PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 12, December 2018, Pages 5157–5167
http://dx.doi.org/10.1090/proc/14071
Article electronically published on September 10, 2018

SEPARABLE REDUCTION OF LOCAL METRIC REGULARITY

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(Communicated by Thomas Schlumprecht)

Dedicated to the memory of our friend and collaborator, Jonathan Michael Borwein

Abstract. We prove that the property of a set-valued mapping $F : X \rightrightarrows Y$ to be locally metrically regular (and consequently, the properties of the mapping to be linearly open or pseudo-Lipschitz) is separably reducible by rich families of separable subspaces of $X \times Y$. In fact, we prove that, moreover, this extends to computation of the functor $\operatorname{reg} F$ that associates with F the rates of local metric regularity of F near points of its graph.

1. Introduction

Our discussions in this note will be centered around the following three principal concepts:

- metric regularity;
- separable reduction;
- rich families of closed separable subsets.

Precise definitions will be given in proper places, and here we just want to mention that

(a) The concept of metric regularity of mappings (in general, set-valued) is one of the most fundamental concepts studied in variational analysis. It takes its roots in the classical regularity concept (the derivative (of a mapping from a Banach space into another) at a given point is a linear operator onto) which is behind a series of basic results of the classical analysis such as the implicit function theorem, the Sard theorem and the Lyusternik–Graves theorem. In variational analysis, metric regularity and/or its geometric equivalent known as linear openness are the main instruments in analysis of the existence (in the absence of compactness) and stability of solutions of inclusions like $0 \in F(x)$. It also offers efficient mechanisms for proving necessary conditions in optimization problems and plays an important part in convergence analysis of various optimization algorithms. We refer to [11] for a state-of-the-art account of the metric regularity theory.

(b) Separable reduction relates to the possibility to reduce the study of a certain property (e.g., of mappings) on a generally nonseparable space to restrictions of the property to separable subspaces. Such a reduction often leads to substantial

Received by the editors October 21, 2017, and, in revised form, December 21, 2017.
2010 *Mathematics Subject Classification.* Primary 46B26, 47H04, 49J53, 58C06.
Key words and phrases. Metric space, local metric regularity, metric subregularity, rich family, separable reduction.
The first author was supported by grant GAČR 17–00941S and by RVO: 67985840.
The research of the third author was partly supported by the Bulgarian National Fund for Scientific Research under the grant DFMI I02/10/2014.

simplification of analysis and in certain cases offers the only way to extend to the general case results otherwise available only in separable spaces. A simple example of a separably reducible property is continuity of a mapping (see [6]). More and much less trivial examples can be found there and earlier in [2, 8, 13].

(c) The main property of rich families of subsets of a metric space is that the intersection of countably many rich families is again a rich family. An immediate consequence of this fact is that, having countably many properties, each satisfied by elements of a rich family, we can be sure that all these properties are simultaneously satisfied on a certain rich family (cf. the Baire category theorem!). The concept of a rich family was introduced by J. M. Borwein and W. B. Moors in [2] and thoroughly discussed in [7, 12] in connection with a number of properties relating to differentiability and subdifferentiability.

It was shown in [10] that the property of a set-valued mapping between a couple of Banach spaces to be metrically regular near a certain point of its graph is separably reducible. Here we prove a much stronger result, namely that given a set-valued mapping F with closed graph from a complete metric space X into another metric space Y, there is a rich family of "rectangular" closed separable subsets of $X \times Y$ (that is, sets of the form $M \times N,\ M \subset X,\ N \subset Y$) such that for any point of the graph of F the property of F to be metrically regular near the point is separably reducible via this family. Moreover, the rates of regularity (that is, the modulus of metric regularity, the rate of surjection, and the Lipschitz modulus) of the mapping at any point can be recovered from the corresponding rates of the restriction of the mapping to elements of the rich family.

This fact is certainly interesting by itself. But there are additional and more practical reasons that make it attractive. The point is that verification of metric regularity for mappings between separable spaces is much easier. The density theorem mentioned in the next section implies that in general the property can be verified only for countably many points of the graph. In the case when both spaces are Banach there is another advantage following the fact that the Dini–Hadamard subdifferential is trusted on any separable normed space. (This means that every lower semicontinuous function on the space is a Dini–Hadamard subdifferentiable at all points of a dense subset of the domain of the function and the standard fuzzy calculus rules are valid; see [5, 11].) This gives a universal mechanism for verification of metric regularity which is the simplest (as the calculation of a Dini–Hadamard subdifferential is simpler than the calculation of any other subdifferential) and the most precise, unless the space is Fréchet smooth and we can use the Fréchet subdifferential. It should be noted, however, that on a separable Asplund (hence Fréchet smooth) space the limiting Fréchet and the limiting Dini–Hadamard subdifferentials of a Lipschitz function coincide (see [1]).

