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ON POLETSKY THEORY OF DISCS IN COMPACT MANIFOLDS

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Abstract. We provide a direct construction of Poletsky discs via local arc approximation and a Runge-type theorem by A. Gournay [Geom Funct. Anal. **22** (2012), pp. 311-351].

Let $\mathbb{D} = \{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$ denote the open unit disc. Given a smooth (almost) complex manifold M and $p \in M$, we denote by $\mathcal{O}(\overline{\mathbb{D}}, M, p)$ the set of smooth maps $u \colon \overline{\mathbb{D}} \to M$ that are (pseudo)holomorphic in some neighborhood of $\overline{\mathbb{D}}$ and satisfy $u(0) = p$. Given $p \in M$, $\epsilon > 0$, and an open set $U \subset M$, we call an element of $u \in \mathcal{O}(\overline{\mathbb{D}}, M, p)$ a *Poletsky disc* (associated to p, ϵ, and U) if most of its boundary lies in U; that is, there exists an exceptional set $E \subset [0, 2\pi)$ of Lebesgue measure $|E| < \epsilon$ and such that $u(e^{it}) \in U$ for $t \notin E$. Such discs were used by E. Poletsky [11] in order to characterize the polynomial hull for compact sets in $\mathbb{C}^n$. Similarly, they can describe the projective hull of a compact set in complex projective spaces [2, 8].

All of the above-mentioned characterizations are based on the following explicit formula for the largest plurisubharmonic minorant of a given upper-semicontinuous function f on M:

$$\hat{f}(p) = \inf \left\{ \int_0^{2\pi} f \circ u(e^{it}) \frac{dt}{2\pi} : \quad u \in \mathcal{O}(\overline{\mathbb{D}}, M, p) \right\}. \tag{1}$$

The formula was proved to be valid on any complex manifold by J. P. Rosay (see [7] for related results), who also observed that if M admits no nonconstant bounded plurisubharmonic function, there exists a Poletsky disc for any $p \in M$, $\epsilon > 0$, and open set $U \subset M$ [12, Corollary 0.2]. Indeed, in this case the minorant of the negative indicator function $f = -\chi_U$ equals $\hat{f} \equiv -1$; hence the existence of the desired discs follows directly from the definition of the infimum in (1).

In this paper we present a new, direct proof of this corollary valid for a certain class of manifolds admitting a Runge-type approximation provided by A. Gournay [5]. In particular, we give a partial answer to Rosay's question raised in [12, Section 5]: given a compact complex manifold, can a Poletsky disc be provided without using (1)? Moreover, our theorem includes some examples of almost complex manifolds.

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Theorem 1. *Let M be a smooth, connected compact manifold equipped with a regular almost complex structure and admitting a doubly tangent property. Given a point $p \in M$, a positive constant $\epsilon > 0$, and any open set $U \subset M$, there exist a disc $u \in \mathcal{O}(\overline{\mathbb{D}}, M, p)$ and a set $E \subset [0, 2\pi)$ such that $|E| < \epsilon$ and $u(e^{it}) \in U$ for $t \notin E$.*

The assumptions in the above theorem are rather technical and should be read as "such that Gournay's approximation result applies" (we further discuss them in §2). However, they are fulfilled for a wide class of compact complex manifolds including complex projective spaces and Grassmannians. Moreover, as mentioned, they are valid for some manifolds equipped with a nonintegrable almost complex structure J (e.g., $\mathbb{C}P^n$ with J tamed by the standard symplectic form) [5, p. 313]. Note that for the above the Poletsky-Rosay formula (1) is proved only in a low-dimensional case [6]. Hence for $\dim_{\mathbb{R}} M \geq 6$ and a nonintegrable J, the theorem is new.

Finally, let us remark that in §1 we present another original statement that will be needed in the proof of the main theorem: based on [1] we provide a Mergelyan-type result for maps defined on smooth arcs (Theorem 5).

