PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 12, December 2018, Pages 5181–5193
https://doi.org/10.1090/proc/14124
Article electronically published on September 10, 2018

ON ASYMPTOTIC VARIANCE OF WHOLE-PLANE SLE

XUAN HIEU HO AND THANH BINH LE

(Communicated by Jeremy T. Tyson)

Dedicated to Professor Michel Zinsmeister on the occasion of his 60th birthday

Abstract. In this paper we rigorously compute the average McMullen asymptotic variance for the logarithmic derivative of the interior whole-plane Schramm–Loewner evolution SLE_2. Combined with some earlier results on the integral mean spectrum by B. Duplantier, Chi T. P. Nguyen, Nga T. T. Nguyen, and M. Zinsmeister (see also B. Duplantier, Hieu X. Ho, Binh T. Le, and M. Zinsmeister (2015 and 2017)), we prove an analogue of McMullen dimension formula.

1. Introduction

1.1. **Schramm–Loewner evolution SLE.** In his seminal paper [13] O. Schramm defined *Stochastic Loewner evolution* (now called *Schramm–Loewner evolution*) by taking a Brownian motion as a driving function in the sense of classical Loewner theory. In the following definition, the stochastic setting is carried out for the radial Loewner equation in the unit disc.

Definition 1.1. The *radial Schramm–Loewner evolution* (of parameter κ), denoted by SLE_κ, in the unit disc is the solution of the stochastic PDE:

$$\frac{\partial f_t(z)}{\partial t} = -z\frac{\partial f_t(z)}{\partial z}\frac{\lambda(t)+z}{\lambda(t)-z}, \quad f_0(z)=z, \tag{1.1}$$

where $\lambda(t) := e^{i\sqrt{\kappa}B_t}$ with B_t being the standard one-dimensional Brownian motion and κ being a nonnegative parameter.

Let $\Omega_t = f_t(\mathbb{D})$, and let $g_t : \Omega_t \to \mathbb{D}$ be the inverse function of f_t; then $g_t(z)$ is the unique solution of the stochastic ODE

$$\frac{\partial g_t(z)}{\partial t} = g_t(z)\frac{\lambda(t)+g_t(z)}{\lambda(t)-g_t(z)}, \quad g_0(z)=z. \tag{1.2}$$

There is another variant of SLE, called *whole-plane Schramm–Loewner evolution*, that is the solution of the equation

$$\frac{\partial f_t(z)}{\partial t} = z\frac{\partial f_t(z)}{\partial z}\frac{\lambda(t)+z}{\lambda(t)-z}, \tag{1.3}$$

with the same random driving function $\lambda(t) := e^{i\sqrt{\kappa}B_t}$. Actually, this variant is the studying object of this paper. An introduction to SLE processes can be found in [14] (see also [8]).

Received by the editors November 14, 2017, and, in revised form, January 30, 2018.
2010 *Mathematics Subject Classification.* Primary 30C99; Secondary 30C62.

Rohde and Schramm [12] proved that the Schamm–Loewner evolution processes are almost surely generated by a curve for $\kappa \neq 8$. In the whole-plane SLE case, it means that there is a continuous curve $\gamma : [0, +\infty) \to \mathbb{C}$, joining a point $\gamma(0)$ in $\mathbb{C}$ to ∞ and not containing 0, such that Ω_t is the unbounded component of $\mathbb{C} \setminus \gamma([t, +\infty))$. Moreover, there are phase transitions in parameter κ, namely, the curve is almost surely simple (does not intersect itself) for $\kappa \leq 4$, the curve has double points for $4 < \kappa < 8$, and when $\kappa \geq 8$ the curve is a space-filling curve. These curves are also proved or conjectured to be the scaling limit of some two-dimensional lattice models in statistical mechanics, for instance, $\kappa = 2$ corresponds to the loop-erased random walk, SLE_κ with $\kappa = 8/3$ is conjectured to be the scaling limit of self-avoiding random walks, $\kappa = 4$ corresponds to the path of the harmonic explorer and contour lines of the Gaussian free field, SLE_κ with $\kappa = 6$ is the scaling limit of critical percolation on the triangular lattice, etc.