In what follows X and Y are metric spaces, with X assumed to be complete in some results, including the main theorem, and $F : X \rightrightarrows Y$ is a set-valued mapping from X into Y. We shall denote the distance in both by the same letter d and hope this will not cause any problem or confusion. By $B(x, \varepsilon)$ we denote the open ball of radius ε around x, and for $Q \subset X$ the symbol $B(Q, \varepsilon)$ stands for the union of all balls $B(x, \varepsilon)$ with $x \in Q$.

2. Metric regularity

Recall that a set-valued mapping $F : X \rightrightarrows Y$ is *metrically regular* near $(\overline{x}, \overline{y}) \in \text{Graph } F$ if there are $K > 0$ and $\varepsilon > 0$ such that

$$d(x, F^{-1}(y)) \leq K d(y, F(x)) \text{ for all } (x, y) \in B(\overline{x}, \varepsilon) \times B(\overline{y}, \varepsilon).$$

Note that *it is possible to add to the definition the requirement that $d(y, F(x)) < \varepsilon$.* This does not imply any change of the property (see [9, Proposition 1 on p. 508]). Denote by $\text{reg}\, F(\overline{x}|\overline{y})$ the infimum of all $K > 0$ for which the inequality above is satisfied for a suitable choice of ε. It is called the *rate (or modulus) of metric regularity of F near $(\overline{x}, \overline{y})$*. The following almost obvious observation plays a crucial role in the subsequent proofs: $\text{reg}\, F(\overline{x}|\overline{y})$ *is the infimum of all rational $K > 0$ with the specified property.* This means that, speaking in what follows about rates of metric regularity, we can deal only with rational K's.

The concept of metric regularity has a clarifying geometric interpretation in terms of the *linear openness* or *covering property* (see, e.g., [11, Proposition 2.10]): $K > \text{reg}\, F(\overline{x}|\overline{y})$ if and only if there is an $\varepsilon > 0$ such that for all $x \in B(\overline{x}, \varepsilon)$ and all positive $t < \varepsilon$ we have with $r = K^{-1}$:

$$B(F(x), rt) \cap B(\overline{y}, \varepsilon) \subset F(B(x, t)).$$

In fact there is no need to verify the inclusion. It is sufficient to show that for any x and t as above the set $F(B(x, t))$ is dense in $B(F(x), rt) \cap B(\overline{y}, \varepsilon)$. This is the content of the *density theorem* mentioned in the introduction ([11, Theorem 2.55]).

As before, let X and Y be metric spaces and let $K > 0$ be given. Let us consider the following metric in $X \times Y$:

$$d_K((x, y), (x'y')) := d(x, x') + K d(y, y'), \quad (x, y),\ (x', y') \in X \times Y.$$ [1]

Further, let $F : X \rightrightarrows Y$ and $(\overline{x}, \overline{y}) \in \text{Graph } F$. It is said that F is *graph regular* near $(\overline{x}, \overline{y})$ if there is a $K > 0$ such that

$$d(x, F^{-1}(y)) \leq d_K((x, y), \text{Graph } F) \tag{1}$$

for all (x, y) in a neighborhood of $(\overline{x}, \overline{y})$.

The following fact will play the key role in our analysis.

Proposition 1. *Let X and Y be metric spaces, let $F : X \rightrightarrows Y$, and let $(\overline{x}, \overline{y}) \in$ Graph F. Then F is regular near $(\overline{x}, \overline{y})$ if and only if it is graph-regular near $(\overline{x}, \overline{y})$. Moreover, $\text{reg}\, F(\overline{x}|\overline{y})$ is the infimum of all rational $K > 0$ for which* (1) *holds for all (x, y) of a neighborhood of $(\overline{x}, \overline{y})$ (depending on K).*

The first part of the result (without the statement concerning rate of metric regularity) was announced by L. Thibault in 1999 in an unpublished note [14]. For the proof of the proposition see [11, Proposition 2.20]. The following regularity criterion will be the key element in the proof of the main result.

Theorem 2 (Criterion for local regularity)**.** *Let X be a complete metric space, let Y be a metric space, let $F : X \rightrightarrows Y$ be a set-valued mapping, with closed graph, and let $(\bar{x}, \bar{y}) \in$ Graph F be given. Suppose that there are $\varepsilon > 0$ and $K > 0$ such that for any positive $\lambda < 1$ and any $(x, y) \in B(\bar{x}, \varepsilon) \times B(\bar{y}, \varepsilon)$ with $y \notin F(x)$ there is a $u \neq x$ satisfying*

$$d_K((u, y), \text{Graph } F) \leq d_K((x, y), \text{Graph } F) - \lambda d(u, x). \tag{2}$$

[1] In [11] the symbol $d_{1,K}$ was used to denote this distance.

Then $\operatorname{reg} F(\overline{x}|\overline{y}) \le K$. *Moreover,* $\operatorname{reg} F(\overline{x}|\overline{y})$ *is the infimum of all rational* $K > 0$ *for which there is a neighborhood of* $(\overline{x}, \overline{y})$ *(that may depend on* K*) such that for any* $(x, y) \notin \text{Graph } F$ *of the neighborhood and any* $\lambda \in (0, 1)$ (2) *holds with some* $u \neq x$.

