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ON THE LOSS OF MAXIMUM PRINCIPLES FOR HIGHER-ORDER FRACTIONAL LAPLACIANS

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Abstract. We study the existence and positivity of solutions to problems involving higher-order fractional Laplacians $(-\Delta)^s$ for any $s > 1$. In particular, using a suitable variational framework and the nonlocal properties of these operators, we provide an explicit counterexample to general maximum principles for $s \in (n, n+1)$ with $n \in \mathbb{N}$ odd, and we mention some particular domains where positivity preserving properties do hold.

1. Introduction

In this paper we show that the so-called *positivity preserving properties* fail in general for the higher-order fractional Laplacian. This is stated in the following result.

Theorem 1.1. *Let $N \in \mathbb{N}$, let $k \in \mathbb{N}$ be odd, let $s \in (k, k+1)$, let $D \subset \mathbb{R}^N$ be an open set, let A be an open ball compactly contained in $\mathbb{R}^N \setminus D$, and let $\Omega := D \cup A$. There are $f \in C^\infty(\overline{\Omega})$ and a sign-changing $u \in C^s(\mathbb{R}^N) \cap C^\infty(\Omega) \cap L^\infty(\mathbb{R}^N)$ such that*

$$(-\Delta)^s u = f > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \mathbb{R}^N \backslash \Omega;$$

in particular $u \lneq 0$ in D and $u > 0$ in A. If Ω is a bounded set, then the solution u is unique.

Before we give a precise definition for the operator $(-\Delta)^s$ (see (1.2) below) and discuss our strategy to prove Theorem 1.1, we motivate the study of higher-order powers of the Laplacian.

In the study of elliptic partial differential equations, most of the analysis has been focused on second-order problems, which effectively describe many natural phenomena. The available results on existence and qualitative properties in this setting have achieved a remarkable degree of sophistication, to a large extent due to very powerful analytic techniques derived from maximum principles, for instance, Harnack inequalities, Hopf lemmas, and sub- and supersolutions methods.

The theory for elliptic higher-order (i.e., higher than 2) operators, on the other hand, is comparatively underdeveloped. Some of the main difficulties that appear in their study is precisely the lack of maximum principles, the fact that the set of solutions is usually larger and more complex, and a much more subtle relationship between regularity of solutions, boundary conditions, and smoothness of the domain.

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Nevertheless, higher-order operators appear in many important models coming, for instance, from continuum mechanics, biophysics, and differential geometry. They appear, for example, in the study of thin elastic plates, stationary surface diffusion flow, Paneitz–Branson equations, Willmore surfaces, suspension bridges, phase-transition, and membrane biophysics; see [13, 23] and the references therein. The study of higher-order operators is also motivated by the understanding of basic questions in the theory of partial differential equations, to identify the key elements which yield existence, uniqueness, qualitative properties, and regularity of solutions.

The paradigmatic higher-order operator is given by powers of the Laplacian $(-\Delta)^m$, $m \in \mathbb{N}$, also known as the *polyharmonic operator*. The validity and characterization of positivity preserving properties in this case is an active field of research, and many basic questions are still open. For example, consider $m = 2$, i.e., the bilaplacian operator $\Delta^2 u = \Delta(\Delta u)$, for which maximum principles are known to be a very delicate issue and do *not* hold in general. To obtain well-posedness in boundary value problems, the bilaplacian requires extra boundary conditions (*b.c.*), for instance *Dirichlet b.c.* $u = \partial_\nu u = 0$ on $\partial\Omega$. The validity of maximum principles for the bilaplacian with Dirichlet b.c. is particularly delicate, and the geometry of the domain plays an essential role. It is known that $\Delta^2 u \geq 0$ in Ω and $u = \partial_\nu u = 0$ on $\partial\Omega$ implies that $u \geq 0$ if Ω is a ball, for example, since the corresponding Green function can be computed explicitly in this case and it is nonnegative. However, if $\Omega \subset \mathbb{R}^2$ is an ellipse with semiaxis 1 and $\frac{1}{5}$, then one can give an elementary counterexample (a polynomial of degree 7) showing that the maximum principle does not hold; see [27]. We also refer to [17] for counterexamples involving even powers of the Laplacian and to [29] for a counterexample to the trilaplacian, which seems to be the only available counterexample for odd powers. See also [13] and the references therein for a survey on positivity preserving properties for boundary value problems involving polyharmonic operators.

In this paper, we study the loss of positivity preserving properties for *higher-order fractional powers* of the Laplacian $(-\Delta)^s$, $s > 1$. Some known results for this operator are the following. General regularity results have been proved in [14], a Pohožaev identity and an integration by parts formula is given in [24], a comparison between different higher-order fractional operators is done in [20], spectral results are obtained in [15], and other aspects of nonlinear problems are considered in [12, 18, 19, 22]. A discussion on the pointwise definition of $(-\Delta)^s$ can be found in [1] and explicit integral representations of solutions in balls in [3]. Furthermore, the operator $(-\Delta)^s$ with $s \geq 1$ appears naturally in geometry, for example, in the prescribed Q-curvature equation $(-\Delta)^{N/2} u = K e^{Nu}$ [8]. We believe that the study of higher-order fractional powers of the Laplacian can be a powerful tool to improve our understanding of the qualitative differences between solutions of second-order and higher-order equations, particularly in the transition between the Laplacian and the bilaplacian, which exhibit very different properties between them.

