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ON THREE-DIMENSIONAL TYPE I κ-SOLUTIONS TO THE RICCI FLOW

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Abstract. κ-solutions are very important to the study of Ricci flow since they serve as the finite-time singularity models. With the help of his profound understanding of κ-solutions, Perelman [11] made the major breakthrough in Hamilton's program. However, three-dimensional κ-solutions are not yet classified until this day. We prove a classification result assuming a Type I curvature bound.

In this short note, we prove that the only simply connected noncompact three-dimensional Type I κ-solution to the Ricci flow is the shrinking cylinder. This work can be regarded as a generalization of Cao, Chow, and Zhang [2], and a complement of Ding [3] and Ni [10]. Up to this point, three-dimensional κ-solutions of Type I are completely classified, and it remains interesting to work further towards Perelman's assertion, that the only remaining possibility of a three-dimensional noncompact κ-solution is the Bryant soliton; see [11]. Brendle [1] is working to that end. The classification of a three-dimensional κ-solution is of importance to the study of four-dimensional Ricci flows because of a possible dimension-reduction procedure.

We remind the reader of the following definition.

Definition 1. An ancient solution to the Ricci flow $(M, g(t))_{t\in(-\infty,0]}$ is called a κ-solution if it is κ-noncollapsed on all scales and has bounded curvature on every time slice. A κ-solution is called Type I if its Riemann curvature tensor satisfies

$$|Rm|(g(t)) \leq \frac{C}{|t|} \tag{1}$$

for all $t \in (-\infty, 0)$, where C is a constant that does not depend on t.

It is well known that every three-dimensional κ-solution has uniformly bounded and nonnegative sectional curvature.

Our main theorem is the following.

Theorem 2. *The only three-dimensional simply connected noncompact Type* I *κ-solution is the shrinking cylinder.*

It is worth mentioning that Ni [10] has proved that a closed Type I κ-solution with positive curvature operator of every dimension is a shrinking sphere or one of its quotients. On the other hand, Theorem 2.4 in Ding [3] implies that the only simply connected noncompact κ-solution that forms a *forward* singularity of Type I is the shrinking cylinder, Cao, Chow, and Zhang [2] gave an alternative proof with

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an additional assumption of backward Type I. Furthermore, the author would like to draw the reader's attention to Hallgren [4], who also classified a three-dimensional Type I κ-solution to the Ricci flow independently, through a more direct approach.

We recall the notion of an ε-neck.

Definition 3. A space-time point (x_0, t_0) in a Ricci flow $(M, g(t))$ is called the center of an ε-neck, where $\varepsilon > 0$, if the Ricci flow $g(t)$ on the space-time neighbourhood $B_{g(t_0)}(x_0, \varepsilon^{-1}R(x_0,t_0)^{-\frac{1}{2}}) \times [t_0 - R(x_0,t_0)^{-1}, t_0]$ is, after parabolic rescaling by the factor $R(x_0,t_0)$, ε-close in the $C^{\lfloor \frac{1}{\varepsilon} \rfloor}$-topology to the corresponding part of a standard shrinking cylinder, or in other words, if there exist diffeomorphisms $\phi_t : \mathbb{S}^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to B_{g(t_0)}(x_0, \varepsilon^{-1}R(x_0,t_0)^{-\frac{1}{2}})$, such that

$$\phi_t^{-1}(x_0) \in \mathbb{S}^2 \times \{0\},$$
$$\Big| R(x_0,t_0)\phi_t^* g(t_0 + tR(x_0,t_0)^{-1}) - g_{cyl}(t) \Big|_{C^{\lfloor \frac{1}{\varepsilon} \rfloor}(\mathbb{S}^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}))} < \varepsilon$$

for any $t \in [-1, 0]$. Here the notation $B_{g(t_0)}(x_0, r)$ stands for the geodesic ball centered at x_0, with radius r, and with respect to the metric $g(t_0)$, and $g_{cyl}(t)$ represents the standard shrinking metric on $\mathbb{S}^2 \times \mathbb{R}$ with $R(g_{cyl}(0)) \equiv 1$.

We remark here that in the above definition, after parabolic scaling, the space-time neighbourhood $B_{g(t_0)}(x_0, \varepsilon^{-1}R(x_0,t_0)^{-\frac{1}{2}}) \times [t_0 - R(x_0,t_0)^{-1}, t_0]$ has time expansion 1, and the scalar curvature at (x_0,t_0) is normalized to be 1. This definition is called the strong ε-neck by Perelman [11], whereas we keep consistency with the definition in Kleiner and Lott [7] and call it an ε-neck.

The following neck stability theorem by Kleiner and Lott is of fundamental importance to our proof. Please refer to Theorem 6.1 in [7].

