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LIFTING DIVISORS WITH IMPOSED RAMIFICATIONS ON A GENERIC CHAIN OF LOOPS

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Abstract. Let C be a curve over an algebraically closed non-archimedean field with non-trivial valuation. Suppose C has totally split reduction and the skeleton Γ is a chain of loops with generic edge lengths. Let P be the rightmost vertex of Γ and let $\mathcal{P} \in C$ be a point that specializes to P. We prove that any divisor class on Γ with imposed ramification at P that is rational over the value group of the base field lifts to a divisor class on C that satisfies the same ramification at $\mathcal{P}$, which extends the result in [Canad. Math. Bull. 58 (2015), 250–262].

1. Introduction

A metric graph which is a generic chain of loops (Definition 2.2) plays a crucial role in connecting classic and tropical Brill-Noether theory. Many properties of these graphs, such as Brill-Noether generality, can be transferred to certain curves with minimal skeleton isometric to them. Related approaches can be found in [CDPR12, JP14, JP16].

Let Γ be a generic chain of loops with or without bridges. Let K be an algebraically closed non-archimedean field with non-trivial value group G and valuation ring R. Let C be a smooth projective curve of genus g over K which has totally split reduction (by which we mean C admits a split semistable R-model as in [BR14, §5] whose special fiber only has rational components) and the skeleton is isometric to Γ. The tropicalization map from $\mathrm{Pic}^d(C)$ to $\mathrm{Pic}^d(\Gamma)$ given by linear expansion of the retraction from C^{an} to Γ maps $W^r_d(C)^{\mathrm{an}}$ to $W^r_d(\Gamma)$ [Bak08], where $W^r_d(C)$ (resp. $W^r_d(\Gamma)$) parametrizes divisor classes on C (resp. Γ) with degree d and rank at least r. It is then proved in [CJP15] that $\mathrm{Trop}(W^r_d(C)) = W^r_d(\Gamma)$ via the classification of divisors in $W^r_d(\Gamma)$, given in [CDPR12]. In other words, every G-rational divisor class on Γ of rank r can be lifted to a divisor class on C of the same rank.

On the other hand, let $\alpha = (\alpha_0, ..., \alpha_r)$ be a *Schubert index* of type (r, d), which is a non-decreasing sequence of non-negative numbers bounded by $d - r$. For an arbitrary chain of loops Γ, Pflueger [Pfl17] provides a straightforward expression for the locus of divisors on Γ with ramification at least α at the rightmost vertex P of Γ:

$$W^{r,\alpha}_d(\Gamma, P) = \{D \in \mathrm{Pic}^d(\Gamma) : r(D - (\alpha_i + i)P) \geq r - i \text{ for } i = 0, 1, ..., r\},$$

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which is locally a union of translates of coordinate planes possibly of different dimensions in $\mathbb{R}^g$. It follows that $W_d^{r,\alpha}(\Gamma, P)$ contains the tropicalization of the corresponding locus on C:

$$W_d^{r,\alpha}(C,\mathcal{P}) = \{\mathcal{D} \in \mathrm{Pic}^d(C) : h^0(\mathscr{O}_C(\mathcal{D} - (\alpha_i + i)P)) \geq r + 1 - i \text{ for } i = 0, 1, ..., r\}.$$

When Γ is a generic chain of loops, [Pfl17] shows that $W_d^{r,\alpha}(\Gamma, P)$ has dimension equal to that of $W_d^{r,\alpha}(C,\mathcal{P})$ and both have pure dimensions $\rho(g, r, d) - \sum_{0\leq i\leq r} \alpha_i$ as expected. We prove an analogue of the lifting result of [CJP15]:

Theorem 1.1. *Let Γ be a generic chain of loops, possibly with bridges, and let C be a smooth projective curve over K which has totally split reduction and skeleton isometric to Γ. Let $\mathcal{P}$ be a point on C that tropicalizes to the rightmost vertex P of Γ. Let α be a Schubert index of type (r, d). Then every G-rational divisor class on Γ with ramification at least α at P lifts to a divisor class on C with ramification at least α at $\mathcal{P}$.*

We now explain the general strategy used in the proof of Theorem 1.1. Let

$$\alpha^j = (\alpha_0, ..., \alpha_j, \alpha_j, ..., \alpha_j)$$

be a Schubert index of type (r, d) whose last $r - j + 1$ coordinates are all α_js. Let $W_j(C) = W_d^{r,\alpha^j}(C,\mathcal{P})$ and $W_j(\Gamma) = W_d^{r,\alpha^j}(\Gamma, P)$. Also let

$$X_j(C) = (\alpha_j + j)\mathcal{P} + W^{r-j}_{d-\alpha_j-j}(C) \text{ and } Y_j(C) = (\alpha_j + j + 1)\mathcal{P} + W^{r-j-1}_{d-\alpha_j-j-1}(C),$$

$$X_j(\Gamma) = (\alpha_j + j)P + W^{r-j}_{d-\alpha_j-j}(\Gamma) \text{ and } Y_j(\Gamma) = (\alpha_j + j + 1)P + W^{r-j-1}_{d-\alpha_j-j-1}(\Gamma).$$

Note that $W_j(C) = W_{j-1}(C) \cap X_j(C)$ and that

$$W_j(\Gamma) = W_{j-1}(\Gamma) \cap X_j(\Gamma) = W_{j-1}(\Gamma) \cap \mathrm{Trop}(X_j(C)).$$

We proceed by induction. At each step, assuming $\mathrm{Trop}(W_{j-1}(C)) = W_{j-1}(\Gamma)$, it suffices to show that

$$\mathrm{Trop}(W_{j-1}(C) \cap X_j(C)) = \mathrm{Trop}(W_{j-1}(C)) \cap \mathrm{Trop}(X_j(C)). \tag{1}$$

Note that $Y_{j-1}(C)$ is locally isomorphic to a *polytopal domain*, which is the preimage of an integral G-affine polytope in $\mathbb{R}^n$ of the tropicalization map $(\mathbb{G}_m^n)^{\mathrm{an}} \to \mathbb{R}^n$ for some n, at points in the relative interior of maximal faces of $Y_{j-1}(\Gamma) = \mathrm{Trop}(Y_{j-1}(C))$ (this equality follows from the main theorem of [CJP15]). Since $W_{j-1}(\Gamma)$ and $X_j(\Gamma) = \mathrm{Trop}(X_j(C))$ (again, by [CJP15]) *intersect properly* in $Y_{j-1}(\Gamma)$, namely, intersect in expected dimension, the problem boils down to lifting proper tropical intersections within a polytopal domain:

Theorem 1.2. *Let $\mathcal{U}_\Delta \subset (\mathbb{G}_m^n)^{\mathrm{an}}$ be the preimage of an integral G-affine polytope Δ in $\mathbb{R}^n$ of dimension n and $\mathcal{X}$ and let $\mathcal{X}'$ be two Zariski closed analytic subspaces of $\mathcal{U}_\Delta$ of pure dimension. Suppose* $\mathrm{Trop}(\mathcal{X})$ *and* $\mathrm{Trop}(\mathcal{X}')$ *intersect properly in Δ. Then we have*

$$\mathrm{Trop}(\mathcal{X}) \cap \mathrm{Trop}(\mathcal{X}') \cap \Delta^\circ = \mathrm{Trop}(\mathcal{X} \cap \mathcal{X}') \cap \Delta^\circ.$$

See Section 4 for a generalized version of this theorem. See also [OP13] for an algebraic counterpart, where the authors proved a lifting theorem for subschemes of an algebraic torus whose tropicalizations intersect properly. See Section 3 for the discussion of local properties of $Y_{j-1}(C)$, which works for all Brill-Noether loci $W^{r'}_{d'}(C)$ on C. The proof of (1) is in Section 5, where we use the notation of ramification imposed by a partition instead of a Schubert index (Section 2).