1. The local arc approximation

Let M be a smooth real manifold of even dimension. A $(1,1)$-tensor field $J\colon TM \to TM$ satisfying $J^2 = -Id$ is called an *almost complex structure*. A differentiable map $u : (M', J') \longrightarrow (M, J)$ between two almost complex manifolds is (J', J)-*holomorphic* if for every $p \in M'$ we have

$$J(u(p)) \circ d_p u = d_p u \circ J'(p). \tag{2}$$

We deal with two simple cases, J-*holomorphic discs* $u\colon \mathbb{D} \to M$ and J-*holomorphic spheres* $u\colon \mathbb{C}P^1 \to M$.

We denote by J_{st} the standard integrable structure on $\mathbb{C}^n$ for any $n \in \mathbb{N}$. In local coordinates $z \in \mathbb{R}^{2n}$ an almost complex structure J is represented by an $\mathbb{R}$-linear operator satisfying $J(z)^2 = -I$; hence (2) equals

$$u_y = J(u)u_x. \tag{3}$$

Further, if $J + J_{st}$ is invertible along u, we have

$$\mathcal{F}(u) = u_{\bar{\zeta}} + A(u)\overline{u_\zeta} = 0, \tag{4}$$

where $\zeta = x + iy$ and $A(z)(v) = (J_{st} + J(z))^{-1}(J(z) - J_{st})(\bar{v})$ is a complex linear endomorphism for every $z \in \mathbb{C}^n$. We call A the *complex matrix of* J and denote by $\mathcal{J}$ the set of all smooth structures on $\mathbb{R}^{2n}$ satisfying the condition $\det(J + J_{st}) \neq 0$.

In [1] the approximation theory was developed for the operator $\mathcal{F}$ defined as in (4) and evaluated in functions admitting a Sobolev weak derivative. In particular, given $\varphi \in W^{1,p}(\mathbb{D})$, $p > 2$, a bounded right inverse Q_φ was constructed for the derivative $d_\varphi \mathcal{F}$ and the following version of the Implicit Function Theorem was applied.

Theorem 2 (Implicit Function Theorem)**.** *Let X and Y be two Banach spaces and consider a map $\mathcal{F} : U \subset X \to Y$ of class $\mathcal{C}^1$ defined on an open set $U \subset X$. Let $x_0 \in U$. Assume that the differential $d_{x_0}\mathcal{F}$ admits a bounded right inverse, denoted by Q_{x_0}. Fix $c_0 > 0$ such that $\|Q\| \leq c_0$ and $\eta > 0$ such that if $\|x - x_0\| < \eta$, then $x \in U$ and*

$$\|d_x\mathcal{F} - d_{x_0}\mathcal{F}\| \leq \frac{1}{2c_0}.$$

Then if $\|\mathcal{F}(x_0)\| < \dfrac{\eta}{4c_0}$, *there exists* $x \in U$ *such that* $\mathcal{F}(x) = 0$ *and*

$$\|x - x_0\| \le 2c_0 \|\mathcal{F}(x_0)\|.$$

In this paper we use the same approach to develop a similar statement for the operator $\mathcal{F}\colon \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}}) \to \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})$. That is, we prove an analogue of [1, Theorem 5] valid for Hölder spaces (we omit the existence part for Q_φ since the proof is the same as in [1, Theorem 2] and [15]). Note that these additional regularity conditions are needed since in the present paper we apply the statement to shrinking neighborhoods of an arc in order to obtain a $\mathcal{C}^0$-approximation. Since the diameter of such sets admits no lower bound, the $W^{1,p}$-result along with the Sobolev embedding theorem does not suffice.

