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A RECURSION THEORETIC PROPERTY OF $\boldsymbol{\Sigma}^1_1$ EQUIVALENCE RELATIONS

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Abstract. Let E be a $\boldsymbol{\Sigma}^1_1$ equivalence relation on 2^ω which does not have perfectly many equivalence classes. For $a \in 2^\omega$, define L^a_E to be the set $\{[x]_E : (\exists y)(y \in [x]_E \text{ and } \omega_1^{\langle a,y\rangle} = \omega_1^a)\}$. For a Turing cone of a's, L^a_E is countable. This is proved assuming $\boldsymbol{\Pi}^1_2$-determinacy.

1. Background

The descriptive set theory of equivalence relations has been an active field of research for about 25 years. Gao [2] is a reference for that field. Our new theorem in that field will be stated in §2 and proved in §3. In §1, we review some background material.

Consider $\boldsymbol{\Sigma}^1_1$ equivalence relations on the space 2^ω. A special case of this is an equivalence relation E' obtained from a $\boldsymbol{\Sigma}^1_1$ equivalence relation E and an E-invariant Borel set B by defining $xE'y$ iff $[xEy$ or $(x \notin B$ and $y \notin B)]$. An even more special case is the orbit equivalence relation of a Polish G-space (that is, a continuous action by a Polish group, G, on 2^ω), restricted (as above) to a Borel set of orbits. And a still more special case is the isomorphism relation for $\mathcal{L}$-structures with universe ω ($\mathcal{L}$ a fixed countable vocabulary), restricted to the models of an $\mathcal{L}_{\omega_1,\omega}$ sentence.

For E an equivalence relation on 2^ω and $x \in 2^\omega$, $[x]_E$ denotes the E-equivalence class of x.

We say that an equivalence relation E on 2^ω has *perfectly many* equivalence classes if there is a nonempty perfect set $P \subset 2^\omega$ such that no two members of P are E-equivalent. If E has perfectly many classes, then it has continuum many classes. For $\boldsymbol{\Sigma}^1_1$ equivalence relations, "perfectly many" is absolute (while "continuum many" is not).

Theorem 1.1 (Burgess; see [2, 9.1.5]). *Let E be a $\boldsymbol{\Sigma}^1_1$ equivalence relation on 2^ω. Exactly one of the following three cases holds:*

(1) *E has perfectly many equivalence classes.*
(2) *E has $\aleph_1$ and not perfectly many equivalence classes.*
(3) *E has countably many equivalence classes.*

Theorem 1.1 is called the *Burgess Trichotomy.*

The Topological Vaught Conjecture and the Vaught Conjecture for $\mathcal{L}_{\omega_1,\omega}$ (which are open) state that in the previously mentioned special cases—orbit equivalence

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relations and isomorphism—case (2) of the Burgess Trichotomy does not occur. As the following example shows, for arbitrary $\mathbf{\Sigma}^1_1$ equivalence relations, case (2) can occur.

Example 1.2. Let xE^*y iff [(neither x nor y is an ordinal code) or (x and y encode the same countable ordinal)].

The above example illustrates another difference between arbitrary $\mathbf{\Sigma}^1_1$ equivalence relations and orbit equivalence relations: all orbits are Borel sets [2, 3.3.2], while there is an E^*-equivalence class which is not Borel.

When a $\mathbf{\Sigma}^1_1$ equivalence relation is Borel, case (2) cannot occur, because of the following theorem.

Theorem 1.3 (Silver; see [2, 5.3.5])**.** *Let E be a Borel equivalence relation on 2^ω. Exactly one of the following two cases holds:*

(1) *E has perfectly many equivalence classes.*
(2) *E has countably many equivalence classes.*

Theorem 1.3 is called the *Silver Dichotomy.*

We next take up some recursion theoretic matters. The space 2^ω plays two roles: it is the space on which the equivalence relation exists; and it is also the space of oracles. We continue to use letters from the end of the alphabet when considering the space of the equivalence relation, and we use letters from the beginning of the alphabet to denote oracles.

In the last few years there have been a number of recursion theoretic results involving counterexamples to Vaught's Conjecture. A well-known example of this trend is the theorem of Montalbán [6] that not having perfectly many isomorphism types is equivalent to "hyperarithmetic is recursive" on a Turing cone. A recent paper of Gregoriades [3] is a part of that trend. That paper is the motivation for this paper, and it will be discussed in §2.