Conventions. Throughout, K will be an algebraically closed non-archimedean field K with non-trivial value group G. For a lattice N we denote by T_N the algebraic torus over K whose lattice of characters, denoted by M, is dual to N. Let $N_{\mathbb{R}} = N \otimes \mathbb{R}$. All polytopes in $N_{\mathbb{R}}$ are assumed integral G-affine.

2. Preliminaries

In this section we recall some notions and techniques which will be useful for later arguments.

2.1. **Special divisors on a generic chain of loops.** Let Γ be a metric graph that is a chain of g loops with or without bridges, and let $\{v_i\}_{1\le i\le g}$ and $\{w_i\}_{1\le i\le g}$ be vertices of Γ as in Figure 1 (with the possibility that $w_i = v_{i+1}$). Let l_i (resp. n_i) be the length of the top (resp. bottom) segment of the ith loop connecting the vertices v_i and w_i. The divisors on Γ with imposed ramification are classified in [Pfl17], we recall some related concepts from [Pfl17].

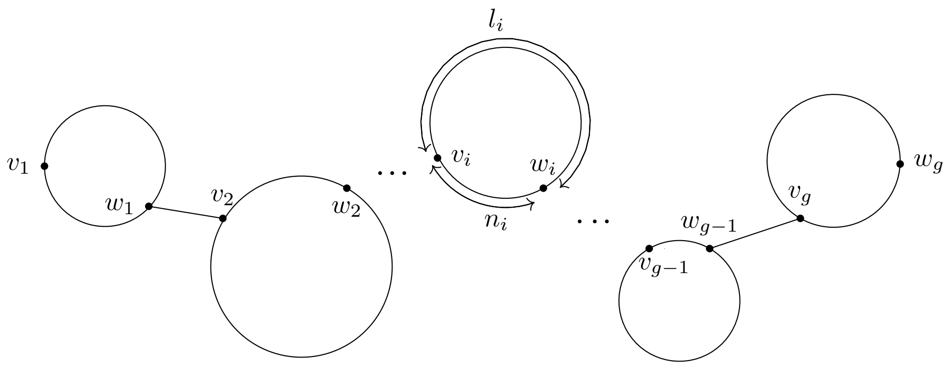

Figure 1. A chain of g loops with bridges.

Definition 2.1. The *torsion profile* of Γ is a sequence $\underline{m} = (m_2, ..., m_g)$ of $g-1$ integers. If l_i/n_i is a rational number, then m_i is the minimum positive integer such that $m_i \cdot l_i$ is an integer multiple of $l_i + n_i$; otherwise $m_i = 0$.

Note that we omit m_1 because it is immaterial to the properties of the divisors of interest. The following notion of a generic chain of loops was introduced in [CDPR12] for constructing Brill-Noether general curves:

Definition 2.2. We say that Γ is *generic* if none of the ratios l_i/n_i are equal to the ratio of two positive integers whose sum does not exceed $2g-2$ or, equivalently, if for each i either $m_i > 2g-2$ or $m_i = 0$.

Let λ be a partition, which is a finite, non-increasing sequence of non-negative integers. As in [Pfl17], we will identify partitions with their Young diagrams in French notation.

Definition 2.3. Let P be a point on Γ. The *Brill-Noether locus* corresponding to a partition λ and the marked graph (Γ, P) is

$$W^{\lambda}(\Gamma, P) = \{D \in \mathrm{Pic}^0(\Gamma) : r(D + d'P) \ge r' \text{ whenever } (g - d' + r', r' + 1) \in \lambda\}.$$

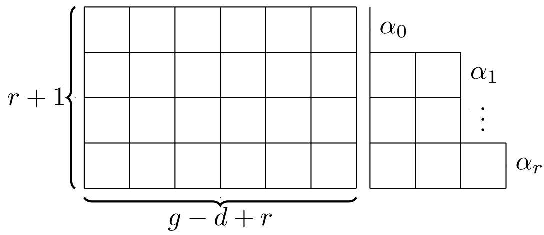

FIGURE 2. The partition as in [Pfl17, Figure 1] associated to g, d, r, α where $r = 3$, $d = g - 3$, and $\alpha = (0, 2, 2, 3)$.

Let $\alpha = (\alpha_0, ..., \alpha_r)$ be a Schubert index of type (r, d) and let $\lambda = (\lambda_0, ..., \lambda_r)$ be the induced partition where $\lambda_i = (g-d+r) + \alpha_{r-i}$ (see Figure 2). Then the locus $W^\lambda(\Gamma, P)$ is isomorphic to $W_d^{r,\alpha}(\Gamma, \mathcal{P})$ under the Abel-Jacobi map with respect to dP. In particular if λ is an $(r+1)\times(g-d+r)$ diagram, then $W^\lambda(\Gamma, P)$ is isomorphic to $W_d^r(\Gamma)$.

We next describe the Brill-Noether locus of a partition when $P = w_g$ is the rightmost vertex of Γ. As in [Pfl17] we identify $\lambda = (\lambda_0, ..., \lambda_r)$ with the set

$$\{(x, y) \in \mathbb{Z}^2_{>0} | 1 \leq x \leq \lambda_{y-1}, 1 \leq y \leq r + 1\}.$$

Definition 2.4. Let λ be a partition, and let $\underline{m} = (m_2, ..., m_g)$ be a $(g-1)$-tuple of non-negative integers. An *$\underline{m}$-displacement tableau* on λ is a function $t\colon \lambda \to \{1, 2, ..., g\}$ satisfying the following properties:

(1) t is strictly increasing in any given row or column of λ.
(2) For any two distinct boxes (x, y) and (x', y') in λ, if $t(x, y) = t(x', y')$, then $x - y \equiv x' - y' \pmod{m_{t(x,y)}}$.

We denote $t \vdash_{\underline{m}} \lambda$ if t is an $\underline{m}$-displacement tableau on λ.

According to [Pfl17, Theorem 1.3 and Corollary 3.8] if Γ is a generic chain of loops and $\underline{m}$ is its torsion profile, then every $\underline{m}$-displacement tableau on λ is injective. In other words, t is a standard Young tableau on λ.

Definition 2.5. Let $\underline{m}$ be the torsion profile of Γ. Let t be an $\underline{m}$-displacement tableau on a partition λ. Denote by $\mathbb{T}(t)$ the set of divisor classes on Γ of the form

$$\sum_{i=1}^{g} \langle \xi_i \rangle_i - g w_g,$$

where $\{\xi_j\}_j$ are real numbers such that $\xi_{t(x,y)} \equiv x - y \pmod{m_{t(x,y)}}$ and the symbol $\langle z \rangle_i$ denotes the point on the ith loop that is located $z \cdot l_i$ units clockwise from w_i.

It follows that $\mathbb{T}(t)$ is a real torus of dimension $d_t = g - |t(\lambda)|$. Moreover, under the identification $\mathrm{Pic}^0(\Gamma) = \prod_{1 \leq i \leq g} \mathbb{R}/(n_i + l_i)\mathbb{Z}$ induced by the Abel-Jacobi map [MI08, §6], the torus $\mathbb{T}(t)$ is the image of a translate of a coordinate d_t-plane in $\mathbb{R}^g$. The following is the description of the Brill-Noether locus of λ ([Pfl17, Theorem 1.4]).