To continue our discussion on $(-\Delta)^s$ for $s > 1$, let us first consider the case $(-\Delta)^\sigma$ with $\sigma \in (0, 1)$, $N \in \mathbb{N}$, and $u \in C_c^\infty(\mathbb{R}^N)$. This operator is known as the *fractional Laplacian* and can be represented via the principal value integral

(1.1)
$$(-\Delta)^\sigma u(x) := c_{N,\sigma} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}}\, dy := c_{N,\sigma} \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}}\, dy$$

for $x \in \mathbb{R}^N$, where $c_{N,\sigma} := 4^\sigma \pi^{-N/2}\sigma(1-\sigma)\frac{\Gamma(\frac{N}{2}+\sigma)}{\Gamma(2-\sigma)}$ is a normalization constant and Γ denotes the Gamma function. This operator is used to model *nonlocal* interactions [5]. Since $(-\Delta)^s$ is a nonlocal operator, boundary value problems are solved by prescribing boundary conditions in the whole complement of the domain (see, e.g., [16]). In this case, as mentioned in [6, Remark 4.2], the maximum principle holds in a weak setting for $\sigma \in (0,1)$ using the Dirichlet-to-Neumann extension from [7] and testing the equation with $u^- := -\min\{u,0\}$. This also follows directly from the nonlocal bilinear form

$$\begin{aligned}\mathcal{E}_\sigma(\varphi,\psi) :&= \frac{c_{N,\sigma}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))}{|x-y|^{N+2\sigma}}\,dx\,dy\\ &= \int_{\mathbb{R}^N}|\xi|^{2\sigma}\mathcal{F}\varphi(\xi)\mathcal{F}\psi(\xi)\,d\xi,\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform; see [5]. In particular, if $\Omega \subset \mathbb{R}^N$ is an open set, u is in the fractional Sobolev space $H^s(\mathbb{R}^N)$, $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$, and $\mathcal{E}_\sigma(u,\varphi) \geq 0$ for all *nonnegative* $\varphi \in H^\sigma(\mathbb{R}^N)$ with $\varphi \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, then $u \geq 0$ in Ω.

For $m \in \mathbb{N}_0$, $\sigma \in (0,1)$, and $s = m+\sigma$, the operator $(-\Delta)^s$ can be defined via finite differences (see [1, equation (1)]), namely,

$$(-\Delta)^s u(x) := \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\frac{\delta_{m+1}u(x,y)}{|y|^{N+2s}}\,dy, \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where

$$\delta_{m+1}u(x,y) := \sum_{k=-m-1}^{m+1}(-1)^k\binom{2(m+1)}{m+1-k}u(x+ky) \qquad \text{for } x,y \in \mathbb{R}^N$$

is a finite difference of order $2(m+1)$ and $c_{N,s}$ is a *positive* normalization constant (for the precise value, see [1, equation (2)]). The Fourier symbol of $(-\Delta)^s$ as given in (1.2) is $|\xi|^{2s}$ (see [25, Lemma 25.3] or [1, Theorem 1.9]); moreover, if $\Omega \subset \mathbb{R}^N$ is an open set and $u \in C^\infty(\Omega) \cap L^\infty(\mathbb{R}^N)$, then $(-\Delta)^s u(x) = (-\Delta)^m(-\Delta)^\sigma u(x)$ for every $x \in \Omega$ (see [1, Corollary 1.3]). In general the fractional Laplacian $(-\Delta)^\sigma$ cannot be interchanged freely with the Laplacian $(-\Delta)$, as this would require extra regularity assumptions on u, particularly across the boundary $\partial\Omega$ (see [1]).

To find solutions for $(-\Delta)^s$, we use the following variational setting. For $\Omega \subset \mathbb{R}^N$ open we define the fractional Sobolev space with zero boundary conditions

$$\mathcal{H}^s_0(\Omega) := \{u \in H^s(\mathbb{R}^N) \ : \ u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega\} \tag{1.3}$$

equipped with the norm $\|u\|_{\mathcal{H}^s_0(\Omega)} := (\sum_{|\alpha|\leq m}\|\partial^\alpha u\|^2_{L^2(\Omega)} + \mathcal{E}_s(u,u))^{\frac{1}{2}}$, where

$$\mathcal{E}_s(u,v) := \begin{cases}\mathcal{E}_\sigma(\Delta^{\frac{m}{2}}u, \Delta^{\frac{m}{2}}v), & \text{if } m \text{ is even},\\ \displaystyle\sum_{k=1}^N \mathcal{E}_\sigma(\partial_k\Delta^{\frac{m-1}{2}}u, \partial_k\Delta^{\frac{m-1}{2}}v), & \text{if } m \text{ is odd},\end{cases} \tag{1.4}$$

for $u,v \in \mathcal{H}^s_0(\Omega)$. For $f \in L^2(\Omega)$ a function $u \in \mathcal{H}^s_0(\Omega)$ is a *weak solution* of

$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega \tag{1.5}$$

if

$$\mathcal{E}_s(u,\varphi) = \int_\Omega f(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in \mathcal{H}^s_0(\Omega). \tag{1.6}$$

The following result shows that maximum principles do not hold in the weak setting, and it is the main tool to show Theorem 1.1.

Theorem 1.2. *Let $N \in \mathbb{N}$, let $D \subset \mathbb{R}^N$ be an open set, let $s \in (k, k+1)$ for some $k \in \mathbb{N}$ odd, and let A be a nonempty ball compactly contained in $\mathbb{R}^N \setminus D$. There is a nonnegative function $f \in L^2(\Omega)$ such that the problem* (1.5) *in $\Omega = D \cup A$ has a sign-changing weak solution $u \in \mathcal{H}^s_0(\Omega) \cap C(\mathbb{R}^N)$ with $u \lneq 0$ in D and $u \gneq 0$ in A.*

Theorem 1.2 is particularly interesting for $s \in (1, \frac{3}{2})$, because in this case [4, Théorème 1] implies that $u^- \in H^s(\Omega)$ if $u \in H^s(\Omega)$. Since this is the main ingredient in the proof of maximum principles for $s \in (0, 1]$—which uses u^- as a test function—it was expected that maximum principles would hold for $s \in (0, \frac{3}{2})$. Theorem 1.2, however, reveals that it is *not* the belonging of u^- to the space of test functions as to the reason why maximum principles hold for $s \in (0, 1]$, and that this positivity preserving property immediately fails to hold in general for $s \in (1, 2)$.

To show Theorem 1.2 we construct an explicit counterexample, which exploits the nonlocal nature of the operator and the fact that the domain is disconnected. For dimensions $N \geq 2$ it is possible to use a perturbation argument to extend Theorem 1.2 to connected domains (see for example [3, Theorem 3.6]). If $N = 1$, then connected domains are intervals and the maximum principle holds, as discussed below.