As was remarked in [8, Remark 4.18 and Section 4.3], if $U : (-\infty, +\infty) \to \mathbb{R}$ is a continuous function, then (1.1) with $\lambda(t) = e^{iU(t)}$ can be solved for time variable $t \in (-\infty, +\infty)$ or, equivalently, (1.2) can be solved for $t \in (-\infty, +\infty)$. In equation (1.1) of the definition of interior radial SLE, one may consider the driving function $\lambda(t) = e^{i\sqrt{\kappa}B_t}$ with a two-side Brownian motion $B_t, -\infty < t < +\infty$. By following the same arguments, one can have an analogue of Lemma 1 in [2] (where the authors deal with the exterior radial SLE defined in the complement of the closed unit disc) for the interior version, i.e., the processes $f_t = g_t^{-1}$ and g_{-t} have the same law (up to conjugation by $e^{i\sqrt{k}B_t}$) (see also [5]). We redefine a radial SLE as

$$\tilde{f}_t(z) := g_{-t}(z) \overset{\text{(law)}}{=} e^{-i\sqrt{k}B_t} g_t^{-1}(e^{i\sqrt{k}B_t}), \quad t \in \mathbb{R}. \tag{1.4}$$

Then the (conjugate, inverse) radial SLE process $\tilde{f}_t$ satisfies the ODE

$$\frac{\partial \tilde{f}_t(z)}{\partial t} = \tilde{f}_t(z) \frac{\tilde{f}_t(z) + \lambda(t)}{\tilde{f}_t(z) - \lambda(t)}, \quad \tilde{f}_0(z) = z. \tag{1.5}$$

The fact that $\tilde{f}$ is a solution of an ODE instead of a PDE is suitable for our calculations in Section 2, the main part of the present paper. In that part, a martingale argument is used. Let us introduce two crucial ingredients of that method. The first one is an important property of the radial SLE process $\tilde{f}_t$, the so-called Markov property, due to the Markov property of Brownian motion B_t.

Lemma 1.2 (*Markov property*)**.**

$$\tilde{f}_t(z) = \lambda(s) \tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s)). \tag{1.6}$$

The second one is a relation between the whole-plane SLE and the radial SLE $\tilde{f}_t$.

Lemma 1.3. *The limit in law, $\lim_{t\to+\infty} e^t \tilde{f}_t(z)$, exists and has the same law as the (time zero) interior whole-plane random map $f_0(z)$:*

$$\lim_{t\to+\infty} e^t \tilde{f}_t(z) \overset{\text{(law)}}{=} f_0(z).$$

These two lemmas above are, respectively, interior versions of Lemmas 2 and 3 in [2] (see also [1]). Their proofs can also be found in [5].

1.2. **McMullen's question and its Minkowski dimension version.** The original question of McMullen is stated for analytic families of conformal maps on the complement of the unit disc. Let us introduce a version where conformal mappings are defined on the unit disc.

Let $(\phi_t), t \in U$, be a general analytic family of conformal maps on the unit disc with $\phi_0 = \mathrm{id}$ and $\phi_t(0) = 0\ \forall t \in U$, where U is a neighborhood of 0. Then one may write

$$\phi_t(z) = \int_0^z e^{\log \phi_t'(u)} du, z \in \mathbb{D}.$$

The function $b(z) = \frac{\partial}{\partial z}\frac{\partial}{\partial t}\phi_t(z)|_{t=0} = \frac{\partial}{\partial t}(\log \phi_t'(z))|_{t=0}$ belongs to the Bloch space $\mathcal{B}$ which is defined by

$$\mathcal{B} = \{\text{b holomorphic in } \mathbb{D}, \sup_{\mathbb{D}}(1-|z|^2)|b'(z)| < \infty\}.$$

McMullen [10] asked under what conditions on the family (ϕ_t) it is true that

$$2\frac{d^2}{dt^2}\dim_H(\phi_t(\partial\mathbb{D}))\bigg|_{t=0} = \sigma^2(b), \tag{1.7}$$

where $\dim_H(\phi_t(\partial\mathbb{D}))$ is the Hausdorff dimension of $\phi_t(\partial\mathbb{D})$ and $\sigma^2(b)$ is McMullen's asymptotic variance of the Bloch function b given by

$$\sigma^2(b) = \limsup_{r\to 1^-} \frac{1}{2\pi|\log(1-r)|}\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta.$$

Another version of McMullen's question was proposed and considered for the first time by THN. Le and M. Zinsmeister [7]. In this version, the Hausdorff dimension is replaced by the Minkovski dimension in (1.7), i.e., they dealt with the relation

$$2\frac{d^2}{dt^2}\dim_M(\phi_t(\partial\mathbb{D}))\bigg|_{t=0} = \sigma^2(b), \tag{1.8}$$

where $\dim_M(\phi_t(\partial\mathbb{D}))$ stands for the Minkovski dimension of $\phi_t(\partial\mathbb{D})$. They considered analytic families (ϕ_t) defined in term of a Bloch function b as

$$\phi_t(z) = \int_0^z e^{tb(u)} du. \tag{1.9}$$

There exists a neighborhood U of 0 such that if $t \in U$, then ϕ_t is a conformal map with quasiconformal extension, and then equation (1.8) is rewritten as

$$2\frac{d^2}{dt^2}\dim_M(\phi_t(\partial\mathbb{D}))\bigg|_{t=0} = \limsup_{r\to 1^-} \frac{1}{2\pi|\log(1-r)|}\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta. \tag{1.10}$$

For the definition and properties of Hausdorff dimension and Minkovski dimension, we refer the reader to any standard textbooks on dimension, for instance [6].