Theorem 4. *For any $\kappa > 0$, there exists a constant $\delta = \delta(\kappa) > 0$, such that for all δ_0, $\delta_1 \le \delta$, there is a $T = T(\delta_0, \delta_1, \kappa) \in (-\infty, 0)$, with the following property. Let $(M^3, g(t))_{t\in(-\infty,0]}$ be a noncompact three-dimensional κ-solution to the Ricci flow that is not the $\mathbb{Z}_2$-quotient of the shrinking cylinder. Let $(x_0, 0) \in M \times \{0\}$ be such that $R(x_0, 0) = 1$. If $(x_0, 0)$ is the center of a δ_0-neck, then for all $t \le T$, (x_0, t) is the center of a δ_1-neck.*

For the remaining of this paper, we fixed a small positive constant

$$\varepsilon < \min\left\{\frac{1}{100}, \delta(\kappa), \varepsilon_0(\kappa)\right\},$$

where $\delta(\kappa)$ is defined in Theorem 4, and ε_0 is the constant given in Corollary 48.1 of Kleiner and Lott [6]. With such ε we are guaranteed that the ε-canonical neighbourhood property holds for all κ-solutions of dimension three. We will use this ε as the small positive constant in the definition of the ε-neck.

The following lemma is inspired by Ding [3] and Ni [10].

Lemma 5. *Let $(M^3, g(t))_{t\in(-\infty,0]}$ be a three-dimensional noncompact Type* I *κ-solution with strictly positive sectional curvature on every time slice. Let p_0 be an arbitrary fixed point on M. Then for every instance $t \in (-\infty, 0]$, there exists a point $p(t) \in M$ such that $(p(t), t)$ is* **not** *the center of an ε-neck. Moreover, $dist_{g(0)}(p_0, p(t)) \to \infty$ as $t \to -\infty$.*

Proof. First of all, such $p(t)$ must exist for every $t \in (-\infty, 0]$. We know from the Gromoll-Meyer theorem that M is diffeomorphic to $\mathbb{R}^3$. By Corollary 48.1 in

Kleiner and Lott [7], such an ancient solution must fall into category B, on which there is always a cap (the so-called M_ε). In particular, since M is diffeomorphic to $\mathbb{R}^3$, the cap is topologically a disk instead of $\mathbb{R}P^3 \setminus B^3$.

Assume by contradiction that there exists $\{t_i\}_{i=1}^{\infty} \subset (-\infty, 0)$, such that $t_i \searrow -\infty$ but $dist_{g(0)}(p_0, p(t_i)) \leq C_1$, where C_1 is a constant. We prove the following claim.

Claim. There exists a constant $C_2 < \infty$, such that

$$dist_{g(t_i)}(p_0, p(t_i)) \leq C_2\sqrt{|t_i|} + C_1 \tag{2}$$

for every i.

Proof of the Claim. We recall Perelman's distance distortion estimate [11]. Suppose on a t_0-slice of a Ricci flow, around two points x_0, x_1 that are not too close to each other, the Ricci curvature tensor is bounded from above, that is, if for some $r > 0$, $dist_{g(t_0)}(x_0, x_1) \geq 2r$ and $Ric \leq (n-1)K$ on $B_{g(t_0)}(x_0, r) \cup B_{g(t_0)}(x_1, r)$, then we have

$$\frac{d}{dt} dist_{g(t)}(x_0, x_1) \geq -2(n-1)\left(\frac{2}{3}Kr + r^{-1}\right) \tag{3}$$

at time $t = t_0$. Applying the curvature bound (1) and $r = |t|^{\frac{1}{2}}$ to (3), we have

$$\frac{d}{dt} dist_{g(t)}(p_0, p(\tau_i)) \geq -4\left(C+1\right)|t|^{-\frac{1}{2}} \tag{4}$$

for every i, whenever $dist_{g(t)}(p_0, p(t_i)) > 2|t|^{\frac{1}{2}}$. Integrating (4) from 0 to $t_i \in (-\infty, 0)$ completes the proof of the claim. □