The proof of Theorem 1.2 also reveals that an essential role is played by the following fact due to integration by parts: for $u \in H^s(\mathbb{R}^N)$, $\phi \in C^\infty_c(\mathbb{R}^N)$, and $u, \phi \geq 0$ with $\operatorname{supp} u \cap \operatorname{supp} \phi = \emptyset$, we have that $\mathcal{E}_s(u, \phi) < 0$ if $s \in (0, 1)$ and $\mathcal{E}_s(u, \phi) > 0$ if $s \in (k, k+1)$ with $k \in \mathbb{N}$ odd. This is the main reason why the proof of maximum principles for $s \in (0, 1)$ cannot be extended to $s \in (1, \frac{3}{2})$; see Remark 4.6. Another consequence of this fact is the following remarkable property.

Proposition 1.3. *Let $m \in \mathbb{N}_0$, let $\sigma \in (0, 1)$, let $s = m + \sigma$, let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, and let $g \in C^\infty_c(\Omega) \backslash \{0\}$ be a nonnegative function. Then $(-1)^{m+1}(-\Delta)^s g > 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$.*

Note that this is a purely nonlocal phenomenon. Another direct consequence of Theorem 1.2 is the following.

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set such that $\mathbb{R}^N \backslash \Omega$ has nonempty interior, and let $s \in (k, k+1)$ for some $k \in \mathbb{N}$ odd. There is a function $u \in H^s(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ satisfying* (1.6) *with a nonnegative function $f \in L^2(\Omega)$ such that $u \lneq 0$ in Ω and $u \geq 0$ in $\mathbb{R}^N \backslash \Omega$.*

In particular, maximum principles for $(-\Delta)^s$ with $s > 1$ cannot hold in general for a notion of a supersolution.

To close this introduction let us mention some particular domains where maximum principles *do* hold for $(-\Delta)^s$ with $s > 1$. If $\Omega = \mathbb{R}^N$, then solutions are given by a convolution with the fundamental solution [25, equation (25.32) and Theorem 26.3] (which is positive in some cases); if Ω is a ball, then the associated Green function is positive [2, 9], which immediately implies a positivity preserving property. Similarly, by [3, Theorem 1.10] the Green function of two disjoint unitary balls is positive if $s \in (k, k+1)$ with $k \in \mathbb{N}$ *even* (this is also the reason why our counterexample cannot be applied for s in this range). Finally, the Green function for the halfspace and for the complement of the ball are also positive, and we give the details in a future work.

Although our approach to prove Theorem 1.2 cannot be used for $s \in (k, k+1)$ with $k \in \mathbb{N}$, even we expect that general maximum principles do not hold for any $s > 1$. However, finding a counterexample for s in this range remains an open problem.

2. Notation

Let $N \in \mathbb{N}$ and $U, D \subset \mathbb{R}^N$ be nonempty measurable sets. For $h \in C(\mathbb{R}^N)$, we say that $h \lneq 0$ (resp., $h \gneq 0$) in U if $h \leq 0$ in U and $\inf_U h < 0$ (resp., $h \geq 0$ in U and $\sup_U h > 0$). We denote by $1_U : \mathbb{R}^N \to \mathbb{R}$ the characteristic function and by $|U|$ the Lebesgue measure. The notation $D \Subset U$ means that $\overline{D}$ is compact and contained in the interior of U. The distance between D and U is given by $\operatorname{dist}(D,U) := \inf\{|x-y| \,:\, x \in D,\, y \in U\}$. For $x \in \mathbb{R}^N$ and $r > 0$ let $B_r(x)$ denote the open ball centered at x with radius r; moreover, we fix $B := B_1(0)$.

If u is in a suitable function space, we use $\mathcal{F}u$ to denote the Fourier transform of u. For any $s \in \mathbb{R}$, we define $H^s(\mathbb{R}^N) := \big\{u \in L^2(\mathbb{R}^N) \,:\, (1+|\xi|^2)^{\frac{s}{2}}\, \mathcal{F}u \in L^2(\mathbb{R}^N)\big\}$; moreover, if U is open, we define $\mathcal{H}^s_0(U)$ as in (1.3) and, if U is smooth, we put $H^s(U) := \{u1_U \,:\, u \in H^s(\mathbb{R}^N)\}$.

For $m \in \mathbb{N}_0$, $\sigma \in (0,1)$, $s = m+\sigma$, and U open, we write $C^s(U)$ (resp., $C^s(\overline{U})$) to denote the space of m-times continuously differentiable functions in U (resp., $\overline{U}$) whose derivatives of order m are σ-Hölder continuous in U. Moreover, for $s \in [0,\infty]$, $C^s_c(U) := \{u \in C^s(\mathbb{R}^N) : \operatorname{supp} u \Subset U\}$ and $C^s_0(U) := \{u \in C^s(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus U\}$, where $\operatorname{supp} u := \overline{\{x \in U \,:\, u(x) \neq 0\}}$ is the support of u.

Recall (1.4). If $m \in \mathbb{N}$ is odd we also use the following vector notation:

$$\mathcal{E}_\sigma(\nabla\Delta^{\frac{m-1}{2}}u, \nabla\Delta^{\frac{m-1}{2}}u) := \sum_{k=1}^{N} \mathcal{E}_\sigma(\partial_k\Delta^{\frac{m-1}{2}}u, \partial_k\Delta^{\frac{m-1}{2}}u) = \mathcal{E}_s(u,u).$$

Let $u : U \to \mathbb{R}$ be a function. We use $u^+ := \max\{u, 0\}$ and $u^- := -\min\{u, 0\}$ to denote the positive and negative part of u, respectively. Finally, Γ denotes the standard *Gamma function*, and if $f : U \times D \to \mathbb{R}$ we write $\Delta_x f(x,y)$ to denote derivatives with respect to x, whenever they exist in some appropriate sense.