We shall have occasion to consider (lightface) Σ^1_1 equivalence relations. Special cases of this include the orbit equivalence relation of a recursive Polish G-space (as defined in [3]) and isomorphism when the vocabulary is recursive.

2. Some theorems and questions

The background material in §1 concerns the three cases of the Burgess Trichotomy applied to *all* the equivalence classes. We now change our point of view and consider the three cases applied only to the equivalence classes that are "low" in the sense that they have members which cannot compute large countable ordinals.

Definition 2.1. Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on 2^ω, and let $a \in 2^\omega$. Define $L^a_E = \{[x]_E : (\exists y)(y \in [x]_E \text{ and } \omega_1^{\langle a,y\rangle} = \omega_1^a)\}$.

In [3], Gregoriades proved Theorems 2.2 and 2.3, below, and asked Question 2.4, below.

Theorem 2.2. *For every recursive Polish G-space with orbit equivalence relation E, the following are equivalent:*

(1) *E does not have perfectly many equivalence classes.*
(2) *For all $a \in 2^\omega$, L^a_E is countable.*

Theorem 2.3. *Let E be an arbitrary Σ^1_1 equivalence relation on 2^ω. If for all $a \in 2^\omega$, L^a_E is countable, then E does not have perfectly many equivalence classes.*

Question 2.4. Is the following statement true? If E is an arbitrary Σ^1_1 equivalence relation on 2^ω which does not have perfectly many equivalence classes, then for all $a \in 2^\omega$, L^a_E is countable.

The (1) implies (2) direction of Theorem 2.2 is proved by establishing that E restricted to L^a_E is a Borel equivalence relation, and then applying Theorem 1.3. In general, the E-equivalence classes need not be Borel (see Example 1.2), so clearly E restricted to L^a_E need not be a Borel equivalence relation.

This paper does not answer Question 2.4. It does, however, provide a positive answer with "for all a" weakened to "for a Turing cone of a's".

Theorem 2.5. *Assume $\mathbf{\Pi}^1_2$-determinacy. Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on 2^ω which does not have perfectly many equivalence classes. There exists a b such that for all $a \geq_T b$, L^a_E is countable.*

Theorem 2.5 will be proved in §3.

Corollary 2.6. *Assume $\mathbf{\Pi}^1_2$-determinacy. Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on 2^ω. The following are equivalent:*

(1) *E does not have perfectly many equivalence classes.*
(2) *There exists a b such that for all $a \geq_T b$, L^a_E is countable.*
(3) *There exists a b such that for all $a \geq_T b$, L^a_E does not have perfectly many equivalence classes.*

Proof. To prove (3) implies (1), suppose (1) is false. Let $P \subset 2^\omega$ be a perfect set with no two members E-equivalent. Let $b \in 2^\omega$ be such that P is the set of branches of a recursive-in-b pruned tree on 2. Let $a \geq_T b$. Then for a comeager subset C of P, for all $y \in C$, $\omega_1^{\langle a,y\rangle} = \omega_1^a$. $\square$

So while a $\mathbf{\Sigma}^1_1$ equivalence relation E need not satisfy the Silver Dichotomy, for a Turing cone of a's, L^a_E does satisfy that dichotomy.

Corollary 2.6 yields a recursion theoretic characterization of counterexamples to the Silver Dichotomy (that is, counterexamples to "Vaught's Conjecture") for $\mathbf{\Sigma}^1_1$ equivalence relations, E: E is a counterexample iff E has uncountably many equivalence classes, but for a Turing cone of a's, L^a_E has only countably many.

Perhaps the most surprising thing in this paper is the use of an axiom as strong as $\mathbf{\Pi}^1_2$-determinacy to prove a theorem about $\mathbf{\Sigma}^1_1$ sets. This leads to an obvious question.

Question 2.7. Is Theorem 2.5 provable from an axiom which is weaker than $\mathbf{\Pi}^1_2$-determinacy?

3. The proof

The rest of this paper consists of a proof of Theorem 2.5. The axiom $\mathbf{\Pi}^1_2$-determinacy is always assumed.