Proof. In the proof and later on we shall often use the simplifying notation

$$\omega^K(x, y) = d_K((x, y), \text{Graph } F)$$

when F is clear from the context. Obviously, ω^K is a Lipschitz function. Note also that $\omega^K(x, y) \le K d(y, F(x))$.

First we shall prove a slightly weaker statement, namely that $\operatorname{reg} F(\overline{x}|\overline{y}) \le K$ if there is an $\varepsilon > 0$ such that for any $(x, y) \in B(\bar{x}, \varepsilon) \times B(\bar{y}, \varepsilon)$ with $y \notin F(x)$ there is a $u \neq x$ satisfying

$$d_K((u, y), \text{Graph } F) \le d_K((x, y), \text{Graph } F) - d(u, x). \tag{3}$$

So let (3) hold with some $K > 0$ and $\varepsilon > 0$. We have to show that there is a $\delta > 0$ such that the inequality $d(x, F^{-1}(y)) \le K d(y, F(x))$ holds for all (x, y) satisfying $d(x, \overline{x}) < \delta$ and $d(y, \overline{y}) < \delta$. Take such a pair (x, y) with $\delta \le \varepsilon/2$. By Ekeland's principle (applied to $f = \omega_K(\cdot, y)$) there is a $\hat{x}$ such that $d(\hat{x}, x) \le \omega^K(x, y)$ and

$$\omega^K(w, y) + d(w, \hat{x}) > \omega^K(\hat{x}, y) \quad \forall\, w \neq \hat{x}. \tag{4}$$

We claim that $y \in F(\hat{x})$. Indeed, otherwise, as $d_K((\hat{x}, y), \text{Graph } F) < \varepsilon$, by the assumption there is a $u \neq \hat{x}$ such that (3) holds with $x = \hat{x}$, that is, $\omega^K(u, y) \le \omega^K(\hat{x}, y) - d(u, \hat{x})$, in contradiction with (4).

Thus $y \in F(\hat{x})$ and therefore

$$d(x, F^{-1}(y)) \le d(x, \hat{x}) \le \omega^K(x, y) \le K d(y, F(x)).$$

As this is true for all (x, y) satisfying the above specified conditions, we deduce that $\operatorname{reg} F(\overline{x}|\overline{y}) \le K$.

Now let there be $K > 0$ and $\varepsilon > 0$ such that (2) holds with some $u \neq x$ whenever $(x, y) \in B(\overline{x}, \varepsilon) \times B(\overline{y}, \varepsilon)$ and $y \notin F(x)$. Let $d_n(x, x')$, $n = 2, 3, \ldots$, stand for the metric $(1 - n^{-1}) d(x, x')$ in X, and let $\operatorname{reg}_n F$ be the rate of metric regularity of F when X is considered with the d_n-metric. It is an easy matter to see that $\operatorname{reg}_n F(x|y) \to \operatorname{reg} F(x|y)$ as $n \to \infty$. If (2) holds with $\lambda = 1 - n^{-1}$, then by what we have just proved $\operatorname{reg}_n F(\overline{x}|\overline{y}) \le K$ for any n, hence $\operatorname{reg} F(\overline{x}|\overline{y}) \le K$. This proves the first statement.

To prove the second statement, take a rational $K > \operatorname{reg} F(\overline{x}|\overline{y})$. As F is graph regular, by Proposition 1 $d(x, F^{-1}(y)) \le d_K((x, y), \text{Graph } F)$ for all (x, y) of a neighborhood of $(\overline{x}, \overline{y})$. Take a pair (x, y) in the neighborhood with $y \notin F(x)$ and a positive $\lambda < 1$ and choose a $u \in F^{-1}(y)$ satisfying $\lambda d(x, u) \le d(x, F^{-1}(y))$. Then

$$\begin{aligned} d_K((u, y), \text{Graph } F) = 0 \;&\le\; d_K((x, y), \text{Graph } F) - d(x, F^{-1}(y)) \\ &\le\; d_K((x, y), \text{Graph } F) - \lambda d(x, u), \end{aligned}$$

which immediately implies the second statement of the theorem as K can be chosen arbitrarily close to $\operatorname{reg} F(\overline{x}|\overline{y})$. □

Corollary 3. *Set for a* $K > 0$

$$\varphi^K(x, y) = \sup_{u \in X \setminus \{x\}} \frac{(\omega^K(x, y) - \omega^K(u, y))^+}{d(x, u)}.$$

Then, under the assumptions of the theorem, $\operatorname{reg} F(\overline{x}|\overline{y})$ *coincides with the infimum of all rational* $K > 0$ *such that* $\varphi^K(x, y) \geq 1$ *for all* $(x, y), y \notin F(x)$ *of a neighborhood of* $(\overline{x}, \overline{y})$.