3. Variational framework

Let $\Omega \subset \mathbb{R}^N$ be an open set, and fix $m \in \mathbb{N}_0 := \{0,1,2,\ldots\}$, $\sigma \in (0,1)$, and $s = m+\sigma$. Recall the space $\mathcal{H}^s_0(\Omega)$ as defined in (1.3) equipped with the bilinear form $\mathcal{E}_s(\cdot,\cdot)$ defined in (1.4). Since the particular variational framework presented in the introduction does not seem to have been used before in the literature, we begin by showing the equivalence between the definition of a weak solution (see (1.6)) and the definition of a solution via the Fourier transform $\mathcal{F}$.

Proposition 3.1. *Let $f \in L^2(\Omega)$. The function $u \in H^s(\mathbb{R}^N)$ is a weak solution of* (1.5) *if and only if*

$$\int_{\mathbb{R}^N} |\xi|^{2s}\mathcal{F}u(\xi)\mathcal{F}\varphi(\xi)\, d\xi = \int_{\mathbb{R}^N} f(x)\varphi(x)\, dx$$

for all $\varphi \in \mathcal{H}^s_0(\Omega)$. Moreover, the operator $(-\Delta)^s : H^{2s}(\mathbb{R}^N) \to L^2(\Omega)$ given by $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s}\mathcal{F}u)$ is well defined and

$$\mathcal{E}_s(u,\phi) = \int_{\mathbb{R}^N} (-\Delta)^s u(x)\varphi(x)\, dx \qquad \textit{for all } \varphi \in H^s(\mathbb{R}^N).$$

Proof. Let $\varphi \in \mathcal{H}^s_0(\Omega)$, and let $u \in H^s(\mathbb{R}^N)$. If m is even, then

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\xi|^{2s}\mathcal{F}u(\xi)\mathcal{F}\varphi(\xi)\, d\xi = \int_{\mathbb{R}^N} |\xi|^{s}\mathcal{F}u(\xi)\cdot|\xi|^{s}\mathcal{F}\varphi(\xi)\, d\xi \\
&= \int_{\mathbb{R}^N} (-\Delta)^{\frac{\sigma}{2}}\Delta^{\frac{m}{2}}u(x)\cdot(-\Delta)^{\frac{\sigma}{2}}\Delta^{\frac{m}{2}}\varphi(x)\, dx \\
&= \frac{c_{N,\sigma}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\Delta^{\frac{m}{2}}u(x)-\Delta^{\frac{m}{2}}u(y))\cdot(\Delta^{\frac{m}{2}}\varphi(x)-\Delta^{\frac{m}{2}}\varphi(y))}{|x-y|^{N+2\sigma}}\, dxdy,
\end{aligned}$$

and if m is odd, then

$$\begin{aligned}
&\frac{2}{c_{N,\sigma}}\int_{\mathbb{R}^N} |\xi|^{2s}\mathcal{F}u(\xi)\mathcal{F}\varphi(\xi)\, d\xi = \frac{2}{c_{N,\sigma}}\int_{\mathbb{R}^N} |\xi|^{s-1}(-i)\xi\mathcal{F}u(\xi)\cdot\ i\xi|\xi|^{s-1}\mathcal{F}\varphi(\xi)\, d\xi \\
&= \frac{2}{c_{N,\sigma}}\int_{\mathbb{R}^N} |\xi|^{s-1}(-i)\xi\mathcal{F}u(\xi)\cdot\overline{(-i\xi|\xi|^{s-1}\mathcal{F}\varphi(\xi))}\, d\xi \\
&= \frac{2}{c_{N,\sigma}}\int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}\nabla\Delta^{\frac{m-1}{2}}u(x)\cdot(-\Delta)^{\sigma/2}\nabla\Delta^{\frac{m-1}{2}}\varphi(x)\, dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\nabla\Delta^{\frac{m-1}{2}}u(x)-\nabla\Delta^{\frac{m-1}{2}}u(y))\cdot(\nabla\Delta^{\frac{m-1}{2}}\varphi(x)-\nabla\Delta^{\frac{m-1}{2}}\varphi(y))}{|x-y|^{N+2\sigma}}\, dxdy.
\end{aligned}$$

This proves the first part. If, in addition, $u \in H^{2s}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} |(-\Delta)^s u(x)|^2\, dx = \int_{\mathbb{R}^N} |\xi|^{4s}\left|\mathcal{F}u(\xi)\right|^2\, d\xi = \mathcal{E}_{2s}(u,u) < \infty,$$

by standard properties of the Fourier transform. Now the last part follows from the above calculations. $\square$

Remark 3.2. If $u \in H^{2s}(\mathbb{R}^N)$, then it follows from the proof of Proposition 3.1 that

$$\begin{aligned}
(-\Delta)^s u &= (-\Delta)^m(-\Delta)^\sigma u = (-\Delta)^\sigma(-\Delta)^m u \\
&= \begin{cases} (-\Delta)^{\frac{m}{2}}(-\Delta)^\sigma(-\Delta)^{\frac{m}{2}}u & \text{for } m \text{ even}, \\ \operatorname{div}(-\Delta)^{\frac{m-1}{2}}(-\Delta)^\sigma(-\Delta)^{\frac{m-1}{2}}\nabla u & \text{for } m \text{ odd}, \end{cases}
\end{aligned}$$

where $(-\Delta)^\sigma$ is defined as in (1.1); see also [1, Corollary 1].

3.1. **Poincaré inequality and principal eigenvalues.** For completeness, we now show that $\mathcal{E}_s$ satisfies a Poincaré-type inequality in bounded domains. This yields that $\mathcal{E}_s$ is a scalar product and that $(\mathcal{H}^s_0(\Omega), \mathcal{E}_s)$ is a Hilbert space. Let $\lambda_{1,s} = \lambda_{1,s}(\Omega)$ and $\lambda_{1,1} = \lambda_{1,1}(\Omega)$ denote the first eigenvalue of $((-\Delta)^s, \mathcal{H}^s_0(\Omega))$ and of $(-\Delta, H^1_0(\Omega))$, respectively.