Now we recall Perelman's reduced distance function $l_{(p_0,0)}(p,t)$ centered at $(p_0, 0)$ and evaluated at (p,t); see [11]. By the estimate of Naber (see Proposition 2.2 in [9]), we have that $l_{(p_0,0)}(p(t_i), t_i) < C_3$, where $C_3 < \infty$ is a constant. From Perelman [11] it follows that there exists a subsequence of $\{(M, |t_i|^{-1}g(|t_i|t), (p(t_i), -1))_{t\in[-2,-1]}\}_{i=1}^{\infty}$ that converges in the pointed smooth Cheeger-Gromov sense to the canonical form of a nonflat shrinking gradient Ricci soliton; see Morgan and Tian [8] and Naber [9] for details. Notice that the time interval of these scaled flows are taken as $[-1, -\frac{1}{2}]$ in Perelman's argument, whereas we take the interval to be $[-2, -1]$, so as to keep consistency with the definition of the ε-neck. This is valid because $\sup_{t\in[2t_i, t_i]} l_{(p_0,0)}(p(t_i), t)$ is bounded uniformly. One may easily verify this bound by using Perelman's differential inequalities for the reduced distance. The only nonflat three-dimensional shrinking gradient Ricci solitons are the shrinking sphere, the shrinking cylinder, and their quotients; see Perelman [12]. The limit shrinking gradient Ricci soliton cannot be flat, since otherwise Perelman's reduced volume is equal to 1 for all time and the Ricci flow is flat; see [13]. The shrinking cylinder is the only one that can arise as the limit of a sequence of Ricci flows that are diffeomorphic to $\mathbb{R}^3$. However, this yields a contradiction, as we have assumed that $(p(t_i), t_i)$ is not the center of an ε-neck. □

We are now ready to present the proof of our main theorem.

Proof of Theorem 2. If $g(t)$ has zero sectional curvature somewhere in space-time, by Hamilton's strong maximum principle [5], $g(t)$ also has zero sectional curvature everywhere in space at more ancient times, and hence splits locally. Since we assume M to be simply connected, it must be the shrinking cylinder. Therefore, henceforth, we assume that $g(t)$ has strictly positive curvature on every time slice.

We fixed an arbitrary time sequence $\{t_i\}_{i=1}^{\infty} \subset (-\infty, 0)$ such that $t_i \searrow -\infty$. For every i, let $p_i \in M$ be such that (p_i, t_i) is ***not*** the center of an ε-neck. By Lemma 5, we have that $dist_{g(0)}(p_i, p_0) \to \infty$. Since by Perelman [11] every three-dimensional noncompact κ-solution splits as a shrinking cylinder at spacial infinity, we can extract from $\{(M, R(p_i, 0)g(tR(p_i, 0)^{-1}), (p_i, 0))_{t\in(-\infty,0]}\}_{i=1}^{\infty}$ a (not relabelled) subsequence that converges in the smooth Cheeger-Gromov sense to the shrinking round cylinder. For the sake of simplicity we denote $g_i(t) := R(p_i, 0)g(tR(p_i, 0)^{-1})$. It follows that for ever i large, $(p_i, 0)$ is the center of an ε-neck. The following claim is an easy consequence of Theorem 4.

Claim.

$$\bar{t}_i := t_i R(p_i, 0) \geq T, \tag{5}$$

for all large i where $T := T(\varepsilon, \varepsilon, \kappa) \in (-\infty, 0)$ as defined in Theorem 4.

Proof of the Claim. Suppose the claim is not true and, by passing to a subsequence, we can assume $\bar{t}_i = t_i R(p_i, 0) < T$ for all i. We consider the scaled Ricci flows $g_i(t)$, and apply Theorem 4 to elements in $\{(M, g_i(t), (p_i, 0))_{t\in(-\infty,0]}\}_{i=1}^{\infty}$. First of all we have that $R_i(p_i, 0) = 1$ because of the scaling factors that we chose. Moreover, as $\{(M, g_i(t), (p_i, 0))_{t\in(-\infty,0]}\}_{i=1}^{\infty}$ converges to the shrinking cylinder, we have that $(p_i, 0)$ is the center of an ε-neck when i is large. It follows that $(p_i, \bar{t}_i)$ is the center of an ε-neck when i is large. However, $g_i(\bar{t}_i) = R(p_i, 0)g(\bar{t}_i R(p_i, 0)^{-1}) = R(p_i, 0)g(t_i)$. By our assumption, on the original Ricci flow $g(t)$ the space-time point (p_i, t_i) is not the center of an ε-neck; this is a contradiction. Notice here that the ε-necklike property is scaling invariant. $\square$

We continue the proof of the theorem. In the following argument we consider the scaled Ricci flows $g_i(t)$ and notice that by our assumption for every i the space-time point $(p_i, \bar{t}_i)$ is not the center of an ε-neck, where $\bar{t}_i$ is defined as (5). Since the limit of the sequence $\{(M, g_i(t), (p_i, 0))_{t\in(-\infty,0]}\}_{i=0}^{\infty}$ is exactly a shrinking round cylinder, we have that for every large $A \in [4|T|, \infty)$, $(B_{g_i(0)}(p_i, A), g_i(t))_{\tau\in[T-A,0]}$ is as close as we like to the correspondent piece of the shrinking cylinder when i is large enough. In particular, $(p_i, \bar{t}_i)$ is the center of an ε-neck since $\bar{t}_i \in [T, 0]$ according to the claim; this is a contradiction. Here we have again taken into account the scaling invariance of the ε-necklike property. $\square$

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