By using a probability argument M. Zinsmeister and THN. Le [7] described a relatively large class of functions in $\mathcal{B}$ for which the family (ϕ_t) defined by (1.9) satisfies (1.10). Namely, they proved that

$$\lim_{p\to 0} \frac{4\beta(p,\phi)}{p^2} = \limsup_{r\to 1^-} \frac{1}{2\pi|\log(1-r)|}\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta \tag{1.11}$$

which is, in this setting, equivalent to (1.10), due to a consequence of Corollary 10.18 in [11]. Here $\beta(p,\phi)$ is the integral means spectrum of ϕ, $\phi'(z) = \exp(b(z))$, defined by

$$\beta(p,\phi) = \limsup_{r\to 1^-} \frac{\log(\int_0^{2\pi} |\phi'(re^{i\theta})|^p d\theta)}{|\log(1-r)|}, p \in \mathbb{R}. \tag{1.12}$$

Let us note that in the same paper, the authors also constructed a Bloch function b for which the McMullen relation (1.10) does not hold.

1.3. **Main results.** In the above setting by THN. Le and M. Zinsmeister, we consider $b := \log f'$, where f is the interior whole-plane SLE$_\kappa$ map at time 0, and show that (1.11) holds in a sense of expectation for $\kappa = 2$. Namely, this paper aims to prove the following theorem.

Theorem 1.4. *Let $f := f_0$ be the interior whole-plane* SLE$_\kappa$ *map at time* 0, *and let $\bar\beta(p)$ be the average integral means spectrum of f defined by*

$$\bar\beta(p) = \limsup_{r\to 1^-} \frac{\log(\int_0^{2\pi} \mathbb{E}(|f'(re^{i\theta})|^p) d\theta)}{|\log(1-r)|}, p \in \mathbb{R}.$$

For $\kappa = 2$,

$$\lim_{p\to 0} \frac{4\bar\beta(p)}{p^2} = \lim_{r\to 1^-} \frac{1}{2\pi|\log(1-r)|} \int_0^{2\pi} \mathbb{E}(|\log f'(re^{i\theta})|^2) d\theta = \frac{4}{9}. \tag{1.13}$$

The number $\frac{4}{9}$ comes from the explicit formula of the average integral mean spectrum $\bar\beta(p)$ for $\kappa = 2$. In fact, this explicit formula was obtained by B. Duplantier, TPC. Nguyen, TTN. Nguyen, M. Zinsmeister [5] (see also [4]) for all nonnegative κ.

Proposition 1.1. *Let $f := f_0$ be the interior whole-plane* SLE$_\kappa$ *map at time* 0, *and let $\bar\beta(p)$ be the average integral means spectrum of f. Then*

$$\bar\beta(p) = \begin{cases} \beta_{tip}(p,\kappa) & p < p_0'(\kappa), \\ \beta_0(p,\kappa) & p_0'(\kappa) \le p < p^*(\kappa), \\ \beta_1(p,\kappa) & p^*(\kappa) \le p \le \min\{\hat p(\kappa), p(\kappa)\}, \end{cases} \tag{1.14}$$

where

$$\beta_{tip}(p,\kappa) = -p - 1 + \frac{1}{4}(4+\kappa - \sqrt{(4+\kappa)^2 - 8\kappa p}), \tag{1.15}$$

$$\beta_0(p,\kappa) = -p + \frac{4+\kappa}{4\kappa}(4+\kappa - \sqrt{(4+\kappa)^2 - 8\kappa p}), \tag{1.16}$$

$$\beta_1(p,\kappa) = 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa p} \tag{1.17}$$

and

$$p_0'(\kappa) = -1 - \frac{3\kappa}{8}, \tag{1.18}$$

$$p^*(\kappa) = \frac{1}{32\kappa}\left(\sqrt{2(4+\kappa)^2+4} - 6\right)\left(\sqrt{2(4+\kappa)^2+4} + 2\right), \tag{1.19}$$

$$\hat p(\kappa) = 1 + \frac{\kappa}{2}, \quad p(\kappa) = \frac{(6+\kappa)(2+\kappa)}{8\kappa}. \tag{1.20}$$