Proposition 2.6. *We have*

$$W^\lambda(\Gamma, w_g) = \bigcup_{t \vdash_{\underline{m}} \lambda} \mathbb{T}(t).$$

In particular, if Γ *is generic, then* $W^\lambda(\Gamma, w_g)$ *is of pure dimension* $g - |\lambda|$.

2.2. **Curves with special skeletons and their tropicalizations.** Let C be a smooth projective curve of genus g over K which has totally split reduction and the skeleton is isometric to Γ. Let $\tau\colon C^{\mathrm{an}} \to \Gamma$ be the retraction map. The Jacobian variety of C is totally degenerate in the sense of [Gub07, §6]. In other words, $\mathrm{Pic}^0(C)^{\mathrm{an}}$ is isomorphic to $(T_N)^{\mathrm{an}}/L$ where N is a lattice of rank g and L is a discrete subgroup of $T_N(K)$ which maps isomorphically onto a complete lattice of $N_{\mathbb{R}}$ under the tropicalization map. Moreover, the induced tropicalization map on $\mathrm{Pic}^0(C)^{\mathrm{an}}$ is compatible with the retraction to its skeleton, which is canonically identified with $\mathrm{Pic}^0(\Gamma)$ ([BR14, §6]):

$$\begin{tikzcd} C^{\mathrm{an}} \arrow[r, "\alpha_{\mathcal{P}}"] \arrow[d, "\tau"] & \mathrm{Pic}^0(C)^{\mathrm{an}} \arrow[d, "\mathrm{Trop}"] & (T_N)^{\mathrm{an}} \arrow[l] \arrow[d, "\mathrm{Trop}"] \\ \Gamma \arrow[r, "\alpha_P"] & \mathrm{Pic}^0(\Gamma) & N_{\mathbb{R}} \arrow[l] \end{tikzcd}$$

where $\alpha_{\mathcal{P}}$ and α_P are the Abel-Jacobi maps associated to $\mathcal{P} \in C$ and $P \in \Gamma$ with $\tau(\mathcal{P}) = P$.

Definition 2.7. Let $\mathcal{P}$ be a point in C. As in Definition 2.3 the Brill-Noether locus corresponding to a partition λ and the marked curve $(C, \mathcal{P})$ is

$$W^\lambda(C,\mathcal{P}) = \{\mathcal{D} \in \mathrm{Pic}^0(C) : h^0(\mathscr{O}_C(\mathcal{D} + d'\mathcal{P})) \geq r' + 1 \quad \text{whenever } (g - d' + r', r' + 1) \in \lambda\}.$$

Let $\alpha = (\alpha_0, ..., \alpha_r)$ be a Schubert index of type (r, d) and let $\lambda = (\lambda_0, ..., \lambda_r)$ be the induced partition as above. Then as in the graph case the locus $W^\lambda(C, \mathcal{P})$ is isomorphic to $W^{r,\alpha}_d(C, \mathcal{P})$ under the Abel-Jacobi map with respect to $d\mathcal{P}$. In particular if λ is an $(r + 1) \times (g - d + r)$ diagram, then $W^\lambda(C, \mathcal{P})$ is isomorphic to $W^r_d(C)$.

The following theorem is a (partial) summary of [Pfl17, Theorems 1.13 and 5.1] and [CJP15, Theorem 1.1].

Theorem 2.8. *Let* C *be as above, let* Γ *be a generic chain of loops, and let* $\mathcal{P}$ *be a point of* C *that tropicalizes to* $P = w_g$*. Then* $\mathrm{Trop}(W^\lambda(C, \mathcal{P})) \subset W^\lambda(\Gamma, P)$ *and* $W^\lambda(C, \mathcal{P})$ *is of pure dimension* $g - |\lambda|$*. Moreover, if* λ *is an* $(r + 1) \times (g - d + r)$ *diagram, then* $\mathrm{Trop}(W^\lambda(C, \mathcal{P})) = W^\lambda(\Gamma, P)$.

In the theorem above, if λ is induced by α, then the dimension of $W^\lambda(\Gamma, P)$ is equal to the expected dimension of $W^\lambda(C, \mathcal{P})$ (or $W^{r,\alpha}_d(C, \mathcal{P})$). It follows that $W^\lambda(C, P)$ is of expected dimension as the tropicalization preserves dimension [Gub07, §6].

2.3. **Intersection multiplicities in a polytopal domain.** Let N be a lattice of rank n. Let $\Delta \subset N_{\mathbb{R}}$ be a polytope. We denote by $\mathcal{U}_\Delta$ the preimage of Δ in $(T_N)^{\mathrm{an}}$ under the tropicalization map. Then $\mathcal{U}_\Delta$ is an affinoid domain in $(T_N)^{\mathrm{an}}$ by [Gub07] and is called a polytopal domain. Denote by $K\langle\mathcal{U}_\Delta\rangle$ the corresponding affinoid algebra, whose basic properties can be found in [Rab12, Proposition 6.9].

Definition 2.9. Let $f_1, ..., f_k \in K\langle\mathcal{U}_\Delta\rangle$. Let Y be a Zariski-closed analytic subspace of $\mathcal{U}_\Delta$ of dimension $n - k$. Let $Y_i = V(f_i)$ and $Z = Y \cap (\bigcap_{1 \leq i \leq k} Y_i)$. The *intersection multiplicity* of Y and Y_i at an isolated point ξ of Z is

$$i(\xi, Y \cdot Y_1 \cdots Y_k; \mathcal{U}_\Delta) = \dim_K(\mathcal{O}_{Z,\xi}).$$

If Z is finite we define the *intersection number* of Y and $Y_1, ..., Y_k$ as

$$i(Y \cdot Y_1 \cdots Y_k; \mathcal{U}_\Delta) = \sum_{\xi \in Z} \dim_K(\mathcal{O}_{Z,\xi}).$$

This definition agrees with [Rab12, Definition 11.4] and is also compatible with the intersection multiplicities of algebraic varieties. We refer to [Rab12, §11] about intersection multiplicities of tropical hypersurfaces in Δ when Δ is of maximal dimension, which is compatible with the stable intersection of tropical cycles in $N_{\mathbb{R}}$.

Theorem 2.10 ([Rab12, Theorem 11.7])**.** *Let Δ be a polytope in $N_{\mathbb{R}}$ and let $f_1, ..., f_n \in K\langle \mathcal{U}_\Delta \rangle$. Let $Y_i = V(f_i)$ for all i and let $w \in \bigcap_{1 \le i \le n} \mathrm{Trop}(Y_i)$ be an isolated point contained in the interior of Δ. Let $Z = \bigcap_{1 \le i \le n} Y_i$. Then*

$$\sum_{\xi \in Z, \mathrm{Trop}(\xi) = w} i(\xi, Y_1 \cdots Y_n; \mathcal{U}_\Delta) = i(w, \mathrm{Trop}(Y_1) \cdots \mathrm{Trop}(Y_n); \Delta).$$