Theorem 3. *Let* $0 < \alpha < 1$. *Let* $J \in \mathcal{J}$ *and let* A *be its complex matrix. We define* $\mathcal{F} : \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}}) \to \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})$ *to be the operator given by*

$$\mathcal{F}(u) = u_{\bar\zeta} + A(u)\overline{u_\zeta}.$$

For every $c_0 > 0$, *there exists* $\delta > 0$ *such that for any* $\varphi \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})$ *satisfying*

$$\|\varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})} \le c_0, \ \ \|Q_\varphi\| \le c_0, \ \ \|\mathcal{F}(\varphi)\|_{\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})} < \delta,$$

there exists a J*-holomorphic disc* $u \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})$ *such that*

$$\|u - \varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})} \le 2c_0 \|\mathcal{F}(\varphi)\|_{\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})}.$$

Proof. The key step is to prove that the derivative of $\mathcal{F}$ is locally Lipschitz under present assumptions. That is, there exists $c > 0$ such that

$$\|d_{\tilde\varphi}\mathcal{F} - d_\varphi\mathcal{F}\| \le c\|\tilde\varphi - \varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})}$$

for any $\tilde\varphi \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})$ in a $\mathcal{C}^{1,\alpha}$-neighborhood of φ, say $\|\tilde\varphi - \varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})} < 1$. If such a statement is valid, then we can simply set

$$\eta = \min\left\{1, \frac{1}{2cc_0}\right\} \quad \text{and} \quad \delta = \frac{\eta}{4c_0}$$

and apply Theorem 2 to $X = \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})$, $Y = \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})$, and $x_0 = \varphi$.

Let $h \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})$ and let $\|\varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})} < c_0$. We need to prove that

$$\|d_{\tilde\varphi}\mathcal{F}(h) - d_\varphi\mathcal{F}(h)\|_{\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})} \le c\|\tilde\varphi - \varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})}\|h\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})}. \tag{5}$$

Note that

$$d_\varphi\mathcal{F}(h) = h_{\bar\zeta} + A(\varphi)\overline{h_\zeta} + d_\varphi A(h)\ \overline{\varphi_\zeta},$$

where $d_\varphi A(h) = \displaystyle\sum_{j=1}^n \frac{\partial A}{\partial z_j}(\varphi)h_j + \frac{\partial A}{\partial \bar z_j}(\varphi)\bar h_j$. We write

$$d_{\tilde\varphi}\mathcal{F}(h) - d_\varphi\mathcal{F}(h) = I + II + III,$$

where

$$\left\{\begin{array}{lcl} I & = & (A(\tilde\varphi) - A(\varphi))\,\overline{h_\zeta}, \\ II & = & (d_{\tilde\varphi}A - d_\varphi A)(h)\overline{\tilde\varphi_\zeta}, \\ III & = & d_\varphi A(h)\left(\overline{\tilde\varphi_\zeta - \varphi_\zeta}\right). \end{array}\right.$$

Let us estimate each of the three parts.

Since $\mathbb{D}$ is convex and bounded, the following embeddings are compact:

$$\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}}) \subset \mathcal{C}^1(\overline{\mathbb{D}}) \subset \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}}).$$

Therefore, we can bound the $\mathcal{C}^{0,\alpha}$-norm of the entries in $A(\tilde{\varphi}) - A(\varphi)$ by their $\mathcal{C}^1$-norm. This implies the existence of a constant $c_1 > 0$ depending on $\|\tilde{\varphi}\|_{\mathcal{C}^1(\overline{\mathbb{D}})} < 1+c_0$ such that

$$\|I\|_{\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})} \leq c_1 \|\tilde{\varphi} - \varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})} \|h\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})}.$$

By a similar argument for $\frac{\partial A}{\partial z_j}(\tilde{\varphi}) - \frac{\partial A}{\partial z_j}(\varphi)$ and $\frac{\partial A}{\partial \bar{z}_j}(\tilde{\varphi}) - \frac{\partial A}{\partial \bar{z}_j}(\varphi)$ we get

$$\|II\|_{\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})} \leq c_0 c_2 \|\tilde{\varphi} - \varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})} \|h\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})}.$$