3. Rich families

Let X be a metric space. By $\mathcal{S}(X)$ we denote the collection of closed separable subspaces of X. If Y is another metric space, then $\mathcal{S}_{\square}(X \times Y)$ is the collection of sets $L \times M$, where $L \in \mathcal{S}(X)$ and $M \in \mathcal{S}(Y)$.

A family $\mathcal{R}$ of elements of $\mathcal{S}(X)$ is *rich* if it has the following two properties:

• $\mathcal{R}$ is *cofinal*, that is, for any $C \in \mathcal{S}(X)$ there is an $S \in \mathcal{R}$ containing C;

• $\mathcal{R}$ is σ-*closed*, that is, for any increasing sequence $S_1, S_2, \ldots$ of elements of $\mathcal{R}$ the closure of the union of S_n also belongs to $\mathcal{R}$.

If the space itself is separable, then the family consisting of X alone is rich and all the statements concerning rich families become trivial. So in what follows we take for granted that the spaces we work with are nonseparable.

The following simple proposition proved by J. M. Borwein and W. Moors [2] describes the main property of rich families. We give a proof of the proposition as it plays a crucial role in subsequent arguments.

Proposition 4. *The intersection of countably many rich families of a metric space* X *is again a rich family.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2, \ldots$ be a sequence of rich families in X. It is immediate to check that the intersection $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2 \cap \cdots$ is σ-closed in $\mathcal{S}(X)$. To prove that $\mathcal{R}$ is cofinal, fix any $S_0 \in \mathcal{S}(X)$. We have to show that there is an $S \in \bigcap \mathcal{R}_n$ containing S_0. This can be done, for instance, as follows.

Let Q_{n1} be an arbitrary element of $\mathcal{R}_n$ containing S_0, and for $k \geq 2$,

$$Q_{nk} \in \mathcal{R}_n, \quad \bigcup \big\{ Q_{mj} : \ j < k, m + j < n + k \big\} \subset Q_{nk}.$$

Clearly such sets Q_{nk} can be found, as the union of finitely many elements of $\mathcal{S}(X)$ also is an element of $\mathcal{S}(X)$. It is also clear that for any n, m, and k there is an ℓ such that $Q_{nk} \subset Q_{m\ell}$. Let S_n stand for the closure of $\bigcup_k Q_{nk}$. Then, as follows from the observation of the previous sentence, the sets S_n coincide, so we can drop the subscript and set $S = S_n$. As $S_n \in \mathcal{R}_n$ belongs to every $\mathcal{R}_n$, it follows that $S \in \mathcal{R}$. ☐

Proposition 5. *Let* X *be a nonseparable metric space and let* f *be an extended-real-valued function on* X. *Then there exists a rich family* $\mathcal{R} \subset \mathcal{S}(X)$ *such that*

$$\forall r > 0 \quad \forall S \in \mathcal{R} \quad \forall x \in S \quad \inf f\big(B(x, r)\big) = \inf g\big(B(x, r) \cap S\big). \tag{5}$$

Proof. Denote by $\mathcal{R}$ the collection of $S \in \mathcal{S}(X)$ for which (5) holds. We have to show that this is a rich family. To begin with, we observe that for any $x \in X$ and any set $P \subset X$ the function $\varphi(r) = \inf f(B(x, r) \cap P)$ is (obviously nonincreasing and) continuous from the left. Indeed, given an $\varepsilon > 0$, take a $u \in B(x, r) \cap P$ such that $f(u) < \varphi(r) + \varepsilon$. As $B(x, r)$ is an open set, there is a $\delta > 0$ such that $u \in B(x, r - \delta)$, and therefore $\varphi(r) \leq \varphi(r - \delta) \leq \varphi(r) + \varepsilon$, as claimed. It follows that φ is fully defined by its values at positive rational r, and therefore (5) may every time be verified only for such r.

Let us next associate with any $x \in X$ and any $r > 0$ a countable set $D(x, r) \subset B(x, r)$ such that $\inf f(B(x, r)) = \inf f(D(x, r))$. We shall first show that $\mathcal{R}$ is cofinal. Take any countable subset C_0 of X and, starting with it, define a sequence (C_n) of countable subsets of X as follows: if a certain C_n has been already defined, we define C_{n+1} as the union of C_n and all sets $D(x,r)$, with $x \in C_n$ and r being a positive rational. Set finally $S = \mathrm{cl}(\cup C_n)$.