On the other hand, for two Zariski closed subspaces $\mathcal{X}$ and $\mathcal{X}'$ of $\mathcal{U}_\Delta$ and an isolated point ξ of $\mathcal{X} \cap \mathcal{X}'$, the intersection multiplicity of $\mathcal{X}$ and $\mathcal{X}'$ at ξ is defined to be

$$i(\xi, \mathcal{X} \cdot \mathcal{X}'; \mathcal{U}_\Delta) = \sum_{i \ge 0} (-1)^i \dim_K \mathrm{Tor}_i^{\mathscr{O}_{\mathcal{U}_\Delta, \xi}}(\mathscr{O}_{\mathcal{X}, \xi}, \mathscr{O}_{\mathcal{X}', \xi})$$

in [OR11, §5]. If $\mathcal{X} \cap \mathcal{X}'$ is finite, the intersection number of $\mathcal{X}$ and $\mathcal{X}'$ is

$$i(\mathcal{X} \cdot \mathcal{X}'; \mathcal{U}_\Delta) = \sum_{\xi \in \mathcal{X} \cap \mathcal{X}'} i(\xi, \mathcal{X} \cdot \mathcal{X}'; \mathcal{U}_\Delta).$$

3. Local properties of $W_d^r(C)^{\mathrm{an}}$

Let C and Γ and T_N be as in Section 2.2 and suppose Γ is generic. We prove in this section that $W_d^r(C)$ is locally analytically isomorphic to a polytopal domain at a "tropically general" point of $W_d^r(\Gamma)$. Before we start, we specify the following notation:

Notation 3.1. Let P be the rightmost vertex of Γ and fix $\mathcal{P} \in C$ that tropicalizes to P. Let $e_1, ..., e_g \in N$ be the standard basis of N and let $e_1', ..., e_g' \in M$ be the dual basis. For each partition λ we write $W^\lambda(C)$ (resp. $W^\lambda(\Gamma)$) instead of $W^\lambda(C, \mathcal{P})$ (resp. $W^\lambda(\Gamma, P)$). For a polytope $\Delta \subset N_{\mathbb{R}}$, denote by $\overline{\Delta}$ the image of Δ in $\mathrm{Pic}^0(\Gamma)$. If Δ maps isomorphically to $\overline{\Delta}$ we denote by $\overline{\mathcal{U}}_\Delta$ the preimage of $\overline{\Delta}$ in $\mathrm{Pic}^0(C)^{\mathrm{an}}$ under the tropicalization map. Let L_Δ be the subspace of $N_{\mathbb{R}}$ that is parallel to Δ and has the same dimension as Δ. Let $N_\Delta = N \cap L_\Delta$. For a given projection $\pi\colon N \to N_\Delta$ we denote by $\widetilde{\mathcal{U}}_\Delta$ the preimage of $\pi(\Delta)$ in $(T_{N_\Delta})^{\mathrm{an}}$. For a pure polyhedral complex γ in $N_{\mathbb{R}}$ denote by $\mathrm{relint}(\gamma)$ the union of relative interiors of all maximal faces of γ.

Now let λ be the $(r+1) \times (g-d+r)$ diagram. As discussed in Section 2 we have $W^\lambda(C)$ isomorphic to $W_d^r(C)$ and $\mathrm{Trop}(W^\lambda(C)) = W^\lambda(\Gamma)$, and $W^\lambda(\Gamma)$ is a union of translates of the images of the coordinate ρ-planes in $N_{\mathbb{R}}$, where $\rho = \rho(g, r, d)$. We next consider the local properties of $W^\lambda(C)$ instead.

Let $\delta \subset N_{\mathbb{R}}$ map isomorphically to a maximal face of $W^\lambda(\Gamma)$. Take a polytope Λ such that $\Lambda \subset \mathrm{relint}(\delta)$ as in Figure 3. Let $\Delta = \Lambda \times I \subset N_{\mathbb{R}}$ where $I_\epsilon = [-\epsilon, \epsilon]^{g-\rho}$ such that Δ maps isomorphically onto its image $\overline{\Delta}$ in $\mathrm{Pic}^0(\Gamma)$. We then have that $\mathcal{U}_\Delta$ is isomorphic to $\overline{\mathcal{U}}_\Delta$. Hence we may consider $W^\lambda(C)^{\mathrm{an}}$ as a Zariski-closed analytic subspace of the polytopal domain $\mathcal{U}_\Delta$. We may assume that L_Λ is generated by $e_1, \ldots, e_\rho$. The canonical projection from N to N_Λ gives rise to a projection $\pi_\Lambda \colon (T_N)^{\mathrm{an}} \to (T_{N_\Lambda})^{\mathrm{an}}$ which is compatible with the tropicalization map. Let $W_\Lambda = W^\lambda(C)^{\mathrm{an}} \cap \mathcal{U}_\Lambda$. The argument in [BPR12, Theorem 4.31] shows that $\pi_\Lambda \colon W_\Lambda \to \widetilde{\mathcal{U}}_\Lambda$ is finite and maps every irreducible component of W_Λ surjectively onto $\widetilde{\mathcal{U}}_\Lambda$. We now prove the following proposition, which gives us the desired property of $W^r_d(C)^{\mathrm{an}}$:

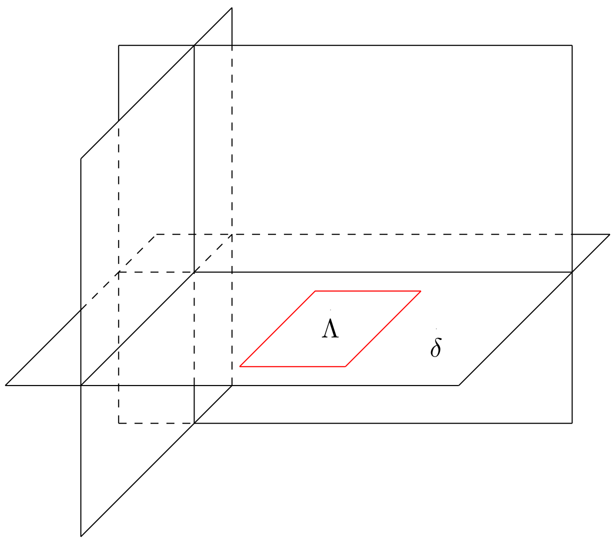

FIGURE 3. A local part of the preimage of $W^\lambda(\Gamma)$ in $N_{\mathbb{R}}$.

Proposition 3.2. *The map $\pi_\Lambda \colon W_\Lambda \to \widetilde{\mathcal{U}}_\Lambda$ is an isomorphism.*

Proof. We first show that π_Λ is of degree one in the sense of [BPR12, §3.27]. Take ρ general translates of theta divisors $\Theta^1_\Gamma = \mathrm{Trop}(\Theta^1_C), \ldots, \Theta^\rho_\Gamma = \mathrm{Trop}(\Theta^\rho_C)$ on $\mathrm{Pic}^0(\Gamma)$ such that $\overline{\Lambda} \cap (\bigcap_i \Theta^i_\Gamma)$ is non-empty and consists of finitely many points, where Θ^i_C are theta divisors on $\mathrm{Pic}^0(C)$. According to [CJP15, §2] we may also assume that $\bigcap_i \Theta^i_\Gamma$ intersects $W^\lambda(\Gamma)$ transversally at m points, where

$$m = g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!} = i(W^\lambda(C) \cdot \Theta^1_C \cdots \Theta^\rho_C; \mathrm{Pic}^0(C)).$$

Since the degree of π_Λ is preserved under flat base change, we may shrink Λ so that $\overline{\Lambda} \cap (\bigcap_i \Theta^i_\Gamma)$ consists of exactly one point. Also, take ϵ small enough such that $\Theta^i_\Gamma \cap \overline{\Delta}$ is of the form $\overline{\widetilde{\Theta}^i_\Gamma \times I_\epsilon}$ where $\widetilde{\Theta}^i_\Gamma$ is a codimension one polyhedral complex in Λ. Note that by [Wil09, §6] we know that $K\langle \mathcal{U}_\Delta \rangle$ is a UFD; hence we can take $f_i \in K\langle \mathcal{U}_\Delta \rangle$ to be the function that defines $(\Theta^i_C)^{\mathrm{an}}$ in $\mathcal{U}_\Delta$.