Finally, we have

$$\|III\|_{\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})} \leq c_3 \|\tilde{\varphi} - \varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})} \|h\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})},$$

where $c_3 > 0$ depends on the $\mathcal{C}^{0,\alpha}$-norm evaluated on the coefficients of $\frac{\partial A}{\partial z_j}(\varphi)$ and $\frac{\partial A}{\partial \bar{z}_j}(\varphi)$. The latter may be bounded by a constant times c_0. $\square$

As stated above, we will apply this approximation result to a shrinking family of sets in $\mathbb{D}$. Let $\{\Omega_m\}_{m\in\mathbb{N}}$ be such a family. Throughout the rest of this section we use the following convention: given $\varphi \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})$, we denote by φ_m its restriction to the set Ω_m and by $\mathcal{F}_m$ the corresponding operator defined as in (4) but mapping from $\mathcal{C}^{1,\alpha}(\overline{\Omega}_m)$ to $\mathcal{C}^{0,\alpha}(\overline{\Omega}_m)$.

Proposition 4. *Let $\{\Omega_m\}_{m\in\mathbb{N}}$ be a shrinking family of sets in $\mathbb{D}$ and let $\varphi \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})$ be a map satisfying the limit condition*

$$\lim_{m\to\infty} \|\mathcal{F}_m(\varphi_m)\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega}_m)} = 0.$$

Then for every $m \in \mathbb{N}$ large enough, there exists a J-holomorphic map $u_m \colon \Omega_m \to \mathbb{R}^{2n}$ such that

$$\lim_{m\to\infty} \|u_m - \varphi_m\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega}_m)} = 0.$$

Proof. We need to verify that the constants $c_0 > 0$ and $\delta > 0$ in Theorem 2 can be chosen independently of the sets Ω_m. For this, we need two bounded linear extension operators $E_k \colon \mathcal{C}^{j,\alpha}(\overline{\mathbb{D}}) \to \mathcal{C}^{j,\alpha}(\overline{\Omega}_m)$, $j = 0, 1$ (see, e.g., [14, Theorem 4, p. 177]). The remarkable fact is that their norms can be bounded by a constant $K_j \geq 1$ that is independent of the sets Ω_m.

For points in Ω_m and $h \in \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})$ we have $d_{\varphi_m}\mathcal{F}_m(h_m) = d_\varphi \mathcal{F}(h)$. Hence we can construct a bounded right inverse Q_m of $d_{\varphi_m}\mathcal{F}_m$ by using the right inverse Q_φ of $d_\varphi\mathcal{F}$. Indeed, given $g_m \in \mathcal{C}^{0,\alpha}(\overline{\Omega}_m)$, we take its extension $g = E_0(g_m)$ and proclaim $h_m = Q_{\varphi_m} g_m$ to be the restriction of $h = Q_\varphi g$ to Ω_m. If $\|Q_\varphi\| < c_0$ and $\|\varphi\|_{\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})} < c_0$, the following estimate is valid:

$$\|Q_{\varphi_m} g_m\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega}_m)} \leq \|Q_\varphi g\|_{\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})} \leq c_0 K_0 \cdot \|g_m\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega}_m)}.$$

Moreover, $\|\varphi_m\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega}_m)} < c_0 \leq c_0 K_0$.

Further, given $m \in \mathbb{N}$, let $\varphi_m, \tilde{\varphi}_m \in \mathcal{C}^{1,\alpha}(\overline{\Omega}_m)$ be such that

$$\|\tilde{\varphi}_m - \varphi_m\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega}_m)} < \frac{1}{K_1}.$$

For $\varphi = E_1(\varphi_m)$ and $\tilde{\varphi} = E_1(\tilde{\varphi}_m)$ we have $\|\tilde{\varphi} - \varphi\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})} < 1$. Hence as in (5) we can conclude that

$$\|d_{\tilde{\varphi}_m}\mathcal{F}_m(h) - d_{\varphi_m}\mathcal{F}_m(h)\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega}_m)} \leq c K_1^2 \|\tilde{\varphi}_m - \varphi_m\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega}_m)} \|h\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega}_m)}.$$

We now get the desired result from Theorem 2 by setting

$$\eta = \min\left\{\frac{1}{K_1}, \frac{1}{2cc_0K_0K_1^2}\right\} \text{ and } \delta = \frac{\eta}{4c_0K_0}. \qquad \square$$

We now prove a local approximation statement that will be used in the proof of the main theorem. We mimic the proof of [3, Lemma 3.5].