We have to verify that $S \in \mathcal{R}$ (which would prove that $\mathcal{R}$ is a cofinal family). To this end we have to check that (5) holds for all $x \in S$. This is obvious if $x \in C_n$ for some n. If on the other hand, $x \in S \setminus \bigcup C_n$, take a sequence of $x_n \in C_n$ converging to x and let $\varepsilon_n = d(x, x_n)$. Then $B(x, r-2\varepsilon) \subset B(x_n, r-\varepsilon) \subset B(x,r)$ if $\varepsilon_n < \varepsilon$. This together with the inequalities (obvious in view of the choice of the sets $D(x,r)$)

$$\inf f(B(x_n, r-\varepsilon) \cap D(x_n, r-\varepsilon)) = \inf f(B(x_n, r-\varepsilon)) \leq \inf f(B(x, r-2\varepsilon))$$

and

$$\inf f(B(x,r) \cap S) \leq \inf f(B(x_n, r-\varepsilon) \cap D(x_n, r-\varepsilon))$$

gives $\inf f(B(x,r) \cap S) \leq \inf f(B(x, r-2\varepsilon))$. The latter immediately implies (5) in view of the established left continuity of $\inf f(B(x,\cdot)$ and $\inf f(B(x,\cdot) \cap S)$ and the obvious inequality $\inf f(B(x,r) \cap S) \geq \inf f(B(x,r))$.

It remains to verify that $\mathcal{R}$ is σ-closed. To this end we only need, given a sequence (S_n) of elements of $\mathcal{R}$, to repeat the arguments of the previous paragraph with C_n and $D(x_n, r)$ replaced by S_n. □

Note that no assumptions have been imposed on the function in the above proposition. Therefore the result remains valid if instead of f we consider the restriction of f to some fixed set $\Omega \subset X$. In other words, the following statement is true.

Corollary 6. *Let X be a nonseparable metric space, let f be an extended-real-valued function on X, and let $\Omega \subset X$. Then there exists a rich family $\mathcal{R} \subset \mathcal{S}(X)$ such that*

$$\forall r > 0 \quad \forall S \in \mathcal{R} \quad \forall x \in S \quad \inf f\big(B(x,r) \cap \Omega\big) = \inf g\big(B(x,r) \cap \Omega \cap S\big). \tag{6}$$

Proposition 7. *Let X be a metric space and let $\Omega \subset X$ be a nonempty set. Then the collection $\mathcal{R}$ of $S \in \mathcal{S}(X)$ such that*

$$\forall x \in S \quad d(x, \Omega) = d(x, \Omega \cap S) \tag{7}$$

is a rich family.

Proof. Given a countable set $C_0 \subset X$, we can construct an increasing sequence $C_0 \subset C_1 \subset C_2 \subset \cdots$ of countable subsets of X such that for any n and any $u \in C_n$

$$d(u, \Omega) = d(u, \Omega \cap C_{n+1}). \tag{8}$$

Indeed, it immediately follows that $S = \mathrm{cl}(\bigcup C_n)$ satisfies (7); hence $\mathcal{R}$ is cofinal. To prove that $\mathcal{R}$ is σ-closed, consider a sequence $S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$ of elements of $\mathcal{R}$ and let $S = \mathrm{cl}(\bigcup S_n)$. Let D_n be a dense countable subset of S_n such that $\Omega \cap \Delta_n$ is dense in $\Omega \cap S$. Then $d(u, \Omega) = d(u, \Omega \cap D_n)$ for any $u \in D_n$. Finally, let C_n be the union of D_k with $k \leq n$. Then obviously $S = \mathrm{cl}(\bigcup C_n)$ and (8) holds for any n and any $u \in C_n$. In other words, S satisfies all conditions defining elements of $\mathcal{R}$. □

In what follows we denote by $\mathcal{S}_{\square}(X \times Y)$ the collection of closed rectangular separable subsets of $X \times Y$:

$$\mathcal{S}_{\square}(X \times Y) := \mathcal{S}(X) \times \mathcal{S}(Y) = \{L \times M : \ L \in \mathcal{S}(X), \ M \in \mathcal{S}(Y)\}.$$

It is obvious that $\mathcal{S}_{\square}(X \times Y)$ *is a rich family* of elements of $\mathcal{S}(X \times Y)$.

Proposition 8. *Let f be a Lipschitzian real-valued function on $X \times Y$ and let some constants $s > r > 0$ be given. Then the family $\mathcal{R}$ of all sets $L \times M \in \mathcal{S}_{\square}(X \times Y)$ such that for any $(x, y) \in (L \times M)$*

$$\begin{aligned} \sup\Big\{\frac{(f(x,y)-f(u,y))^+}{d(x,u)} :\ u \in X,\ r < d(x,u) < s\Big\} \\ = \sup\Big\{\frac{(f(x,y)-f(u,y))^+}{d(x,u)} :\ u \in L, r < d(x,u) < s\Big\} \end{aligned} \tag{9}$$

is rich.

Proof. First consider the case when f does not depend on y. Let C_0 be an arbitrary countable subset of X. Starting with C_0, we construct a sequence (C_n) of countable subsets of X such that $C_0 \subset C_1 \subset C_2 \subset \cdots$, and for any $x \in C_n$,

$$\begin{aligned} \sup\Big\{\frac{(f(x)-f(u))^+}{d(x,u)} :\ u \in X,\ r < d(x,u) < s\Big\} \\ = \sup\Big\{\frac{(f(x)-f(u))^+}{d(x,u)} :\ u \in C_{n+1},\ r < d(x,u) < s\Big\}. \end{aligned}$$

To this end, for any $x \in X$ we take a countable set $D(x) \subset X$ that realizes the supremum in the left-hand part of the above equality and, assuming that C_n has been already defined, we define C_{n+1} as the union of C_n and all sets $D(x)$ corresponding to $x \in C_n$.