According to [Rab12, §8] there is a Laurent polynomial f'_i which is a sum of monomials in f_i such that $\mathrm{Trop}(V(f'_i)) \cap \Delta = \mathrm{Trop}(V(f_i)) = \widetilde{\Theta}^i_\Gamma \times I_\epsilon$. Moreover for all $w \in \Delta$ the monomials in f_i that obtain the minimal w-weight are the same as those in f_i'. Let $A = \{u_1, ..., u_k\} \subset M$ be the set of vertices of the Newton complex of f_i' corresponding to the maximal faces of Δ, whose polyhedral complex structure is induced by $\widetilde{\Theta}^i_\Gamma \times I_\epsilon$. We must have that A is contained in a ρ-dimensional plane in $M_\mathbb{R}$ that is parallel to the one generated by $e_1', ..., e_\rho'$. We may assume that A is contained in the sublattice generated by $e_1', ..., e_\rho'$. Consequently, if $g_i = \sum x^{u_i}$ and $h_i = f_i - g_i$, then for every $a \in K$ with $\mathrm{val}(a) \geq 0$ we have $\mathrm{Trop}(V(g_i + ah_i)) = \mathrm{Trop}(V(f_i))$ (with the same multiplicities, which are all ones by [CJP15, Theorem 3.1]). Moreover, g_i is contained in $K\langle\widetilde{\mathcal{U}}_\Lambda\rangle$.

In Lemma 3.3 below, let $W = W_\Lambda$ and $l_i = g_i + th_i$. It follows that (set $t = 0$)

$$i(W_\Lambda \cdot \prod_{i=1}^{\rho} (\Theta^i_C)^{\mathrm{an}}; \mathcal{U}_\Delta) = i(W_\Lambda \cdot \prod_{i=1}^{\rho} V(f_i); \mathcal{U}_\Delta) = i(W_\Lambda \cdot \prod_{i=1}^{\rho} V(g_i); \mathcal{U}_\Delta)$$
$$= m_\Lambda \cdot i(\prod_{i=1}^{\rho} V(g_i); \widetilde{\mathcal{U}}_\Lambda),$$

where the last equation is the projection formula in [Gub98, Proposition 2.10] and m_Λ is the degree of π_Λ. By Theorem 2.10 we have

$$i(\prod_{i=1}^{\rho} V(g_i); \widetilde{\mathcal{U}}_\Lambda) = i(\prod_{i=1}^{\rho} \mathrm{Trop}(V(g_i)); \Lambda) = 1;$$

therefore $i(W_\Lambda \cdot \prod_{i=1}^{\rho} (\Theta^i_C)^{\mathrm{an}}; \mathcal{U}_\Delta) = m_\Lambda$.

Now for all $w_j \in W^\lambda(\Gamma) \cap (\bigcap_i \Theta^i_\Gamma)$ where $1 \leq j \leq m$ we pick a polytope Λ_j as above and get a degree m_{Λ_i} of the corresponding projection map, which yields

$$\sum_{j=1}^{m} m_{\Lambda_j} = i(W^\lambda(C) \cdot \Theta^1_C \cdots \Theta^\rho_C; \mathrm{Pic}^0(C)) = m.$$

Note that the first equality follows from the fact that the K-dimension of the local ring of $W^\lambda(C) \cap (\bigcap_i \Theta^i_C)$ at a point is equal to that of $W^\lambda(C)^{\mathrm{an}} \cap (\bigcap_i (\Theta^i_C)^{\mathrm{an}})$). Hence we must have $m_{\Lambda_j} = 1$ for all j. Therefore π_Λ is of degree one.

It follows that W_Λ is irreducible and generically reduced. However, W_Λ is Cohen-Macaulay since $W^\lambda(C)$ is, so it is everywhere reduced, hence integral. Now π_Λ induces a finite morphism of degree one between integral domains whose source $K\langle\widetilde{\mathcal{U}}_\Lambda\rangle$ is normal, so it must be an isomorphism. □

Lemma 3.3. *Let $\mathbb{B}^1_K$ be the unit ball in $(\mathbb{G}_m)^{\mathrm{an}}_K$ with coordinate ring $K\langle t\rangle$. Let W be a Cohen-Macaulay Zariski-closed analytic subspace of $\mathcal{U}_\Delta$ of pure codimension k. Let $l_1, ..., l_k \in K\langle\mathcal{U}_\Delta\rangle \times K\langle t\rangle$ be global sections on $\mathcal{U}_\Delta \times \mathbb{B}^1_K$. Let Y_i be the subspace of $\mathcal{U}_\Delta \times \mathbb{B}^1_K$ defined by l_i. For $t \in \mathbb{B}^1_K$ let $Y_i(t) = \pi^{-1}(t) \cap Y_i$ where π is the projection from $\mathcal{U}_\Delta \times \mathbb{B}^1_K$ to $\mathbb{B}^1_K$. Suppose $\mathrm{Trop}(W) \cap (\bigcap_i \mathrm{Trop}(Y_i(t)))$ is finite and contained in Δ° for all i and $t \in |\mathbb{B}^1_K|$. Then the intersection number $i(W \cdot Y_1(t) \cdots Y_k(t); \mathcal{U}_\Delta)$ is constant on $|\mathbb{B}^1_K|$.*

Proof. We proceed by showing that the analytic space $\widetilde{W} = (W \times \mathbb{B}^1_K) \cap (\bigcap_i Y_i)$ is finite and flat over $\mathbb{B}^1_K$; hence every fiber has the same length.

It is obvious that $\widetilde{W}$ has finite fiber. To show it is proper over $\mathbb{B}^1_K$ we use the ideas in [OR11, §4.9]. Since all analytic space appearing here are affinoid, hence compact Hausdorff, we have that π is compact on $\widetilde{W}$ and separated. On the other hand, let $\pi'\colon \mathcal{U}_\Delta \times \mathbb{B}^1_K \to \mathcal{U}_\Delta$ be the other projection. By [OR11, Lemma 4.14] we have

$$\widetilde{W} \subset (\mathrm{Trop} \circ \pi')^{-1}(\Delta^\circ) \subset \mathrm{Int}(\mathcal{U}_\Delta \times \mathbb{B}^1_K/\mathbb{B}^1_K).$$

According to the sequence of morphisms $\widetilde{W} \to \mathcal{U}_\Delta \times \mathbb{B}^1_K \to \mathbb{B}^1_K$ we have

$$\mathrm{Int}(\widetilde{W}/\mathbb{B}^1_K) = \mathrm{Int}(\widetilde{W}/\mathcal{U}_\Delta \times \mathbb{B}^1_K) \cap \mathrm{Int}(\mathcal{U}_\Delta \times \mathbb{B}^1_K/\mathbb{B}^1_K) = \mathrm{Int}(\widetilde{W}/\mathcal{U}_\Delta \times \mathbb{B}^1_K) = \widetilde{W}.$$

Hence π is boundaryless on $\widetilde{W}$. Consequently π is proper and hence finite on $\widetilde{W}$.