Theorem 5. *Let $J \in \mathcal{J}$. Given $\epsilon > 0$, a smoothly embedded arc $\Gamma \subset \mathbb{C}$, and a $\mathcal{C}^2$-map $\varphi\colon \Gamma \to \mathbb{R}^{2n}$, there exists a neighborhood U of Γ and a J-holomorphic map $u\colon U \to \mathbb{R}^{2n}$ such that $\|u - \varphi\|_{\mathcal{C}^{1,\alpha}(\Gamma)} < \epsilon$.*

Proof. Without loss of generality we can assume that $\Gamma \subset \mathbb{D} \cap \mathbb{R}$. By (3) the J-holomorphicity condition equals $u_y = J(u)u_x$. Hence we may extend φ to a function that is quadratic in y and whose $\bar\partial_J$-derivative vanishes up to the first order along Γ. In particular, for

$$\Omega_m = \left\{z \in \mathbb{C} :\ \operatorname{dist}(z, \Gamma) < \frac{1}{m}\right\}$$

there exist $m_0 > 0$ and $C_\alpha > 0$ such that $\Omega_m \subset \mathbb{D}$ and

$$\left\|(\varphi_m)_{\bar\zeta} + A(\varphi_m)\overline{(\varphi_m)_\zeta}\right\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega}_m)} < \frac{C_\alpha}{m}$$

for every $m \geq m_0$. The rest follows from Proposition 4 above. $\square$

2. The Runge-type approximation

As mentioned in the introduction, our result is an application of the following Runge-type theorem provided by A. Gournay [5].

Theorem 6. *Let M be a smooth compact manifold equipped with a regular almost complex structure and admitting a doubly tangent property. Suppose we are given $\epsilon > 0$, a compact Riemann surface Σ, an open set $U \subset \Sigma$, a J-holomorphic map $\varphi\colon U \to M$, and a compact set $K \subset U$. Then, provided that there is a $\mathcal{C}^0$-extension of φ to Σ, there exists a J-holomorphic map $u\colon \Sigma \to M$ such that $\|u - \varphi\|_{\mathcal{C}^0(K)} < \epsilon$.*

We include below a brief discussion on the proof in order to explain why the statement can be applied in the present case.

Provided that there are no topological obstructions, we can define a new map extending the initial data $\varphi|_K$ to Σ in a $\mathcal{C}^\infty$-fashion. Let us denote it by φ again. Of course such a map need not be holomorphic on $\Sigma \setminus K$, and we can express this locally. Fix $\zeta_0 \in \Sigma \setminus K$ and the following two charts: a chart ψ on $\Omega_0 \subset \Sigma$ with $\psi(\zeta_0) = 0$ and a chart ϕ on M taking $q = \varphi(\zeta_0) \in M$ to $0 \in \mathbb{R}^{2n}$ and satisfying $\phi^*(J)(0) = J_{st}$. There exist $a, b \in \mathbb{R}^{2n}$ such that

$$\psi \circ \varphi \circ \phi^{-1}(z) = az + b\bar z + O(|z|^2),$$

where $b \neq 0$ is equivalent to $\bar\partial_J \varphi \neq 0$.