It follows that the development of $\bar{\beta}$ at $p = 0$ is

$$\bar{\beta}(p) = \frac{2\kappa}{(4+\kappa)^2}p^2 + o(p^2). \tag{1.21}$$

2. Asymptotic variance of whole-plane SLE

The aim of this section is to prove Theorem 1.4. The average integral means spectrum of the interior whole-plane SLE_κ has the development at $p = 0$ as (1.21). In particular, for $\kappa = 2$,

$$\lim_{p\to 0} \frac{4\bar{\beta}(p)}{p^2} = \frac{4}{9}. \tag{2.1}$$

Therefore, in order to prove Theorem 1.4, we will show that the right-hand side of (1.13) takes the value $\frac{4}{9}$. For this purpose, we will proceed using the two following steps: First, we use martingale techniques to derive an equation satisfied by $\mathbb{E}(|\log f'(z)|^2)$ for all $\kappa > 0$. Then we solve the equation obtained in the first step for $\kappa = 2$ by a consideration of the series form of the solution. An explicit expression of $\mathbb{E}(|\log f'(z)|^2)$ will be obtained to verify the relation (1.13).

2.1. Logarithmic expectation of whole-plane SLE.

In this section, we present results concerning the logarithmic expectation $F(z) := \mathbb{E}(\log f'(z))$. These results include a differential equation satisfied by F and the formula of the derivative of F (hence F). They will be used in the next sections to derive an equation obeyed by the second logarithmic moment $\mathbb{E}(|\log f'(z)|^2)$ and to find the solutions of this equation.

Proposition 2.1. *Let $f := f_0$ be the interior whole-plane* SLE_κ *map at time* 0, *and let $F(z) = \mathbb{E}(\log f'(z))$. Then F satisfies the equation*

$$-\frac{\kappa}{2}z^2\partial^2_{zz}F + z\Big(\frac{z+1}{z-1} - \frac{\kappa}{2}\Big)\partial_z F + 2\Big(1 - \frac{1}{(z-1)^2}\Big) = 0. \tag{2.2}$$

It follows that

$$\partial_z F(z) = \mathbb{E}\Big(\frac{f''(z)}{f'(z)}\Big) = \frac{4}{\kappa}\frac{(1-z)^{\frac{\kappa}{4}}}{z^{\frac{2}{\kappa}+1}} \int_0^z \frac{u^{\frac{2}{\kappa}}(u-2)}{(1-u)^{\frac{4}{\kappa}+2}}du. \tag{2.3}$$

Proof. Let us first introduce the time-dependent, auxiliary function

$$\tilde{F}(z,t) := \mathbb{E}(\log \tilde{f}_t(z)), \tag{2.4}$$

where $\tilde{f}_t$ is the reverse radial SLE_κ process (1.4). As a consequence of Lemma 1.3, the function F is the limit

$$\lim_{t\to+\infty}(t + \tilde{F}(z,t)) = F(z). \tag{2.5}$$

We now consider the conditional expectation

$$\mathcal{M}_s := \mathbb{E}(\log \tilde{f}'_t|\mathcal{F}_s), \tag{2.6}$$

where $\mathcal{F}_s$ denotes the σ-algebra generated by $\{B_\tau : \tau \le s\}$. Thanks to the Markov property of SLE, $\log \tilde{f}'_t(z)$ may be written as

$$\log \tilde{f}'_t(z) = \log \tilde{f}'_s(z) + \log \tilde{f}'_{t-s}(z_s), \qquad z_s := \frac{\tilde{f}_s(z)}{\lambda(s)}. \tag{2.7}$$

Then

$$\mathcal{M}_s = \log \tilde{f}'_s(z) + \tilde{F}(\tau, z_s), \tag{2.8}$$

where $\tau := t - s$ and $\tilde{F}(t, z) := \mathbb{E}(\log \tilde{f}'_t(z))$. We know that $(\mathcal{M}_s)_{t \geq s \geq 0}$ is a martingale. This fact implies that the ds-drift term of the Itô derivative of $\mathcal{M}_s$ vanishes, which permits us to obtain an equation satisfied by $\tilde{F}$. We now calculate that ds-drift term.

By regarding $\mathcal{M}_s$ as a stochastic process governed by the two processes $\log \tilde{f}'_s(z)$ and z_s, the Itô derivative of $\mathcal{M}_s$ is determined by

$$d\mathcal{M}_s = d \log \tilde{f}'_s(z) - \partial_\tau \tilde{F} ds + \partial_{z_s} \tilde{F} dz_s + \frac{1}{2} \partial^2_{z_s z_s} \tilde{F} dz_s dz_s. \tag{2.9}$$