Further, let L be the closure of the union of all (C_n). Since f is a Lipschitzian function, for any x,

$$\begin{aligned} \lim_{\varepsilon\to 0} \sup\Big\{\frac{(f(x)-f(u))^+}{d(x,u)} :\ u \in X,\ r+\varepsilon < d(x,u) < s-\varepsilon\Big\} \\ = \sup\Big\{\frac{(f(x)-f(u))^+}{d(x,u)} :\ u \in X,\ r < d(x,u) < s\Big\}. \end{aligned}$$

On the other hand, if $x \in L$ and a sequence of $x_n \in C_n$ converges to x, then (as evenually $r < d(x_n, u) < s$ whenever $\varepsilon > 0$ and $r + \varepsilon < d(x, u) < s - \varepsilon$)

$$\begin{aligned} \sup\Big\{\frac{(f(x)-f(u))^+}{d(x,u)} :\ u \in X,\ r+\varepsilon < d(x,u) < s-\varepsilon\Big\} \\ \le \lim_{n\to\infty} \sup\Big\{\frac{(f(x_n)-f(u))^+}{d(x_n,u)} :\ u \in X,\ r < d(x_n,u) < s\Big\} \\ = \lim_{n\to\infty} \sup\Big\{\frac{(f(x_n)-f(u))^+}{d(x_n,u)} :\ u \in L,\ r < d(x_n,u) < s\Big\} \\ = \sup\Big\{\frac{(f(x)-f(u))^+}{d(x,u)} :\ u \in L,\ r < d(x,u) < s\Big\}. \end{aligned} \tag{10}$$

The fact that $\mathcal{R}$ is rich easily follows from these two relations. Indeed, as (10) holds for any $\varepsilon > 0$, (9) follows. As we can choose any countable subset of X to be C_0, this means that $\mathcal{R}$ is cofinal. Further, let (L_n) be an increasing sequence of elements of $\mathcal{R}$, $L = \mathrm{cl}(\cup L_n)$ and $x \in L$. If x belongs to some of L_n, (9) obviously holds. If on the other hand, $x \in L \setminus (\bigcup L_n)$, we get (9) by repeating word for word but with L_n instead of C_n the arguments that led to (10). Thus L is also σ-closed, and hence rich.

Let us return to the general case. For any $y \in Y$, we can consider the set $\mathcal{R}_X(y)$ of all $L \in \mathcal{S}(X)$ such that (9) holds for all $x \in L$. As we have just proved, $\mathcal{R}_X(y)$ is a rich family. Now let $M \in \mathcal{S}(Y)$, and let D be a dense countable subset of M.

By Proposition 4 the intersection of all $\mathcal{R}_X(y)$ for $y \in D$ is also a rich family in $\mathcal{S}(X)$. However, as f is Lipschitz in y, the equality in (9) actually holds for any $y \in M$. Thus the collection $\mathcal{R}_X(M)$ of all $L \in \mathcal{S}(X)$ such that (9) holds for all $x \in L$ and $y \in M$ is a rich family. Finally, let $\mathcal{R}$ be the collection of all $L \times M$ such that $M \in \mathcal{S}(Y)$ and $L \in \mathcal{R}_X(M)$. We have to show that this is a rich family in $\mathcal{S}(X \times Y)$.

Let $Q \in \mathcal{S}(X \times Y)$ and let M be the closure of the canonical projections of Q onto Y. As $\mathcal{R}_X(M)$ is a rich family, there is an $L \in \mathcal{R}_X(M)$ containing the projection of Q onto X. Therefore $L \times M$ contains Q, and hence $\mathcal{R}$ is cofinal. To prove that $\mathcal{R}$ is σ-closed consider an increasing sequence $L_n \times M_n$ of elements of $\mathcal{R}$. Set $L = \operatorname{cl}(\bigcup L_n)$, $M = \operatorname{cl}(\bigcup M_n)$. To complete the proof, we have to verify that $L \times M \in \mathcal{R}$. So let $x \in L$, $y \in M$. If x belongs to some L_n, take $y_k \in M_k$ converging to y. If $k \geq n$, then $x \in L_k$, and therefore (9) holds for $y = y_k$. Again, as f is Lipshitzian in y, we deduce that (9) holds for the given x and y. If, on the other hand, $x \notin \bigcup L_n$, we take a sequence $(x_n, y_n) \in L_n \times M_n$ converging to (x, y). If $k \geq n$, then $y_n \in M_k$ and therefore (9) holds with $y = y_n$ and any $x \in L_k$. By definition $L_k \in \mathcal{R}_X(M_k)$, and therefore $L_k \in \mathcal{R}_X(M_n)$ for all $k \geq n$. As $\mathcal{R}_X(M_n)$ is a rich family, it follows that $L \in \mathcal{R}_X(M_n)$, and therefore again we see that (9) holds with y replaced by y_n for any n and hence also for the y itself. □