The first of the two key assumptions in the method of A. Gournay is that the manifold enjoys the *doubly tangent property.* That is, for almost every $q \in M$ and almost every pair $a, b \in \mathbb{R}^{2n}$, there exists a J-holomorphic sphere $H^r_{a,b}\colon \mathbb{C}P^1 \to M$ whose local (Laurent) expansion equals

$$\psi \circ H^r_{a,b} \circ \phi^{-1}(z) = az + br^2/z + O(r^{1+\epsilon}).$$

Here, $\epsilon > 0$ and $r > 0$ are such that for $\frac{r}{(1+r^\epsilon)} < |z| < r(1+r^\epsilon)$ we have $\psi^{-1}(z) \in \Omega_0$. Hence, using an appropriate cut-off function, the map φ may be replaced by $H^r_{a,b}$ in the vicinity of ζ_0. Moreover, since $H^r_{a,b}$ is holomorphic and almost agrees with φ on $|z| = r$, such a surgery diminishes the "size" of the $\bar{\partial}_J$-derivative. Gournay calls such a procedure *grafting*, and he repeats it finitely many times until reaching the desired bounds (see [5, §3.1]).

Once an approximate solution is constructed (we denote it by $\varphi\colon \Sigma \to M$ again), the $\bar{\partial}_J$-equation can be solved similarly as in §1. Let us briefly explain this. The nonlinear $\bar{\partial}_J$-operator, now defined globally, may be linearized at a compact curve u so that the corresponding Fredholm operator D_u maps from $\mathcal{C}^\infty(\Sigma, \varphi^*TM)$ to $\mathcal{C}^\infty(\Sigma, \Lambda^{0,1}\varphi^*TM)$ (see [9, (3.1.4.)]). The notion of regularity refers to its surjectivity. In particular, for us the structure J is *regular* when D_u is onto for every J-holomorphic sphere. The idea is to find a bounded right inverse for D_φ. That is, given $\eta \in \mathcal{C}^\infty(\Sigma, \Lambda^{0,1}\varphi^*TM)$, we seek a solution $\xi \in \mathcal{C}^\infty(\Sigma, \varphi^*TM)$ of $D_\varphi \xi = \eta$ with bounds.

In [5] the above is obtained in two steps. First, it is shown in [5, §3.2] that it suffices to solve local equations $D_{\varphi_j}\xi_j = \eta_j$, where $\{(\varphi_j, \xi_j, \eta_j)\}_{j\geq 0}$ stands for slightly perturbed data (φ, ξ, η) restricted either to the original surface $\Sigma_0 = \Sigma$ or to one of the finitely many grafts $\Sigma_j = \mathbb{C}P^1$, $j > 0$. Second, it is proved in [5, §3.3] that though the local equations for $j > 0$ interact with the one for $j = 0$, the iteration starting at $\xi_0 = 0$ is indeed contractible. Here the regularity of the structure is crucial since it ensures that the local equation is always solvable along the grafts. In contrast, the inversion of the linear equation for $j = 0$ is very subtle [5, §3.3].

Finally, it is worth mentioning that the norms in question are not the ones associated with Sobolev or Hölder spaces. The reason lies in the fact that each graft increases the L^p-norm of $d\varphi$ by a quantity that is a priori significant. Furthermore, the number of surgeries is not bounded in general. Hence the local Lipschitz constant grows with $\partial\varphi$ when D_φ is treated as a map from $W^{1,p}(\Sigma, \Lambda^{0,1}\varphi^*TM)$ to $L^p(\Sigma, \varphi^*TM)$, $p > 2$. This makes it impossible to use the Implicit Function Theorem. Hence a certain sup-norm introduced by C. Taubes [16] is used (see also [4, §4]).

We now state the corollary that will be used in the proof of Theorem 1.

Corollary 7. *Let (M, J) be as in Theorem 6. Let $K \subset \partial\mathbb{D}$ be a compact set and let the map φ be continuous near $\overline{\mathbb{D}}$ and J-holomorphic near $K \cup \{0\}$. Given $\epsilon > 0$, there exists $u \in \mathcal{O}(\overline{\mathbb{D}}, M, \varphi(0))$ such that $\|u - \varphi\|_{\mathcal{C}^0(K)} < \epsilon$.*

Proof. First note that the map φ can be continuously extended to $\Sigma = \mathbb{C}P^1$. Hence the Runge-type theorem guarantees the existence of a J-holomorphic map approximating φ on $K \cup \{0\}$. It remains to explain why the above proof can be adopted slightly in order to obtain $u(0) = \varphi(0)$.