Since the Itô differentials $d \log \tilde{f}'$ and dz_s are written in term of ds and dB_s as

$$d \log \tilde{f}'_s(z) = \Big(1 - \frac{2}{(z_s - 1)^2}\Big) ds, \tag{2.10}$$

$$dz_s, = z_s \Big(\frac{z_s + 1}{z_s - 1} - \frac{\kappa}{2}\Big) ds - i\sqrt{\kappa} z_s dB_s, \tag{2.11}$$

we obtain the coefficient of the drift term of $d\mathcal{M}_s$:

$$\tilde{\mathcal{P}}_{sing}(D)(\tilde{F}) := 1 - \frac{2}{(z_s - 1)^2} - \partial_\tau \tilde{F} + \Big(\frac{z_s + 1}{z_s - 1} - \frac{\kappa}{2}\Big) z_s \partial_{z_s} \tilde{F} - \frac{\kappa}{2} z_s^2 \partial^2_{z_s z_s} \tilde{F}. \tag{2.12}$$

The vanishing of this quantity gives us the equation

$$\tilde{\mathcal{P}}_{sing}(D)(\tilde{F}) = 1 - \frac{2}{(z_s - 1)^2} - \partial_\tau \tilde{F} + \Big(\frac{z_s + 1}{z_s - 1} - \frac{\kappa}{2}\Big) z_s \partial_{z_s} \tilde{F} - \frac{\kappa}{2} z_s^2 \partial^2_{z_s z_s} \tilde{F} = 0. \tag{2.13}$$

Since $\mathbb{E}(\log e^\tau \tilde{f}'_\tau(z)) = \tau + \tilde{F}(z)$, let us rewrite (2.13) as

$$\begin{aligned} &2 - \frac{2}{(z_s - 1)^2} - \partial_\tau \mathbb{E}(\log e^\tau \tilde{f}'_\tau) + \Big(\frac{z_s + 1}{z_s - 1} - \frac{\kappa}{2}\Big) z_s \partial_{z_s} \mathbb{E}(\log e^\tau \tilde{f}'_\tau) \\ &\quad - \frac{\kappa}{2} z_s^2 \partial^2_{z_s z_s} \mathbb{E}(\log e^\tau \tilde{f}'_\tau) = 0. \end{aligned} \tag{2.14}$$

We pass to the limit in (2.14) as τ tends to $+\infty$ by using (2.5) and then derive an equation obeyed by F:

$$2\Big(1 - \frac{1}{(z - 1)^2}\Big) + \Big(\frac{z + 1}{z - 1} - \frac{\kappa}{2}\Big) z \partial_z F - \frac{\kappa}{2} z^2 \partial_{zz} F = 0. \tag{2.15}$$

In the above, we followed an argument used in [4] (see [4, proof of Preposition 2.1]). The exchange of the $\tau \to +\infty$ limit and partial derivation of $\mathbb{E}(\log e^\tau \tilde{f}'_\tau)$ with respect to z is justified by the fact that the τ-family $\log e^\tau \tilde{f}'_\tau$ is normal. This family is thus equicontinuous and so, together with the existence of the limit (2.5), is uniformly convergent on compact sets of $\mathbb{D}$. In addition, in order to obtain (2.15), we has also used the fact $\lim_{\tau \to +\infty} \partial_\tau \mathbb{E}(\log e^\tau \tilde{f}'_\tau) = 0$. The Schramm–Loewner equation (1.5) gives us the following:

$$\frac{\partial}{\partial \tau} \log e^\tau \tilde{f}'_\tau(z) = \frac{2\tilde{f}_\tau(z) - 4\tilde{f}_\tau(z)\lambda(\tau)}{(\tilde{f}_\tau(z) - \lambda(\tau))^2}. \tag{2.16}$$

By means of classical Koebe theorems for the function $z \mapsto e^\tau \tilde{f}_\tau(z)$ of class $\mathcal{S}$, the right-hand side is bounded by $C(z)e^{-\tau}$, with C defined in $\mathbb{D}$. This allows us to exchange the expectation and the τ-derivative as well as justify the vanishing of the above limit.

After eliminating the factor z in (2.15), it yields an equivalent equation

$$-\frac{\kappa}{2}z\partial^2_{zz}F+\Big(\frac{z+1}{z-1}-\frac{\kappa}{2}\Big)\partial_z F+\frac{2(z-2)}{(z-1)^2}=0. \tag{2.17}$$

This equation is an ODE of order one of $\partial_z F$ with the initial condition $\partial_z F(0)=\frac{-8}{2+\kappa}$, obtained by substituting $z=0$ into (2.17). One may use the integrating factor method to solve this equation and get the expression (2.3) of $\partial_z F$. $\square$