4. Main result

Now, we are armed enough to prove our main result. To begin with, we have to agree on some notation and terminology. Let $F : X \rightrightarrows Y$ be a set-valued mapping and let $Q \subset X \times Y$. We shall denote by F_Q the set-valued mapping whose graph is $(\text{Graph } F) \cap Q$. For a rectangular $Q = L \times M$ with $L \subset X$ and $M \subset Y$ we shall write simply F_{LM} (rather than $F_{L \times M}$). We observe the following obvious fact:

$$\forall (x, y) \in L \times M \quad (x, y) \notin \text{Graph } F \iff (x, y) \notin \text{Graph } F_{LM}.$$

We shall use the word *functor* for operations that transform mappings or functions of a certain class to other mappings or functions. Thus the operation $\operatorname{reg} F$ that associates with a set-valued mapping $F : X \rightrightarrows Y$ the function $X \times Y \ni (x, y) \longmapsto \operatorname{reg} F(x|y)$ is a functor on the class of set-valued mappings between metric spaces.

Finally we shall say that the functor $\operatorname{reg} F$ is *separably reducible* on a certain class of metric spaces if for any X, Y of the class and any $F : X \rightrightarrows Y$ with closed graph there is a cofinal family $\mathcal{R}$ of separable subsets of $X \times Y$ such that for any $Q \in \mathcal{R}$ the equality $\operatorname{reg} F(x|y) = \operatorname{reg} F_Q(x|y)$ holds for any $(x, y) \in Q$.

Theorem 9. *The functor $F \longmapsto \operatorname{reg} F(\cdot|\cdot)$ is separably reducible on the class of pairs of metric spaces (X, Y) with the domain space X being complete. Specifically, given a complete metric space X, a metric space Y, and a set-valued mapping $F : X \rightrightarrows Y$, with closed graph, there exists a rich family $\mathcal{R} \subset \mathcal{S}_{\square}(X \times Y)$ such that*

$$\forall L \times M \in \mathcal{R} \quad \forall (x, y) \in \text{Graph } F_{LM}, \quad \operatorname{reg} F(x|y) = \operatorname{reg} F_{LM}(x|y).$$

Proof. In the proof we consider positive rational K, and as before we set for brevity

$$\omega^K(x, y) = d_K((x, y), \text{Graph } F), \ (x, y) \in X \times Y;$$
$$\omega^K_{LM}(x, y) = d_K((x, y), \text{Graph } F_{LM}), \ (x, y) \in L \times M.$$

As $\mathcal{S}_{\square}(X \times Y)$ is a rich family, Proposition 7 (jointly with Proposition 4) implies that for every $K > 0$ there is a rich family $\mathcal{R}^K_1 \subset \mathcal{S}_{\square}(X \times Y)$ such that for any

$L \times M \in \mathcal{R}_1^K$ and any $(x, y) \in L \times M$,

$$\omega^K(x, y) = \omega^K_{LM}(x, y). \tag{11}$$

Then $\mathcal{R}_1 := \bigcap \{\mathcal{R}_1^K : K > 0, K \text{ rational}\}$ is also a rich family by Proposition 4.

Furthermore, from Proposition 8 we deduce that for every $K > 0$ and every $0 < r < s$ the family $\mathcal{R}_2^{K,r,s}$ consisting of all rectangles $L \times M \in \mathcal{S}_{\square}(X \times Y)$ such that the equality

$$\begin{aligned}\sup\left\{\frac{(\omega^K(x,y) - \omega^K(u,y))^+}{d(x,u)} : u \in X,\ r < d(x,u) < s\right\}\\ = \sup\left\{\frac{(\omega^K(x,y) - \omega^K(u,y))^+}{d(x,u)} : u \in L,\ r < d(x,u) < s\right\}\end{aligned}$$

holds for any $(x, y) \in L \times M$ is rich. Hence by Proposition 4 for every $0 < r < s$ the family $\mathcal{R}_3^{r,s} := \bigcap\{\mathcal{R}_2^{K,r,s} : K > 0, K \text{ rational}\}$ is again rich. This allows us to conclude, taking (11) into account, that

$$\begin{aligned}\sup\left\{\frac{(\omega^K(x,y) - \omega^K(u,y))^+}{d(x,u)} : u \in X,\ r < d(x,u) < s\right\}\\ = \sup\left\{\frac{(\omega^K(x,y) - \omega^K(u,y))^+}{d(x,u)} : u \in L,\ r < d(x,u) < s\right\}\\ = \sup\left\{\frac{(\omega^K_{LM}(x,y) - \omega^K_{LM}(u,y))^+}{d(x,u)} : u \in L,\ r < d(x,u) < s\right\}\end{aligned}$$

for *every* rational $K > 0$, every $0 < r < s$, every $L \times M \in \mathcal{R}_3^{r,s} := \mathcal{R}_1 \cap \mathcal{R}_2^{r,s}$, and every $(x, y) \in L \times M$.