Proposition 3.3 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary. For all $u \in \mathcal{H}^s_0(\Omega)$ we have that*

$$\mathcal{E}_s(u,u) \geq \lambda_{1,s}\|u\|^2_{L^2(\Omega)}$$

and

$$\mathcal{E}_s(u,u) \geq \begin{cases} \lambda_{1,\sigma}\|\Delta^{\frac{m}{2}}u\|^2_{L^2(\Omega)} & \text{if } m \text{ is even}, \\ \lambda_{1,\sigma}\|\nabla\Delta^{\frac{m-1}{2}}u\|^2_{L^2(\Omega)} & \text{if } m \text{ is odd}, \end{cases}$$

where

$$\lambda_{1,s} = \lambda_{1,s}(\Omega) := \min_{u \in \mathcal{H}^s_0(\Omega)\setminus\{0\}} \frac{\mathcal{E}_s(u,u)}{\|u\|^2_{L^2(\Omega)}} > 0, \tag{3.1}$$

$\lambda_{1,s} \geq \lambda_{1,1}^{\frac{m}{2}}\lambda_{1,\sigma}$ *if* m *is even, and* $\lambda_{1,s} \geq \lambda_{1,1}^{\frac{m+1}{2}}\lambda_{1,\sigma}$ *if* m *is odd. In particular,* $\lim_{r\to 0}\inf_{|\Omega|=r} \lambda_{1,s}(\Omega) = \infty$. *Moreover,* $(\mathcal{H}^s_0(\Omega), \mathcal{E}_s(\cdot,\cdot))$ *is a Hilbert space.*

Proof. Let $u \in \mathcal{H}^s_0(\Omega)$, and let m even. By standard estimates we have

$$\mathcal{E}_\sigma((-\Delta)^{\frac{m}{2}}u, (-\Delta)^{\frac{m}{2}}u) \geq \lambda_{1,\sigma}\|(-\Delta)^{\frac{m}{2}}u\|^2_{L^2(\Omega)} \geq \lambda_{1,1}^{\frac{m}{2}}\lambda_{1,\sigma}\|u\|^2_{L^2(\Omega)}.$$

Clearly this also implies that $\mathcal{E}_{1+\sigma}$ is a scalar product and (3.1) follows. The case m odd is analogous.

We now prove that $\mathcal{H}^s_0(\Omega)$ is complete with respect to $\mathcal{E}_s$. Let $(u_n)_n \subset \mathcal{H}^s_0(\Omega)$ be a Cauchy sequence with respect to $\mathcal{E}_s$. Hence by the above inequality it follows that $u_n \to u \in L^2(\Omega)$ for $n \to \infty$, where we use $L^2(\Omega) = \{u \in L^2(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \setminus \Omega\}$. Thus there is a subsequence $(u_{n_k})_k$ such that $u_{n_k} \to u$ a.e. in Ω as $k \to \infty$. By Fatou's Lemma we have

$$\mathcal{E}_s(u,u) \leq \liminf_{k\to\infty} \mathcal{E}_s(u_{n_k}, u_{n_k}) \leq \sup_{k\in\mathbb{N}} \mathcal{E}_s(u_{n_k}, u_{n_k}) < \infty$$

so that $u \in \mathcal{H}^s_0(\Omega)$. Again by Fatou's Lemma we have for any $k \in \mathbb{N}$

$$\begin{aligned}\mathcal{E}_s(u - u_{n_k}, u - u_{n_k}) &\leq \liminf_{j\to\infty} \mathcal{E}_s(u_{n_j} - u_{n_k}, u_{n_j} - u_{n_k}) \\ &\leq \sup_{j \geq k} \mathcal{E}_s(u_{n_j} - u_{n_k}, u_{n_j} - u_{n_k}) < \infty,\end{aligned}$$

which gives $u_{n_k} \to u$ in $\mathcal{H}^s_0(\Omega)$ for $k \to \infty$ since $(u_{n_k})_k$ is a Cauchy sequence with respect to $\mathcal{E}_s$. This shows the completeness. ☐

Remark 3.4. The assumption on the Lipschitz regularity of the boundary in Proposition 3.3 can be removed if one argues instead with the Sobolev embedding of $H^m_0(\Omega)$ into $L^2(\Omega)$, but in this case the estimates for $\lambda_{1,s}$ are not clear, since they rely on integration by parts.

Remark 3.5. For Ω smooth and $m = 1$ we have the strict inequality $\lambda_{1,s} = \lambda_{1,1+\sigma} > \lambda_{1,1}\lambda_{1,\sigma}$. Indeed, let $A_s u := \sum_{i\in\mathbb{N}} a_i(u)\lambda^s_{i,1} e_i$ denote the *spectral* fractional Laplacian, where e_i and $\lambda_{i,1} > 0$ are the eigenfunctions and eigenvalues of $-\Delta$ in $H^1_0(\Omega)$ and $a_i(u) := \int_\Omega u e_i \, dx$ is the projection of u in the direction e_i; see [20, 26]. We also introduce the following associated quadratic forms as in [20]:

$$\begin{aligned} Q^D_s[u] &:= \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 \, d\xi, \qquad && u \in \mathrm{Dom}(Q^D_s), \\ Q^N_s[u] &:= \sum_{j\in\mathbb{N}} \lambda^s_{j,1} a_i(u)^2, && u \in \mathrm{Dom}(Q^N_s), \end{aligned}$$

where $\mathrm{Dom}(Q^D_s) := \{u \in \mathcal{S}'(\mathbb{R}^N) : Q^D_s[u] < \infty, \ \mathrm{supp}(u) \subset \overline{\Omega}\}$, $\mathrm{Dom}(Q^N_s) := \{u \in \mathcal{S}'(\mathbb{R}^N) : Q^N_s[u] < \infty\}$, and $\mathcal{S}'$ denotes the space of tempered distributions. Then, by [20, Theorem 1 and Lemma 2] we have that $Q^D_s[u] > Q^N_s[u]$ and $\mathrm{Dom}(Q^D_s) \subset \mathrm{Dom}(Q^N_s)$ for $s \in (1, 2)$. Thus

$$\lambda_{1,s} = \inf_{u\in\mathrm{Dom}(Q^D_s)} Q^D_s[u] \geq \inf_{u\in\mathrm{Dom}(Q^N_s)} Q^N_s[u] = \lambda^s_{1,1},$$

since the first eigenvalue of A_s is given by $\lambda_{1,1}^s$, as is easily seen from the definition of A_s. Furthermore, $\lambda_{1,\sigma} < (\lambda_{1,1})^\sigma$ for $\sigma \in (0,1)$ by [26, Theorem 1]. Thus, if $s = 1+\sigma$ we have that $\lambda_{1,s} \geq (\lambda_{1,1})^s = \lambda_{1,1}(\lambda_{1,1})^\sigma > \lambda_{1,1}\lambda_{1,\sigma}$, as claimed.