From Proposition 2.1, simple formulas of $\partial_z F(z)$ can be obtained for particular cases. For example, in the case of $\kappa=2$,

$$\partial_z F(z)=-\frac{4}{3}+\frac{2}{3}\frac{1}{z-1}, \tag{2.18}$$

for $\kappa=1$,

$$\partial_z F(z)=\frac{7}{15}z-\frac{28}{15}+\frac{4}{5}\frac{1}{z-1}, \tag{2.19}$$

and for $\kappa=4$, by putting $w:=\sqrt{z}$,

$$\partial_z F(z)=\frac{w^2-1}{8w^3}\Big(\frac{10w-6w^3}{(w^2-1)^2}+5\log\frac{1-w}{1+w}\Big). \tag{2.20}$$

The explicit formula of F is obtained for certain values of κ by integrating both sides of (2.3). Let us note that with $\kappa=2$

$$F(z)=-\frac{4}{3}z+\frac{2}{3}\log(1-z). \tag{2.21}$$

Remark 2.1. If $\kappa=\frac{2}{n}$, then

$$F(z)=P_n(z)+\frac{2n}{2n+1}\log(1-z), \tag{2.22}$$

where P_n is a polynomial of degree n.

Remark 2.2. For general κ,

$$\partial_z F(z)\sim\frac{4}{4+\kappa}\frac{1}{z-1}, z\to 1, \tag{2.23}$$

so

$$F(z)\sim\frac{4}{4+\kappa}\log(1-z), z\to 1. \tag{2.24}$$

2.2. **Second logarithmic moment of whole-plane SLE.** In order to obtain the value of the average asymptotic variance of the Bloch function $\log f'$, we need the integral means on circles $\{|z|=r\}$ of the second moduli logarithmic moment $\mathbb{E}(|\log f'|^2)$. For this purpose, we continue following the martingale argument to arrive at an equation satisfied by $G(z):=\mathbb{E}(|\log f'(z)|^2)$. Next, we rewrite G as

$$G(z,\bar z)=F(z)\overline{F(z)}+R(z,\bar z) \tag{2.25}$$

and show that R is the solution of a differential equation which may, intuitively, be easier to deal with than one satisfied by G.

We now consider the martingale $(\mathcal{N}_s)_{t\geq s\geq 0}$ defined by

$$\mathcal{N}_s:=\mathbb{E}(|\log \tilde f_t'|^2|\mathcal{F}_s). \tag{2.26}$$

Recall that the martingale argument is based on the fact that the ds-drift term in the Itô derivative of a martingale vanishes. As in the preceding section, we first calculate the ds term of $\mathcal{N}_s$ to find an equation of the auxiliary function $\tilde{G}$ which is defined as following

$$\tilde{G}(t,z,\bar{z}) := \mathbb{E}(|\log \tilde{f}_t'(z)|^2), \tag{2.27}$$

where $\tilde{f}_t$ is the reverse radial SLE_κ process (1.4).

We rewrite $\mathcal{N}_s$ by using the Markov property of SLE as

$$\mathcal{N}_s = |\log \tilde{f}_s'(z)|^2 + \tilde{G}(\tau, z_s, \bar{z}_s) + \log \tilde{f}_s'(z)\overline{\tilde{F}(\tau, z_s)} + \overline{\log \tilde{f}_s'(z)}\tilde{F}(\tau, z_s). \tag{2.28}$$

$\tilde{F}$ is defined in the preceding section by (2.4).

For reasons of concision, we hereafter denote by $\mathcal{C}_{ds}(X_s)$ the coefficient of ds in the Itô derivative dX_s of a stochastic processe X_s. Thanks to the linearity of the Itô derivative, one can perform the ds term of $d\mathcal{N}_s$ as the sum of those of the three terms on the right-hand side of (2.28). Since the coefficients of ds in $d|\log \tilde{f}_s'(z)|^2$, $d\log \tilde{f}_s'(z)\overline{\tilde{F}(\tau, z_s)}$, and $d\overline{\log \tilde{f}_s'(z)}\tilde{F}(\tau, z_s)$ are, respectively,

$$\begin{aligned}
&\partial_s \log \tilde{f}_t'(z)\overline{\log \tilde{f}_t'(z)} + \log \tilde{f}_t'(z)\partial_s\overline{\log \tilde{f}_t'(z)},\\
&\partial_s \log \tilde{f}_t'(z)\overline{\tilde{F}(\tau, z_s)} + \log \tilde{f}_t'(z)\mathcal{C}_{ds}\big(\overline{\tilde{F}(\tau, z_s)}\big),\\
&\partial_s\overline{\log \tilde{f}_t'(z)}\tilde{F}(\tau, z_s) + \overline{\log \tilde{f}_t'(z)}\mathcal{C}_{ds}(\tilde{F}(\tau, z_s)),
\end{aligned}$$