Now the flatness of π follows from [Liu02, Exercise 1.2.12] and induction. Note that the finiteness of fibers of π and the Cohen-Macaulayness of W ensures that each l_i is not a zero divisor on $W \cap Y_1(t) \cap \cdots \cap Y_{i-1}(t)$ for all t. □

Remark 3.4. An algebraic analogue of Proposition 3.2 is that a reduced (or Cohen-Macaulay) closed subscheme Z of T_N is locally analytically isomorphic to a torus at a point that tropicalizes to the relative interior of a maximal face of $\mathrm{Trop}(Z)$ of multiplicity one; see for example [He16, Lemma 6.2].

4. Lifting tropical intersections in a polyhedral domain

Let N be an arbitrary lattice of rank n as in Theorem 1.2 and let T_N be the induced torus. Let Δ be a polytope of maximal dimension in $N_\mathbb{R}$. In this section we use Osserman and Rabinoff's continuity theorem [OR11, §5] of analytic intersection numbers to prove a generalized version of Theorem 1.2. Let $\mathcal{U}_0 \subset T_N^{\mathrm{an}}$ be the preimage of the origin in $N_\mathbb{R}$. Then $\mathcal{U}_0$ acts on $\mathcal{U}_\Delta$. Denote the action by $\mu\colon \mathcal{U}_0 \times \mathcal{U}_\Delta \to \mathcal{U}_\Delta$ and let $\pi\colon \mathcal{U}_0 \times \mathcal{U}_\Delta \to \mathcal{U}_0$ be the projection. This gives an isomorphism:

$$(\pi, \mu)\colon \mathcal{U}_0 \times \mathcal{U}_\Delta \to \mathcal{U}_0 \times \mathcal{U}_\Delta.$$

The following lemma is a consequence of [OR11, Proposition 5.8]:

Lemma 4.1. *Let π be as above. Let $\mathcal{Y}, \mathcal{Y}' \subset \mathcal{U}_0 \times \mathcal{U}_\Delta$ be Zariski-closed subspaces, flat over $\mathcal{U}_0$, such that $\mathcal{Y} \cap \mathcal{Y}'$ is finite over $\mathcal{U}_0$. Then the map*

$$s \mapsto i(\mathcal{Y}_s \cdot \mathcal{Y}'_s; \mathcal{U}_\Delta) \quad : \quad |\mathcal{U}_0| \to \mathbb{Z}$$

is constant on $\mathcal{U}_0$, where $\mathcal{Y}_s$ and $\mathcal{Y}'_s$ are fibers of π.

We refer to [Duc11] or [OR11, §5] about the notion of flatness for analytic spaces which is preserved under composition and change of base. Any analytic space is flat over K. We now prove the following theorem:

Theorem 4.2. *Let Σ be a polyhedral complex of maximal dimension in $N_\mathbb{R}$ consisting of integral G-affine polyhedra, and let $\mathcal{U}_\Sigma$ be the preimage of $|\Sigma|$ in $(\mathbb{G}_m^n)^{\mathrm{an}}$. Let $\mathcal{X}_1, ..., \mathcal{X}_m$ be Zariski-closed subspaces of $\mathcal{U}_\Sigma$ of pure dimensions whose tropicalizations intersect properly. Then*

$$\mathrm{Trop}(\mathcal{X}_1) \cap \cdots \cap \mathrm{Trop}(\mathcal{X}_m) \cap |\Sigma|^\circ = \mathrm{Trop}(\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_m) \cap |\Sigma|^\circ.$$

In particular, we have

$$\mathrm{Trop}(\mathcal{X}_1) \cap \cdots \cap \mathrm{Trop}(\mathcal{X}_m) = \mathrm{Trop}(\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_m)$$

if $\mathcal{X}_1, ..., \mathcal{X}_m$ are Zariski-closed analytic subspaces of T_N^{an} with proper tropical intersections.

Proof. As the statement is local, we may assume that $\Sigma = \Delta$ is a polytope. We first prove the equality for $\mathcal{X}_1, ..., \mathcal{X}_m$ such that $\mathrm{codim}(\mathcal{X}_1) + \cdots + \mathrm{codim}(\mathcal{X}_m) = n$. By the argument in [OP13, §5.2], it is enough to consider the case $m = 2$. Now suppose $\mathcal{X}, \mathcal{X}' \subset \mathcal{U}_\Delta$ are Zariski closed analytic subspaces such that $\mathrm{codim}(\mathcal{X}) + \mathrm{codim}(\mathcal{X}') = n$; hence $\mathcal{X} \cap \mathcal{X}'$ is finite. As the statement is local, we may also assume that $\mathrm{Trop}(\mathcal{X}) \cap \mathrm{Trop}(\mathcal{X}')$ contains only one point w that lies in Δ°. It suffices to show that $\mathcal{X} \cap \mathcal{X}'$ is non-empty.

In Lemma 4.1, let $\mathcal{Y} = (\pi, \mu)(\mathcal{U}_0 \times \mathcal{X})$ and $\mathcal{Y}' = \mathcal{U}_0 \times \mathcal{X}'$ where π and μ are as above. Then both $\mathcal{Y}$ and $\mathcal{Y}'$ are flat over $\mathcal{U}_0$. On the other hand, the argument in Lemma 3.3 shows that $\mathcal{Y} \cap \mathcal{Y}'$ is finite over $\mathcal{U}_0$, as $\mathrm{Trop}(\mathcal{Y}_s) \cap \mathrm{Trop}(\mathcal{Y}'_s) = \mathrm{Trop}(\mathcal{X}) \cap \mathrm{Trop}(\mathcal{X}')$ is finite for every $s \in |\mathcal{U}_0|$. Therefore $i(\mathcal{Y}_s \cdot \mathcal{Y}'_s; \mathcal{U}_\Delta)$ is constant on $|\mathcal{U}_0|$ by Lemma 4.1. Take $\xi \in |\mathcal{X}|$ and $\xi' \in |\mathcal{X}'|$ such that $\mathrm{Trop}(\xi) = \mathrm{Trop}(\xi') = w$. Take also $t \in |\mathcal{U}_0|$ such that $t(\xi) = \xi'$. Then $\mathcal{Y}_t \cap \mathcal{Y}'_t = t(\mathcal{X}) \cap \mathcal{X}'$ contains ξ', hence $i(\mathcal{Y}_t \cdot \mathcal{Y}'_t; \mathcal{U}_\Delta) > 0$. Thus $i(\mathcal{Y}_s \cdot \mathcal{Y}'_s; \mathcal{U}_\Delta) > 0$ for all $s \in |\mathcal{U}_0|$. Taking s to be the identity in $\mathcal{U}_0$ implies that $\mathcal{X} \cap \mathcal{X}' = \mathcal{Y}_s \cap \mathcal{Y}'_s$ is non-empty. Thus $w \in \mathrm{Trop}(\mathcal{X}) \cap \mathrm{Trop}(\mathcal{X}')$ and the proof for $\mathrm{codim}(\mathcal{X}_1) + \cdots + \mathrm{codim}(\mathcal{X}_m) = n$ is completed.

