phi}{\phi}G-2\frac{|\nabla\phi|^2}{\phi^2}G\\ \geq&2\phi\Big[C_1(n,\alpha)\frac{|\nabla h|^4}{h^2}-C_2(n,\alpha)\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2\Big]\\ &+\frac{\Delta\phi}{\phi}G-2\frac{|\nabla\phi|^2}{\phi^2}G\\ =&2C_1(n,\alpha)\frac{G^2}{\phi h^2}+2C_2(n,\alpha)\frac{G}{\phi}\nabla\phi\frac{\nabla h}{h}-2KG+\frac{\Delta\phi}{\phi}G-2\frac{|\nabla\phi|^2}{\phi^2}G,\end{aligned} \tag{2.30}$$

where, in the second inequality, the estimate (2.27) is used. Multiplying both sides of (2.30) by $\frac{\phi}{G}$ yields

$$2C_1(n,\alpha)\frac{G}{h^2}\leq -2C_2(n,\alpha)\nabla\phi\frac{\nabla h}{h}+2\phi K-\Delta\phi+2\frac{|\nabla\phi|^2}{\phi}. \tag{2.31}$$

Inserting the Cauchy inequality

$$\begin{aligned}-2C_2(n,\alpha)\nabla\phi\frac{\nabla h}{h}\leq&2C_2(n,\alpha)|\nabla\phi|\frac{|\nabla h|}{h}\\ \leq&\frac{C_2^2(n,\alpha)}{C_1(n,\alpha)}\frac{|\nabla\phi|^2}{\phi}+C_1(n,\alpha)\frac{G}{h^2}\end{aligned}$$

into (2.31) yields

$$C_1(n,\alpha)\frac{G}{h^2}\leq 2\phi K-\Delta\phi+\Big(2+\frac{C_2^2(n,\alpha)}{C_1(n,\alpha)}\Big)\frac{|\nabla\phi|^2}{\phi}. \tag{2.32}$$

Hence, for $x\in B_p(R)$, we have

$$\begin{aligned}C_1(n,\alpha)G(x)\leq&C_1(n,\alpha)G(x_0)\\ \leq&h^2(x_0)\left[2K+\frac{C(n,\alpha)}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right].\end{aligned} \tag{2.33}$$

It shows that

$$|\nabla u|^2(x)\leq C(n,\alpha)M^2\left[K+\frac{1}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right] \tag{2.34}$$

and, hence,

$$|\nabla u(x)|\leq C(n,\alpha)M\sqrt{K+\frac{1}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)}. \tag{2.35}$$

We complete the proof of Theorem 1.4.

3. Appendix

Here we give a sketch of the proof of Theorem 1.2. The interested readers can consult Yang's paper [10] for details. Assume that $u(x)$ is a positive solution to (1.3) with $c>0$ and $\alpha\leq 1$. Let $f=\log u$. Then we have

$$\Delta f=-|\nabla f|^2-cu^{\alpha-1}. \tag{3.1}$$

Let $F = |\nabla f|^2 + cu^{\alpha-1}$. Then we have $\Delta f = -F$, and by the well-known Weitzenbock–Bochner formula we have

$$\Delta|\nabla f|^2 = 2\nabla f\nabla\Delta f + 2|\nabla^2 f|^2 + 2\mathrm{Ric}(\nabla f, \nabla f),$$

where $\nabla^2 f$ is the Hessian of f. Since $c > 0$ and $\alpha \leq 1$, we obtain by the above two inequalities

$$\begin{aligned}\Delta F =&\Delta|\nabla f|^2 + c\Delta u^{\alpha-1}\\ =&-2\nabla f\nabla F + 2|\nabla^2 f|^2 + 2\mathrm{Ric}(\nabla f, \nabla f)\\ &+ c(1-\alpha)u^{\alpha-1}F + c(1-\alpha)^2 u^{\alpha-1}|\nabla f|^2\\ \geq&-2\nabla f\nabla F + \frac{2}{n}F^2 - 2KF\end{aligned}$$

on $B_p(2R)$, where we used the fact that $|\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2$. Then following Yang's proof line by line we finish the proof of Theorem 1.2.

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References

[1] Kevin Brighton, *A Liouville-type theorem for smooth metric measure spaces*, J. Geom. Anal. **23** (2013), no. 2, 562–570. MR3023849

[2] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), no. 4, 525–598. MR615628

[3] Zongming Guo and Juncheng Wei, *Hausdorff dimension of ruptures for solutions of a semilinear elliptic equation with singular nonlinearity*, Manuscripta Math. **120** (2006), no. 2, 193–209. MR2234248

[4] Guangyue Huang and Bingqing Ma, *Gradient estimates and Liouville type theorems for a nonlinear elliptic equation*, Arch. Math. (Basel) **105** (2015), no. 5, 491–499. MR3413924

[5] Guangyue Huang and Bingqing Ma, *Hamilton's gradient estimates of porous medium and fast diffusion equations*, Geom. Dedicata **188** (2017), 1–16. MR3639621

[6] G. Y. Huang, Z. Li, *Liouville type theorems of a nonlinear elliptic equation for the V-Laplacian*, Anal. Math. Phys., **8** (2018), no. 1, 123-134. DOI 10.1007/s13324-017-0168-6.

[7] Jiayu Li, *Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equations on Riemannian manifolds*, J. Funct. Anal. **100** (1991), no. 2, 233–256. MR1125225

[8] Bin Qian, *Yau's gradient estimates for a nonlinear elliptic equation*, Arch. Math. (Basel) **108** (2017), no. 4, 427–435. MR3627401

[9] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu; Translated from the Chinese by Ding and S. Y. Cheng; Preface translated from the Chinese by Kaising Tso, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994. MR1333601

[10] Yun Yan Yang, *Gradient estimates for the equation $\Delta u + cu^{-\alpha} = 0$ on Riemannian manifolds*, Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 6, 1177–1182. MR2644055

[11] Shing Tung Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228. MR0431040

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