the ds-coefficient of $d\mathcal{N}_s$ is obtained as

$$\begin{aligned}
&\mathcal{C}_{ds}(\tilde{G}(\tau, z_s, \bar{z}_s)) + \partial_s \log \tilde{f}_t'(z)\overline{\tilde{F}(\tau, z_s)} + \partial_s\overline{\log \tilde{f}_t'(z)}\tilde{F}(\tau, z_s)\\
&+ \overline{\log \tilde{f}_t'(z)}(\partial_s \log \tilde{f}_t'(z) + \mathcal{C}_{ds}(\tilde{F}(\tau, z_s))\\
&+ \log \tilde{f}_t'(z)(\partial_s\overline{\log \tilde{f}_t'(z)} + \mathcal{C}_{ds}\big(\overline{\tilde{F}(\tau, z_s)}\big).
\end{aligned} \tag{2.29}$$

Note that $\partial_s \log \tilde{f}_t'(z) + \mathcal{C}_{ds}(\tilde{F}(\tau, z_s))$ and $\partial_s\overline{\log \tilde{f}_t'(z)} + \mathcal{C}_{ds}\big(\overline{\tilde{F}(\tau, z_s)}\big)$ vanish because the first is the coefficient of the ds-drift term in the Itô derivative of the martingale $(\mathcal{M}_s)_{t\geq s\geq 0}$ defined by (2.6) while the second is the complex conjugate of the first one. The ds-drift term coefficient of $d\mathcal{N}_s$ is thus reduced as

$$\mathcal{C}_{ds}(\tilde{G}(\tau, z_s, \bar{z}_s)) + \partial_s \log \tilde{f}_t'(z)\overline{\tilde{F}(\tau, z_s)} + \partial_s\overline{\log \tilde{f}_t'(z)}\tilde{F}(\tau, z_s). \tag{2.30}$$

We now expand the terms that appear in (2.30). Namely,

$$\begin{aligned}
\partial_s \log \tilde{f}_s' &= \frac{\partial_z\left[\tilde{f}_s\frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)}\right]}{\tilde{f}_s'} = \frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)} - \frac{2\lambda(s)\tilde{f}_s}{(\tilde{f}_s-\lambda(s))^2}\\
&= 1 - \frac{2}{(1-z_s)^2},\\
\partial_s\overline{\log \tilde{f}_s'} &= 1 - \frac{2}{(1-\bar{z}_s)^2}.
\end{aligned}$$

In addition, by again applying the Itô formula to $\tilde{G}(\tau, z_s, \bar{z}_s)$, we also have

$$\mathcal{C}_{ds}(\tilde{G}(\tau, z_s, \bar{z}_s)) = \frac{z_s+1}{z_s-1}z_s\partial_{z_s}\widetilde{G} + \frac{\bar{z}_s+1}{\bar{z}_s-1}\bar{z}_s\bar{\partial}_{\bar{z}_s}\widetilde{G} - \partial_\tau\widetilde{G} - \frac{\kappa}{2}(z_s\partial_{z_s} - \bar{z}_s\,\bar{\partial}_{\bar{z}_s})^2\widetilde{G}. \tag{2.31}$$

Let us denote the differential operator on the right-hand side of (2.31) by $\mathcal{P}_{prin}(D)$. The quantity (2.30) then equals

$$\mathcal{P}_{prin}(D)(\tilde{G}) + \Big(1 - \frac{2}{(1-z_s)^2}\Big)\overline{\tilde{F}(\tau, z_s)} + \Big(1 - \frac{2}{(1-\bar{z}_s)^2}\Big)\tilde{F}(\tau, z_s). \tag{2.32}$$

We recall that this quantity vanishes since $\mathcal{N}_s$ is a martingale. One thus obtains an equation satisfied by $\tilde{G}$:

$$\mathcal{P}_{prin}(D)(\tilde{G}) + \Big(1 - \frac{2}{(1-z_s)^2}\Big)\overline{\tilde{F}(\tau, z_s)} + \Big(1 - \frac{2}{(1-\bar{z}_s)^2}\Big)\tilde{F}(\tau, z_s) = 0. \tag{2.33}$$

We continue by passing to the limit in (2.33) as τ tends to $+\infty$ and using Lemma 1.3 to arrive at an equation obeyed by G. Before doing that, let us rewrite $|\log \tilde{f}_\tau'|^2$ as follows:

$$|\log \tilde{f}_\tau'|^2 = |\log e^\tau \tilde{f}_\tau'|^2 - \tau(\log \tilde{f}_\tau' + \overline{\log \tilde{f}_\tau'}) - \tau^2. \tag{2.34}$$