Now assume $\mathrm{codim}(\mathcal{X}_1) + \cdots + \mathrm{codim}(\mathcal{X}_m) < n$. Again by [OP13, §5.2] we may only consider $m = 2$ and take $\mathcal{X}, \mathcal{X}' \subset \mathcal{U}_\Delta$ such that $\mathrm{Trop}(\mathcal{X}) \cap \mathrm{Trop}(\mathcal{X}')$ has dimension $l > 0$. For any G-rational point $v \in \mathrm{Trop}(\mathcal{X}) \cap \mathrm{Trop}(\mathcal{X}') \cap \Delta^\circ$ we can find a Zariski-closed subspace $\mathcal{Z}$ of $\mathcal{U}_\Delta$ of codimension l such that $\mathrm{Trop}(\mathcal{Z})$ contains v and intersects properly with $\mathrm{Trop}(\mathcal{X}) \cap \mathrm{Trop}(\mathcal{X}')$ near v. Hence the argument above implies that

$$v \in \mathrm{Trop}(\mathcal{Z} \cap \mathcal{X} \cap \mathcal{X}') \subset \mathrm{Trop}(\mathcal{X} \cap \mathcal{X}').$$

As G-rational points are dense in $\mathrm{Trop}(\mathcal{X}) \cap \mathrm{Trop}(\mathcal{X}')$ this implies that $\mathrm{Trop}(\mathcal{X}) \cap \mathrm{Trop}(\mathcal{X}') \cap \Delta^\circ = \mathrm{Trop}(\mathcal{X} \cap \mathcal{X}') \cap \Delta^\circ$. $\square$

It is necessary to consider only the interior of Δ in Theorem 1.2 or Theorem 4.2. See the example below.

Example 4.3. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two curves in $(K^*)^2$ defined by $x+y+1 = 0$ and $x+y = 0$ respectively. Let $\Delta = [0,1]^2$ as in Figure 4. Then $\mathrm{Trop}(\mathcal{X}_1) \cap \mathrm{Trop}(\mathcal{X}_2) \cap \Delta$ is the origin; hence $\mathrm{Trop}(\mathcal{X}_1)$ and $\mathrm{Trop}(\mathcal{X}_2)$ intersect properly in Δ. But $\mathcal{X}_1 \cap \mathcal{X}_2 \cap \mathcal{U}_\Delta$ is empty.

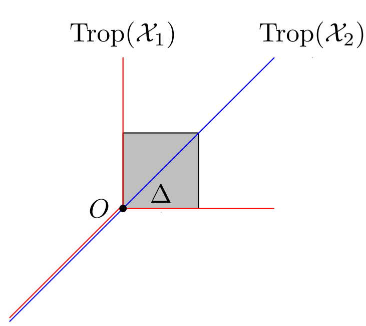

FIGURE 4

5. Lifting divisors with imposed ramification

In this section we prove Theorem 1.1. We will use the notation in Notation 3.1. Let $\alpha = (\alpha_0, ..., \alpha_r)$ be a Schubert index of type (d, r). Let $\lambda = (\lambda_0, ..., \lambda_r)$ be the induced partition. Hence $\lambda_i = g - d + r + \alpha_{r-i}$. See Figure 5.

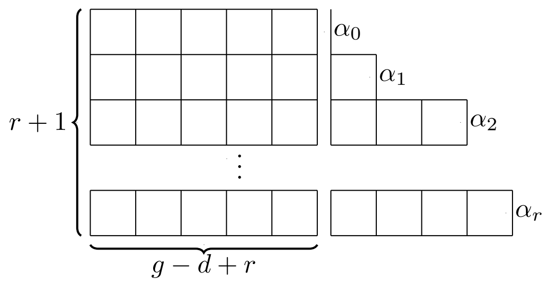

Figure 5

After translating every divisor class on C (resp. Γ) of degree d to its image in $\mathrm{Pic}^0(C)$ (resp. $\mathrm{Pic}^0(\Gamma)$) under the Abel-Jacobi map induced by $d\mathcal{P}$ (resp dP) we may assume that the ramification is imposed by λ (instead of α). Now Theorem 1.1 is equivalent to the following:

Theorem 5.1. *We have* $\mathrm{Trop}(W^\lambda(C)) = W^\lambda(\Gamma)$.

The strategy of proof is to write $W^\lambda(C)$ as an intersection in $\mathrm{Pic}^0(C)$ of Brill-Noether loci $W^{\lambda_i}(C)$, which is isomorphic to some $W^{r_i}_{d_i}(C)$, such that $W^\lambda(\Gamma)$ is equal to the intersection of $\mathrm{Trop}(W^{\lambda_i}(C)) = W^{\lambda_i}(\Gamma)$. Then use the commutativity of tropicalization with intersection (Theorem 4.2) to get the equality.

Let λ_j be the partition corresponding to the $(r + 1 - j) \times (g - d + r + \alpha_j)$ diagram (see Figure 6). Then $W^{\lambda_j}(C)$ is isomorphic to $W^{r-j}_{d-\alpha_j-j}(C)$ as well as to $X_j(C)$ in Section 1, and $W^\lambda(C) = \bigcap_{0\le i\le r} W^{\lambda_i}(C)$, while $W^{\lambda_j}(\Gamma)$ is isomorphic to $W^{r-j}_{d-\alpha_j-j}(\Gamma)$ as well as to $X_j(\Gamma)$ in Section 1, and $W^\lambda(\Gamma) = \bigcap_{0\le i\le r} W^{\lambda_i}(\Gamma)$.

Note that $W^{\lambda_j}(C)$ does not satisfy the condition of Theorem 4.2; namely, the intersection of all $W^{\lambda_j}(\Gamma)$ is not proper in $\mathrm{Pic}^0(\Gamma)$. To resolve this issue, let λ^j be the union of $\lambda_1, ..., \lambda_j$ and $W_j(C) = \bigcap_{0\le i\le j} W^{\lambda_i}(C) = W^{\lambda^j}(C)$ and $W_j(\Gamma) = \bigcap_{0\le i\le j} W^{\lambda_i}(\Gamma) = W^{\lambda^j}(\Gamma)$ for $0 \le j \le r$. We consider inductively $W_j(C)$ and $W^{\lambda_{j+1}}(C)$ as subspaces of a locus $W^{\mu_j}(C)$, isomorphic to some $W^{r'_j}_{d'_j}(C)$, of suitable dimension. Then the local property in Section 3 and Theorem 4.2 would ensure that their tropicalizations commute with intersection.

Let μ_j be the partition corresponding to the $(r - j) \times (g - d + r + \alpha_j)$ diagram (this is the intersection of λ^j and λ_{j+1}). As above we have $W^{\mu_j}(C)$ isomorphic to $W^{r-j-1}_{d-\alpha_j-j-1}(C)$ as well as to $Y_j(C)$ in Section 1, and $W^{\mu_j}(\Gamma)$ isomorphic to $W^{r-j-1}_{d-\alpha_j-j-1}(\Gamma)$ as well as to $Y_j(\Gamma)$ in Section 1. Moreover, we have $W_j(C) \subset W^{\mu_j}(C)$ and $W^{\lambda_{j+1}}(C) \subset W^{\mu_j}(C)$ and $W_j(\Gamma) \subset W^{\mu_j}(\Gamma)$ and $W^{\lambda_{j+1}}(\Gamma) \subset W^{\mu_j}(\Gamma)$.