Fix a $\mathbf{\Sigma}^1_1$ equivalence relation E on 2^ω with uncountably many but not perfectly many equivalence classes. We prove Theorem 2.5 for this E, via a sequence of lemmas.

Lemma 3.1. *There exists a prewellordering, $\trianglelefteq$, of 2^ω satisfying the following three properties:*

(a) *$\trianglelefteq$ has length ω_1.*

(b) *The levels of* $\trianglelefteq$ *are precisely the* E*-equivalence classes.*
(c) *The relation* $\{(x,y) : x \trianglelefteq y\}$ *is* $\boldsymbol{\Delta}^1_2$.

Proof. This is a theorem of Burgess [1], proved under the assumption that sharps exist. By Harrington [4], the existence of sharps follows from $\boldsymbol{\Pi}^1_1$-determinacy. $\square$

Fix $\trianglelefteq$ satisfying Lemma 3.1. For $x \in 2^\omega$, let $||x||$ denote the ordinal which is the level of x in $\trianglelefteq$.

Let WO be the set of codes for infinite countable ordinals. For $w \in \mathrm{WO}$, $|w|$ denotes the ordinal encoded by w, and $\leq_w$ denotes the wellordering of ω corresponding to w.

Lemma 3.2. *There exists a* $b \in 2^\omega$ *satisfying the two conditions below:*
(a) E *is* $\Sigma^1_1(b)$.
(b) *For any* $w \in \mathrm{WO}$*, for any* $\xi < |w|$*, there exists an* $x \in 2^\omega$ *such that* $||x|| = \xi$ *and* $x \leq_T \langle b, w\rangle$.

Proof. Consider the following game. Player I plays $w_{\mathrm{I}} \in 2^\omega$ and Player II plays $w_{\mathrm{II}} \in 2^\omega$ and $y \in 2^\omega$, where y encodes an infinite sequence $\langle y_i\rangle$ from 2^ω in the usual way. Player I must play $w_{\mathrm{I}} \in \mathrm{WO}$; if not, he loses. Assuming $w_{\mathrm{I}} \in \mathrm{WO}$, Player II wins the round of the game iff:

$$[w_{\mathrm{II}} \in \mathrm{WO} \ \text{ and } \ |w_{\mathrm{II}}| \geq |w_{\mathrm{I}}| \ \text{ and } \ (\forall i, j \in \omega)(i \leq_{w_{\mathrm{II}}} j \leftrightarrow y_i \trianglelefteq y_j)$$
$$\text{and } \ (\forall i \in \omega)(\forall z \in 2^\omega)(\text{if } \ z \trianglelefteq y_i \ \text{ then } \ (\exists j \in \omega)(||z|| = ||y_j||))].$$

This is a Solovay game: the Boundedness Theorem for WO implies that Player I cannot have a winning strategy. By Lemma 3.1(c), the payoff set for Player II is $\boldsymbol{\Pi}^1_2$, so the game is determined. Let σ be a winning strategy for Player II. Let b be of large enough Turing degree that $\sigma \leq_T b$ and E is $\Sigma^1_1(b)$. $\square$

Fix b satisfying Lemma 3.2. We prove that this b satisfies Theorem 2.5 for E.

Let $\mathcal{L}$ be the language with one binary relation symbol and consider the logic action for $\mathcal{L}$: $S_\infty \curvearrowright X_{\mathcal{L}}$. (For details, see [2, 3.6].) For $v \in X_{\mathcal{L}}$, if v encodes a linear ordering of ω, then $\leq_v$ denotes that linear ordering. Given an orbit $\mathcal{O}$, we abuse the language and say that $C \subset \mathcal{O}$ is a "comeager subset of $\mathcal{O}$" when we mean that for some (equivalently, for any) $v \in \mathcal{O}$, $\{g \in S_\infty : g \cdot v \in C\}$ is a comeager subset of S_∞.

An ordinal is *b-admissible* if it is ω_1^c for some $c \geq_T b$. The order-type of the rationals is denoted η.

Definition 3.3. Let $S = \{v \in X_{\mathcal{L}}$: there exists a b-admissible ordinal α such that v encodes a linear ordering of order-type $\alpha(1+\eta)\}$.

Lemma 3.4. S *is* $\Sigma^1_1(b)$.