The simplest way to do this is by adding an "unnecessary graft" at the center. That is, we start by replacing the map φ with an appropriate graft $H^r_{a,b}$ near $0 \in \mathbb{D}$. Since $\bar{\partial}_J\varphi(0) = 0$, we have $b = 0$ and $\varphi(0) = H^r_{a,b}(0)$ here. We index this graft with $j = 1$ and then proceed with the usual grafting procedure for $j > 1$. Moreover, we add a pointwise restriction $\xi(0) = 0$ each time when solving the local linear equation $D_{\varphi_1}\xi_1 = \eta_1$. Since J is regular, this does not object to the surjectivity of D_{φ_1}. Indeed, check [9, §3.4]. Hence all the key estimates remain fulfilled. In particular, [5, Corollary 2.5.5.] can be used in the iterative scheme from [5, §3.3]. ☐

3. Proof of Theorem 1

The direct construction of a Poletsky disc follows from Theorem 5 and Corollary 7. Indeed, as in [13], we prove the following stronger statement.

Theorem 8. *Let (M, J) be as in Theorem 1 and equipped with some Riemannian metric. Given a point $p \in M$, a positive constant $\epsilon > 0$, and a $\mathcal{C}^2$-map $\lambda\colon \partial\mathbb{D} \to M$, there exist a disc $u \in \mathcal{O}(\overline{\mathbb{D}}, M, p)$ and a set $E \subset [0, 2\pi)$ such that $|E| < \epsilon$ and $\mathrm{dist}(u(e^{it}), \lambda(e^{it})) < \epsilon$ for $t \in [0, 2\pi) \setminus E$.*

Proof. As pointed out above, the direct method consists of two steps. We first make a piecewise holomorphic approximation of λ. Then we use the Runge-type theorem to extend this map to the whole disc. The second step can be understood as adding finitely many poles (grafts).

Fix $e^{it} \in \partial\mathbb{D}$. Let $\phi_t\colon V_t \to \mathbb{B}$ be a local chart mapping a neighborhood of $\lambda(e^{it})$ into a neighborhood of the origin in $\mathbb{R}^{2n}$ and satisfying $\phi_t^*(J) \in \mathcal{J}$. We define $\Gamma'_t \subset \partial\mathbb{D}$ to be the largest connected subarc including e^{it} and satisfying $\lambda(\Gamma'_t) \subset \lambda(\partial\mathbb{D}) \cap V_t$. By compactness, there are points $t_1, \ldots, t_k \in [0, 2\pi)$ such that the the union $\bigcup_{j=1}^k \Gamma'_{t_j}$ covers the whole $\partial\mathbb{D}$. Moreover, we can choose smaller pairwise disjoint subarcs $\Gamma_{t_j} \Subset \Gamma'_{t_j}$ satisfying

$$|\partial\mathbb{D} \setminus \bigcup_{j=1}^{k} \Gamma_{t_j}| < \epsilon.$$

By Theorem 5 there exist J-holomorphic maps $u_j\colon U_j \to V_{z_j}$ that are defined on pairwise disjoint neighborhoods and $\mathcal{C}^{1,\alpha}$-close to $\phi_j \circ \lambda$ on Γ_{t_j}. Moreover, by the classical Nijenhuis-Woolf theorem [10] there exists a small J-holomorphic disc u_0 centered at p. Since M is connected, we can join these pieces into a continuous map φ defined on a neighborhood of $\overline{\mathbb{D}}$ and satisfying $\varphi(0) = u_0(0) = p$. The rest follows from Corollary 7 applied to the compact set $K = \bigcup_{j=1}^k \Gamma_{t_j}$. ☐

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