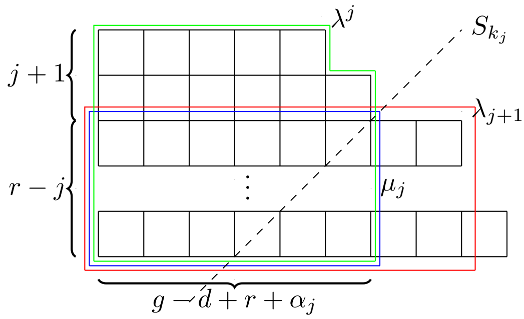

FIGURE 6

We first show the following lemma:

Lemma 5.2. *$W_j(\Gamma)$ and $W^{\lambda_{j+1}}(\Gamma)$ intersect properly in $W^{\mu_j}(\Gamma)$, and there is an open dense subset U_j of $W_j(\Gamma) \cap W^{\lambda_{j+1}}(\Gamma) = W_{j+1}(\Gamma)$ which is contained in* $\mathrm{relint}(W^{\mu_j}(\Gamma))$.

Proof. The properness follows directly from dimension counting (Proposition 2.6), as λ^{j+1} is the union of λ^j and λ_{j+1} while μ_j is the intersection of λ^j and λ_{j+1}. More precisely, we have

$$\dim(W_j(\Gamma)) = g - \big(\sum_{i=0}^{j-1} g - d + r + \alpha_i\big) - (r+1-j)(g-d+r+\alpha_j),$$
$$\dim(W^{\lambda_{j+1}}(\Gamma)) = g - (r-j)(g-d+r+\alpha_{j+1}),$$
$$\dim(W^{\mu_j}(\Gamma)) = g - (r-j)(g-d+r+\alpha_j),$$

and

$$\dim(W_{j+1}(\Gamma)) = g - \big(\sum_{i=0}^{j} g - d + r + \alpha_i\big) - (r-j)(g-d+r+\alpha_{j+1}).$$

Straightforward calculation shows that

$$\dim(W_j(\Gamma)) + \dim(W^{\lambda_{j+1}}(\Gamma)) = \dim(W^{\mu_j}(\Gamma)) + \dim(W_{j+1}(\Gamma)).$$

For the second conclusion it suffices to show that every real torus in $W_{j+1}(\Gamma)$ is contained in exactly one torus in $W^{\mu_j}(\Gamma)$. Take two tori $\mathbb{T}(t) \subset W_{j+1}(\Gamma)$ and $\mathbb{T}(t') \subset W^{\mu_j}(\Gamma)$ such that $\mathbb{T}(t) \subset \mathbb{T}(t')$, where t and t' are standard Young tableaux on λ^{j+1} and μ_j respectively. We claim that $t' = t|_{\mu_j}$.

It is easy to see that $t'(\mu_j) \subset t(\lambda^{j+1})$. On the other hand, let $S_k = \{(x,y)|x-y = k\}$ for all $k \in \mathbb{Z}$. If $t(x,y) = t'(x',y')$, then $x - y = x' - y'$ by the construction of $\mathbb{T}(t)$ and $\mathbb{T}(t')$. It follows that $t'(\mu_j \cap S_k) \subset t(\lambda^{j+1} \cap S_k)$ for all k. In particular, let $k_j = g - d + \alpha_j + j$; then S_{k_j} is as in Figure 6, and we have

$$t'(\mu_j \cap S_{k_j}) = t(\lambda^{j+1} \cap S_{k_j})$$

since $\mu_j \cap S_{k_j} = \lambda^{j+1} \cap S_{k_j}$. Therefore $t|_{\mu_j \cap S_{k_j}} = t'|_{\mu_j \cap S_{k_j}}$ as both t and t' are strictly increasing along rows and columns.

We now prove by induction that $t|_{\mu_j\cap S_k} = t'|_{\mu_j\cap S_k}$ for all $k \le k_j$. Suppose this is true for $k = m+1$. We have $t(r-j+m+l, r-j+l) \notin t'(\mu_j \cap S_m)$ for all $l \ge 1$, since this number is bigger than all numbers in $t(\mu_j \cap S_{m+1}) = t'(\mu_j \cap S_{m+1})$, thus greater than the numbers in $t'(\mu_j \cap S_m)$. It then follows that $t(\mu_j \cap S_m) = t'(\mu_j \cap S_m)$; therefore $t|_{\mu_j\cap S_m} = t'|_{\mu_j\cap S_m}$.

The same argument as above shows that $t|_{\mu_j\cap S_k} = t'|_{\mu_j\cap S_k}$ for all $k \ge k_j$. Thus $t' = t|_{\mu_j}$. □

Proof of Theorem 5.1. We prove by induction that $\mathrm{Trop}(W_k(C)) = W_k(\Gamma)$ for all $0 \le k \le r$. The $k = 0$ case is in Theorem 2.8. Now assume $\mathrm{Trop}(W_j(C)) = W_j(\Gamma)$; we need to show that

$$\mathrm{Trop}(W_j(C) \cap W^{\lambda_{j+1}}(C)) = \mathrm{Trop}(W_{j+1}(C)) = W_{j+1}(\Gamma). \tag{2}$$

Let U_j be as in Lemma 5.2 and fix $w \in U_j$. As we only care about the local geometry near w, we may assume all Brill-Noether loci corresponding to $(C, \mathcal{P})$ (resp. (Γ, P)) are contained in a polytopal domain (resp. polytope) in T_N (resp. $N_{\mathbb{R}}$). We may also assume that w is the origin. Take a polytope $\Lambda \subset \mathrm{relint}(W^{\mu_j}(\Gamma))$ such that $w \in \mathrm{relint}(\Lambda)$. According to Proposition 3.2 we have the following commutative diagram:

$$\begin{array}{ccc} W^{\mu_j}(C)^{\mathrm{an}} \cap \mathcal{U}_\Lambda & \xrightarrow{\mathrm{Trop}} & \Lambda \\ \big\downarrow{\scriptstyle \pi_\Lambda} & & \big\downarrow{\scriptstyle \pi} \\ \widetilde{\mathcal{U}}_\Lambda & \xrightarrow{\mathrm{Trop}} & \pi(\Lambda) \end{array}$$

where both vertical arrows are isomorphisms induced by the natural projection from N to N_Λ as in Proposition 3.2.

Let

$$W_\Lambda^{\lambda_{j+1}} = W^{\lambda_{j+1}}(C)^{\mathrm{an}} \cap \mathcal{U}_\Lambda \qquad \text{and} \qquad W_{\Lambda,j} = W_j(C)^{\mathrm{an}} \cap \mathcal{U}_\Lambda.$$

According to Lemma 5.2 $\mathrm{Trop}(\pi_\Lambda(W_\Lambda^{\lambda_{j+1}}))$ and $\mathrm{Trop}(\pi_\Lambda(W_{\Lambda,j}))$ intersect properly in $\pi(\Lambda)$, which is a polytope of maximal dimensional in $(N_\Lambda)_{\mathbb{R}}$ that contains $\pi(w)$ as an interior point. Hence Theorem 4.2 implies that

$$\pi(w) \in \mathrm{Trop}(\pi_\Lambda(W_\Lambda^{\lambda_{j+1}}) \cap \pi_\Lambda(W_{\Lambda,j}))$$

and that

$$w \in \mathrm{Trop}(W_\Lambda^{\lambda_{j+1}} \cap W_{\Lambda,j}) \subset \mathrm{Trop}(W_j(C) \cap W^{\lambda_{j+1}}(C)).$$

As U_j is dense in $W_{j+1}(\Gamma)$ and $U_j \subset \mathrm{Trop}(W_j(C) \cap W^{\lambda_{j+1}}(C))$, we have $W_{j+1}(\Gamma) \subset \mathrm{Trop}(W_j(C) \cap W^{\lambda_{j+1}}(C))$. This proves (2), as the other direction of containment is trivial. □

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