Proof. It is a well-known theorem, originally due to Harrison [5], that $v \in S$ iff

$$(\exists c \geq_T b)(\exists u \in X_{\mathcal{L}})(u \text{ is in the orbit of } v \text{ and } u \leq_T c \text{ and } u$$
$$\text{encodes a linear ordering of } \omega \text{ and no terminal segment of } \leq_u$$
$$\text{is a wellordering and } \leq_u \text{ has no } \Delta^1_1(c) \text{ descending sequences}). \quad \square$$

Definition 3.5.

(a) Let

$$T = \{(v,x) \in X_{\mathcal{L}} \times 2^\omega : v \in S \text{ and (for a comeager set of elements } v' \text{ in the orbit of } v)(\exists x' \in 2^\omega)(x' \leq_T \langle b, v'\rangle \text{ and } x'Ex)\}.$$

(b) For all $v \in S$, $T_v = \{x : (v,x) \in T\}$.

(c) For all $y \in 2^\omega$, $T^y = \{v : (v,y) \in T\}$.

Lemma 3.6.

(a) *T is $\Sigma_1^1(b)$.*

(b) *For all $v, v' \in S$, for all $x, x' \in 2^\omega$, if $v' \in S_\infty \cdot v$ and xEx' and $(v,x) \in T$, then $(v', x') \in T$.*

(c) *For all $v \in S$, T_v includes only countably many E-equivalence classes.*

(d) *For all $y \in 2^\omega$, there exists a $v \in S$ such that $y \in T_v$.*

Proof.

(a) The definition of T in Definition 3.5(a) establishes that T is $\Sigma_1^1(b)$, using Lemmas 3.2(a) and 3.4, and the fact that the pointclass $\Sigma_1^1(b)$ is closed under category quantifiers (essentially [2, 3.2.9]).

(b) Obvious.

(c) Let $v \in S$. Suppose $[x]_E \subset T_v$. Then there is a basic neighborhood N of S_∞ and an $i \in \omega$ such that for a comeager-in-N set of g, the recursive-in-$\langle b, g \cdot v\rangle$ partial function with index i is a total function x' from ω into 2 (that is, a point in 2^ω) and xEx'. For any N and i there is at most one such E-equivalence class.

(d) Let $y \in 2^\omega$. By Lemma 3.2(b), for any $w \in \mathrm{WO}$ such that $|w| > ||y||$, there exists an x such that $x \leq_T \langle b, w\rangle$ and $||x|| = ||y||$; by Lemma 3.1(b), xEy. By Lemma 3.1(a), there is a countable b-admissible ordinal $\alpha > ||y||$. If v is any element of the orbit $\alpha(1+\eta)$, then $\leq_v$ has an initial segment isomorphic to an ordinal greater than $||y||$. Hence $y \in T_v$. $\square$

For α a b-admissible ordinal, let $T_\alpha = T_v$ for some (equivalently by Lemma 3.6(b), for all) v in the orbit $\alpha(1+\eta)$.

To complete the proof of Theorem 2.5, let $a \geq_T b$. We show that L_E^a is countable.

Let $M = \{[x]_E : (\exists \alpha \leq \omega_1^a)(\alpha \text{ is } b\text{−admissible and } x \in T_\alpha)\}$. By Lemma 3.6(c), M contains only countably many E-equivalence classes. Therefore, it will suffice to show that $L_E^a \subset M$. To prove this, fix $y \in 2^\omega$ with $[y]_E \notin M$; we show that $\omega_1^{\langle a,y\rangle} > \omega_1^a$.

Part (d) of Lemma 3.6 tells us that $T^y \neq \emptyset$ and part (a) of that lemma tells us that T^y is $\Sigma_1^1(a,y)$. So by the Gandy Basis Theorem [2, A.1.4], there exists a $v \in T^y$ with $\omega_1^{\langle a,y,v\rangle} = \omega_1^{\langle a,y\rangle}$. By definition of T, v is in the orbit $\beta(1+\eta)$ for some b-admissible ordinal β; and by definition of M, $\beta > \omega_1^a$. So clearly the linear ordering $\leq_v$ has a wellordered initial segment of order-type greater than ω_1^a. Thus

$$\omega_1^{\langle a,y\rangle} = \omega_1^{\langle a,y,v\rangle} \geq \omega_1^v > \omega_1^a.$$

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