An immediate consequence of Proposition 3.3 and Remark 3.4 is the following.

Corollary 3.6. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Then for any $f \in L^2(\Omega)$ there is a unique weak solution $u \in \mathcal{H}^s_0(\Omega)$ of $(-\Delta)^s u = f$ in Ω.*

Proof. The statement follows from the Riesz Theorem, since $\mathcal{E}_s$ is a scalar product on $\mathcal{H}^s_0(\Omega)$ by Proposition 3.3 and Remark 3.4. □

4. Counterexample to general maximum principles

We begin with some auxiliary results.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$ be open. Then $C_c^{s+\epsilon}(\Omega) \subset \mathcal{H}^s_0(\Omega)$ for every $\epsilon > 0$.*

Proof. Let m be even, and without loss of generality assume that $\epsilon \in (0, 1-\sigma)$. Let $f \in C_c^{m,\sigma+\epsilon}(\Omega)$ and $D := \operatorname{supp}(f)$. There is $C > 0$ such that

$$|(-\Delta)^{\frac{m}{2}} f(x) - (-\Delta)^{\frac{m}{2}} f(y)|^2 \leq C|x-y|^{2\sigma+2\epsilon}$$

and $|f(x)|^2 \leq C$ for all $x, y \in \mathbb{R}^N$. Let $R > 0$ so that $D \Subset U := B_R(0)$ and $\operatorname{dist}(D, \mathbb{R}^N \setminus U) \geq 1$. Then

$$\begin{aligned}\mathcal{E}_\sigma((-\Delta)^{\frac{m}{2}} f, (-\Delta)^{\frac{m}{2}} f) \leq C \int_U \int_U |x-y|^{2\epsilon - N}\, dxdy \\ + 2C \int_D \int_{\mathbb{R}^N \setminus U} |x-y|^{-N-2\sigma}\, dxdy < \infty.\end{aligned}$$

The case m is odd follows similarly. □

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^N$ be open, and let $u \in C_c^{2m+2}(\Omega)$. Then*

$$\mathcal{E}_s(u,v) = \int_\Omega (-\Delta)^s u(x) v(x)\, dx \qquad \text{for all } v \in \mathcal{H}^s_0(\Omega).$$

Proof. This is a consequence of Proposition 3.1 and Lemma 4.1. A direct proof can also be done using integration by parts if Ω has a Lipschitz boundary. □

Corollary 4.3. *For every $f \in C_c^{2m+2}(\mathbb{R}^N)$ there exists a positive constant $C = C(N, m, \sigma, f) > 0$ such that $\mathcal{E}_s(f, \varphi) \leq C \int_{\mathbb{R}^N} \varphi(y)\, dy$ for all nonnegative $\varphi \in H^s(\mathbb{R}^N)$ and*

$$\|(-\Delta)^s f\|_{L^\infty(\mathbb{R}^N)} \leq C.$$

Proof. Note that by Lemma 4.2 we have $\mathcal{E}_s(f,\varphi) = \int_{\mathbb{R}^N} (-\Delta)^s f(x)\varphi(x)\, dx$. Moreover, since $f \in C_c^{2m+2}(\mathbb{R}^N)$ we have $(-\Delta)^m f \in C_c^2(\mathbb{R}^N)$, and thus there is $C > 0$ such that (see, e.g., [28]) $\|(-\Delta)^s f\|_{L^\infty(\mathbb{R}^N)} \leq C$. Hence $\mathcal{E}_s(f,\varphi) \leq C \int_{\mathbb{R}^N} \varphi(y)\, dy$ as claimed. □

Lemma 4.4. *Let $U, D \subset \mathbb{R}^N$ be open sets with Lipschitz boundary and let $\operatorname{dist}(U, D) > 0$, let $\phi \in \mathcal{H}^s_0(U)$, and let $g \in \mathcal{H}^s_0(D)$. Then*

(4.1)
$$\mathcal{E}_s(g,\varphi) = \frac{C_{N,s}}{2} \int_U \int_D \frac{\varphi(x) g(y)}{|x-y|^{N+2s}}\, dxdy, \qquad \text{where} \quad C_{N,s} = \frac{2^{2s}\,\Gamma(N/2+s)}{\pi^{N/2}\,\Gamma(-s)}.$$

Proof. Let g, ϕ be as stated. If m is even, we have using Green's formula

$$\begin{aligned}
\mathcal{E}_s(g,\varphi) &= -\frac{c_{N,\sigma}}{2}\int_U\int_D \frac{(-\Delta)^{\frac{m}{2}}\phi(x)(-\Delta)^{\frac{m}{2}}g(y)}{|x-y|^{N+2\sigma}}\,dydx\\
&= -\frac{c_{N,\sigma}}{2}\int_U\phi(x)\int_D(-\Delta)^{\frac{m}{2}}g(y)(-\Delta)_x^{\frac{m}{2}}|x-y|^{-N-2\sigma}\,dydx\\
&= -\frac{c_{N,\sigma}}{2}\int_U\phi(x)\int_D g(y)(-\Delta)_y^{\frac{m}{2}}(-\Delta)_x^{\frac{m}{2}}|x-y|^{-N-2\sigma}\,dydx\\
&= -\frac{c_{N,\sigma}}{2}\int_U\phi(x)\int_D g(y)(-\Delta)_y^{m}|x-y|^{-N-2\sigma}\,dydx,
\end{aligned}$$

where we used $(-\Delta)_y^{\frac{m}{2}}|x-y|^{-N-2\sigma} = (-\Delta)_x^{\frac{m}{2}}|x-y|^{-N-2\sigma}$.