Finally, setting $\mathcal{R}_3 := \bigcap \{\mathcal{R}_3^{r,1/r} : 0 < r < 1, r \text{ rational}\}$, we find that this is also a rich family by Proposition 4, and for every $K > 0$, every $L \times M \in \mathcal{R}_3$, and every $(x, y) \in L \times M$,

$$\sup_{u \in X \setminus \{x\}} \frac{(\omega^K(x,y) - \omega^K(u,y))^+}{d(x,u)} = \sup_{u \in L \setminus \{x\}} \frac{(\omega^K_{LM}(x,y) - \omega^K_{LM}(u,y))^+}{d(x,u)}. \tag{12}$$

Recall that

$$\varphi^K(x, y) := \sup_{u \in X \setminus \{x\}} \frac{(\omega^K(x,y) - \omega^K(u,y))^+}{d(x,u)}, \quad (x, y) \in X \times Y. \tag{13}$$

We also set

$$\varphi^K_{LM}(x, y) := \sup_{u \in L \setminus \{x\}} \frac{(\omega^K_{LM}(x,y) - \omega^K_{LM}(u,y))^+}{d(x,u)}, \quad (x, y) \in L \times M. \tag{14}$$

Thus (12) looks as follows:

$$\varphi^K(x, y) = \varphi^K_{LM}(x, y) \tag{15}$$

for all rational $K > 0$, all $L \times M \in \mathcal{R}_3$, and all $(x, y) \in L \times M$. Applying Corollary 6 with $\Omega := (X \times Y) \setminus \text{Graph } F$ (and of course X replaced by $X \times Y$) jointly with Proposition 4, we find for every $K > 0$ a rich family $\mathcal{R}_4^K \subset \mathcal{S}_{\square}(X \times Y)$ such that for every $L \times M \in \mathcal{R}_4^K$, every $(x, y) \in L \times M$, and every $\varepsilon > 0$ we have

$$\begin{aligned}&\inf \varphi^K\big(B(x,\varepsilon) \times B(y,\varepsilon) \setminus \text{Graph } F\big)\\ &\quad = \inf \varphi^K\big((B(x,\varepsilon) \times B(y,\varepsilon) \setminus \text{Graph } F\big) \cap (L \times M))\\ &\quad = \inf \varphi^K\big((B(x,\varepsilon) \cap L) \times (B(y,\varepsilon) \cap M) \setminus \text{Graph } F_{LM}\big).\end{aligned} \tag{16}$$

Put $\mathcal{R}_4 = \bigcap \{\mathcal{R}_4^K : \ K > 0, \ K \text{ rational}\}$. Hence combining (16) with (15), we get that for every rational $K > 0$, for every $L \times M \in \mathcal{R}_3 \cap \mathcal{R}_4$, every $(x, y) \in L \times M$, and every $\varepsilon > 0$ we have

$$\begin{aligned} &\inf \varphi^K \big(B(x,\varepsilon) \times B(y,\varepsilon) \setminus \text{Graph } F\big) \\ &\qquad = \inf \varphi^K_{LM}\big((B(x,\varepsilon) \cap L) \times (B(y,\varepsilon) \cap M) \setminus \text{Graph } F_{LM}\big). \end{aligned} \tag{17}$$

The equality (17) together with Corollary 3 immediately gives that

$$\forall \ L \times M \in \mathcal{R}_3 \cap \mathcal{R}_4 \ \forall \ (x, y) \in L \times M, \quad \operatorname{reg} F(x|y) = \operatorname{reg} F_{LM}(x|y).$$

It remains to set $\mathcal{R} = \mathcal{R}_3 \cap \mathcal{R}_4$ to complete the proof. □

5. A remark concerning subregularity

Subregularity is a weakened version of the metric regularity property: $F : X \rightrightarrows Y$ is *subregular* at $(\overline{x}, \overline{y}) \in \text{Graph } F$ if there are $K > 0$ and $\varepsilon > 0$ such that

$$d(x, F^{-1}(\overline{y})) \leq K d(\overline{y}, F(x)) \quad \forall \ x \in B(\overline{x}, \varepsilon).$$

The infimum of such K's usually denoted by subreg$F(\overline{x}|\overline{y})$ is called the *rate* or *modulus of metric subregularity* of F at $(\overline{x}, \overline{y})$.

Although subregularity lacks some good properties of metric regularity, in certain important situations it is subregularity that is needed for analysis (see [4, 11]). Therefore it is natural to ask whether the results of this note can be extended to the subregularity property and the rate of subregularity.

The answer is positive. Indeed, Proposition 1 remains valid if we fix $y = \overline{y}$ in (1) and in the statement of the proposition (see [11, Proposition 2.61]), and on the other hand, an attentive look at the proofs of both theorems in this paper shows that the proofs go through without any change if we fix $y = \overline{y}$.

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