If m is odd we have by integration by parts

$$\begin{aligned}
\mathcal{E}_s(g,\varphi) &= -\frac{c_{N,\sigma}}{2}\int_U\int_D \frac{\nabla(-\Delta)^{\frac{m-1}{2}}\phi(x)\nabla(-\Delta)^{\frac{m-1}{2}}g(y)}{|x-y|^{N+2\sigma}}\,dydx\\
&= \frac{c_{N,\sigma}}{2}\int_U(-\Delta)^{\frac{m-1}{2}}\phi(x)\int_D\nabla(-\Delta)^{\frac{m-1}{2}}g(y)\nabla_x|x-y|^{-N-2\sigma}\,dydx\\
&= -\frac{c_{N,\sigma}}{2}\int_U(-\Delta)^{\frac{m-1}{2}}\phi(x)\int_D\nabla(-\Delta)^{\frac{m-1}{2}}g(y)\nabla_y|x-y|^{-N-2\sigma}\,dydx\\
&= -\frac{c_{N,\sigma}}{2}\int_U(-\Delta)^{\frac{m-1}{2}}\phi(x)\int_D(-\Delta)^{\frac{m-1}{2}}g(y)(-\Delta_y)|x-y|^{-N-2\sigma}\,dydx\\
&= -\frac{c_{N,\sigma}}{2}\int_U\phi(x)\int_D g(y)(-\Delta_y)^m|x-y|^{-N-2\sigma}\,dydx,
\end{aligned}$$

where the last step follows as in the case m is even. Hence to finish the proof, note that for $x\in U$, $y\in D$ and $k>0$ we have $(-\Delta)_y|y-x|^{-k}\,dy = k(N-k-2)|y-x|^{-k-2}$, which gives

$$\begin{aligned}
(-\Delta)_y^m|y-x|^{-N-2\sigma} &= -(N+2\sigma)(2\sigma+2)(-\Delta)_y^{m-1}|y-x|^{-N-2\sigma-2}\\
&= (-1)^m\prod_{i=0}^{m-1}(N+2\sigma+2i)(2\sigma+2(i+1))|y-x|^{-N-2\sigma-2m}.
\end{aligned}$$

We now calculate the constant in (4.1). The above computations give

$$\begin{aligned}
(4.2)\quad C_{N,s} &= (-1)^{m+1}c_{N,\sigma}\prod_{i=0}^{m-1}(N+2\sigma+2i)(2\sigma+2(i+1))\\
&= (-1)^m\frac{2^{2\sigma}\,\Gamma(N/2+\sigma)}{\pi^{N/2}\,\Gamma(-\sigma)}\prod_{i=0}^{m-1}(N+2\sigma+2i)(2\sigma+2(i+1)).
\end{aligned}$$

Recall that $c_{N,\sigma}$ is the constant appearing in (1.1). We are now going to extensively use the basic property of the Gamma function $t\Gamma(t) = \Gamma(t+1)$. We have

$$\begin{aligned} C_{N,s} &= (-1)^m \frac{2^{2\sigma}\,\Gamma(N/2+\sigma)}{\pi^{N/2}\,\Gamma(-\sigma)} \prod_{i=0}^{m-1} (N+2\sigma+2i)(2\sigma+2i+2) \\ &= (-1)^m \frac{2^{2s}\,\Gamma(N/2+\sigma)}{\pi^{N/2}\,\Gamma(-\sigma)} \prod_{i=0}^{m-1} (N/2+\sigma+i)(\sigma+i+1) \\ &= \frac{2^{2s}\,\Gamma(N/2+s)}{\pi^{N/2}\,\Gamma(-\sigma)} \prod_{i=0}^{m-1} (-\sigma-i-1) = \frac{2^{2s}\,\Gamma(N/2+s)}{\pi^{N/2}\,\Gamma(-s)}, \end{aligned}$$

and the proof is complete. $\square$

Using the calculations in [10, Table 3, p. 549] (see also [24, Lemma 2.2] or [9, Corollary 9]) we have the following.

Corollary 4.5. *Let $r > 0$, let $x_0 \in \mathbb{R}^N$, let $s = m + \sigma$ with $m \in \mathbb{N}_0$, and let $\sigma \in (0,1]$. Then the unique weak solution $\psi_{r,x_0} \in \mathcal{H}^s_0(B_r(x_0))$ of $(-\Delta)^s\psi_{r,x_0} = 1$ in $B_r(x_0)$ and $\psi_{r,x_0} = 0$ on $\mathbb{R}^N \setminus B_r(x_0)$ is given for $x \in B_r(x_0)$ by*

$$\psi_{r,x_0}(x) = \begin{cases} \gamma_{N,s}(r^2 - |x-x_0|^2)^s, & \text{if} \quad |x-x_0| < r, \\ 0, & \text{if} \quad |x-x_0| \geq r, \end{cases}$$

where $\gamma_{N,s} = \frac{\Gamma(\frac{N}{2})4^{-s}}{\Gamma(s+1)\Gamma(\frac{N}{2}+s)}$.

We are now ready to construct the counterexample.

Proof of Theorem 1.2. Let $m \in \mathbb{N}$ be odd, let $\sigma \in (0,1)$, let $s := m + \sigma$, and let $D \subset \mathbb{R}^N$ be an open set such that $\mathbb{R}^N \setminus D$ has nonempty interior, A be an open ball compactly contained in the interior of $\mathbb{R}^N \setminus D$. Let $g \in C^\infty_c(D)\backslash\{0\}$ be a nonnegative function, and let $\psi \in \mathcal{H}^s_0(A)$ be the weak solution given by Corollary 4.5; in particular, $\psi \geq 0$ in $\mathbb{R}^N$ and $\mathcal{E}_s(\psi,\phi) = \int_A \phi\, dx$ for all $\phi \in \mathcal{H}^s_0(A)$.

Let $\Omega = D \cup A$, $C = C(N,m,\sigma) > 0$ be the constant given by Lemma 4.4, and let $f : \overline{\Omega} \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} a - C \displaystyle\int_D g(y)|x-y|^{-N-2s}\, dy & \text{for } \ x \in A, \\ aC \displaystyle\int_A \psi(y)|x-y|^{-N-2s}\, dy - (-\Delta)^s g(x) & \text{for } \ x \in D, \end{cases} \tag{4.3}$$

where $a > 0$ is chosen large enough such that $f > 0$ in $\overline{\Omega}$, which is possible by Corollary 4.3 and because $\operatorname{dist}(D,A) > 0$. Observe that $f \in L^2(\Omega)$ since

$$|f(x)| \leq a1_A(x) + \Big(aC\|\psi\|_{L^\infty(A)} \int_A |x-y|^{-N-2s}\, dy + \widetilde{C}\frac{\|(-\Delta)^m g\|_{C^2(D)}}{1+|x|^{N+2\sigma}}\Big)1_D(x)$$

for $x \in \Omega$ and for some $\widetilde{C} = \widetilde{C}(N,\sigma,g)$ by [11, Lemma 2.1] and Remark 3.2. Let $u \in \mathcal{H}^s_0(\Omega)$ be given by

$$u(x) := a\psi(x) - g(x) \qquad \text{for } x \in \mathbb{R}^N. \tag{4.4}$$

We now show that u is a sign-changing weak solution of

$$(-\Delta)^s u = f > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega. \tag{4.5}$$

Let $\varphi \in \mathcal{H}^s_0(\Omega)$. Then $\varphi = \varphi_D + \varphi_A$ for some $\varphi_D \in \mathcal{H}^s_0(D)$ and $\varphi_A \in \mathcal{H}^s_0(A)$. Since m is odd we have

$$\begin{aligned}\mathcal{E}_s(u, \phi_D) &= a\, \mathcal{E}_s(\psi, \phi_D) - \mathcal{E}_s(g, \phi_D)\\ &= a\, C \int_D \int_A \frac{\phi_D(x)\psi(y)}{|x-y|^{N+2s}}\, dy\, dx - \int_D (-\Delta)^s g\ \phi_D\ dx,\end{aligned}$$

by Lemma 4.4 and Remark 3.2. Thus $\mathcal{E}_s(u, \phi_D) = \int_D f(x)\phi_D(x)\ dx$. Analogously,

$$\mathcal{E}_s(u, \phi_A) = a\, \mathcal{E}_s(\psi, \phi_A) - \mathcal{E}_s(g, \phi_A) = a \int_A \phi_A\ dx - C \int_A \int_D \frac{\phi_A(x) g(y)}{|x-y|^{N+2s}}\, dy\, dx,$$

which yields that $\mathcal{E}_s(u, \phi_A) = \int_A f(x)\phi_A(x)\ dx$. Therefore $\mathcal{E}_s(u, \phi) = \mathcal{E}_s(f, \phi)$ for all $\varphi \in \mathcal{H}^s_0(\Omega)$ and u is a sign-changing weak solution of (4.5) as claimed. $\square$

Proof of Theorem 1.1. Let $N \in \mathbb{N}$, let $k \in \mathbb{N}$ be odd, let $s \in (k, k+1)$, let $D \subset \mathbb{R}^N$ be an open set, let A be a nonempty ball compactly contained in $\mathbb{R}^N \setminus D$, and let $\Omega := D \cup A$. Let $f \in C^\infty(\overline{\Omega}) \cap L^\infty(\Omega)$ be given by (4.3), and let $u \in C^s(\mathbb{R}^N) \cap C^\infty(\Omega) \cap L^\infty(\mathbb{R}^N)$ be given by (4.4). Then,

$$\begin{aligned}\int_\Omega f(x)\varphi(x)\ dx = \mathcal{E}_s(u, \varphi) &= \int_\Omega u(x)(-\Delta)^s \varphi(x)\ dx\\ &= \int_\Omega (-\Delta)^s u(x)\varphi(x)\ dx \qquad \text{for } \varphi \in C^\infty_c(\Omega),\end{aligned}$$

where the first equality follows from Theorem 1.2, the second equality from Lemma 4.2, and the third equality can be argued by the Fourier transform or by [1, Lemma 1.5], which uses only direct calculations. Then, by the fundamental lemma of calculus of variations, $(-\Delta)^s u(x) = f(x)$ for all $x \in \Omega$. Finally, if Ω is bounded, the uniqueness of u follows from Corollary 3.6. $\square$

Proof of Proposition 1.3. Let $\varphi \in \mathcal{H}^s_0(\mathbb{R}^N \setminus \Omega)\backslash\{0\}$ be nonnegative. Then, by Lemmas 4.2 and 4.4,

$$\begin{aligned}(-1)^{m+1} \int_{\mathbb{R}^N \setminus \Omega} (-\Delta)^s g(x)\, \varphi(x)\ dx &= (-1)^{m+1} \mathcal{E}_s(g, \varphi)\\ &= C \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} \frac{\varphi(x) g(y)}{|x-y|^{N+2s}}\ dx\, dy > 0.\end{aligned}$$

Since φ is arbitrarily chosen, we obtain that $(-1)^{m+1}(-\Delta)^s g > 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$. $\square$

Remark 4.6. If $u \in H^s(\mathbb{R}^N)$ and $s \in (0, \frac{3}{2})$, then $u^\pm \in H^s(\mathbb{R}^N)$, by [4, Théorème 1]. Hence $\mathcal{E}_s(|u|, |u|) = \mathcal{E}_s(u, u) + 4\mathcal{E}_s(u^+, u^-)$, where $|\mathcal{E}_s(u^+, u^-)| < \infty$. In fact, [21, Theorem 1] implies that $\mathcal{E}_s(u^+, u^-)$ is nonnegative. In particular, if $u \not\equiv |u|$ in $\mathbb{R}^N$, then $\mathcal{E}_s(|u|, |u|) > \mathcal{E}_s(u, u) > 0$, which is the main reason why the proof of the maximum principle for $s \in (0, 1)$ cannot be applied to $s \in (1, \frac{3}{2})$.

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