This identity implies that

$$\partial_{z_s}\tilde{G} = \partial_{z_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) - \tau\partial_{z_s}\tilde{F}(\tau, z_s), \tag{2.35}$$

$$\partial_{\bar{z}_s}\tilde{G} = \partial_{\bar{z}_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) - \tau\partial_{\bar{z}_s}\overline{\tilde{F}(\tau, z_s)}, \tag{2.36}$$

$$\partial^2_{z_s z_s}\tilde{G} = \partial^2_{z_s z_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) - \tau\partial^2_{z_s z_s}\tilde{F}(\tau, z_s), \tag{2.37}$$

$$\partial^2_{\bar{z}_s \bar{z}_s}\tilde{G} = \partial^2_{\bar{z}_s \bar{z}_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) - \tau\partial^2_{\bar{z}_s \bar{z}_s}\overline{\tilde{F}(\tau, z_s)}, \tag{2.38}$$

$$\partial^2_{z_s \bar{z}_s}\tilde{G} = \partial^2_{z_s \bar{z}_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2), \tag{2.39}$$

$$\partial_\tau \tilde{G} = \partial_\tau \mathbb{E}(|\log e^\tau \tilde{f}_t'|^2) - \mathbb{E}(\log e^\tau \tilde{f}_\tau') - \tau\partial_\tau \tilde{F}(\tau, z_s) - \overline{\mathbb{E}(\log e^\tau \tilde{f}_\tau')} - \tau\partial_\tau\overline{\tilde{F}(\tau, z_s)}. \tag{2.40}$$

In (2.33), by replacing the terms that appear on the left-hand side of the above identities by their corresponding right-hand side terms, we arrive at

$$\begin{aligned}
&\mathcal{P}_{prin}(D)(\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2)) \\
&\quad + 2\Big(1 - \frac{1}{(1-\bar{z}_s)^2}\Big)\mathbb{E}(\log e^\tau \tilde{f}_\tau') + 2\Big(1 - \frac{1}{(1-z_s)^2}\Big)\overline{\mathbb{E}(\log e^\tau \tilde{f}_\tau')} \\
&\quad - \tau\tilde{\mathcal{P}}_{sing}(D)(\tilde{F})(\tau, z_s) - \tau\overline{\tilde{\mathcal{P}}_{sing}(D)(\tilde{F})(\tau, z_s)} = 0.
\end{aligned} \tag{2.41}$$

Because of the fact that $\tilde{\mathcal{P}}_{sing}(\tilde{F})(\tau, z_s)$ is the ds-term coefficient in $\mathrm{d}\mathcal{M}_s$, which vanishes, one can get rid of the second line of (2.41) and obtain

$$\begin{aligned}
\mathcal{P}_{prin}(D)(\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) &+ 2\Big(1 - \frac{1}{(1-\bar{z}_s)^2}\Big)\mathbb{E}(\log e^\tau \tilde{f}_\tau') \\
&+ 2\Big(1 - \frac{1}{(1-z_s)^2}\Big)\overline{\mathbb{E}(\log e^\tau \tilde{f}_\tau')} = 0.
\end{aligned} \tag{2.42}$$

Finally, Lemma 1.3 is used to derive an equation satisfied by G.

Proposition 2.2. *Let $f := f_0$ be the interior whole-plane* SLE_κ *map at time* 0*, and let $G(z, \bar{z}) = \mathbb{E}(|\log f'(z)|^2)$. Then G satisfies the equation*

$$\mathcal{P}_{prin}(D)(G) + 2\Big(1 - \frac{1}{(\bar{z}-1)^2}\Big)F(z) + 2\Big(1 - \frac{1}{(z-1)^2}\Big)\overline{F(z)} = 0, \tag{2.43}$$

where

$$\mathcal{P}_{prin}(D) = \frac{z+1}{z-1}z\partial_z + \frac{\bar{z}+1}{\bar{z}-1}\bar{z}\partial_{\bar{z}} - \frac{\kappa}{2}(z\partial_z - \bar{z}\partial_{\bar{z}})^2.$$

We now consider the function G of the form

$$G(z,\bar{z}) = F(z)\overline{F(z)} + R(z,\bar{z}), \tag{2.44}$$

where F is the solution of (2.2) and $\overline{F(z)}$ is its complex conjugate. By replacing $G(z,\bar{z})$ by $F(z)\overline{F(z)} + R(z,\bar{z})$, the left-hand side of (2.43) becomes

$$\begin{aligned}&\mathcal{P}_{prin}(D)(R) + \kappa z\bar{z}\partial_z F \partial_{\bar{z}} \bar{F}\\ &+ \left[-\frac{\kappa}{2} z^2 \partial^2_{zz} F + z\Big(\frac{z+1}{z-1} - \frac{\kappa}{2}\Big)\partial_z F + 2\Big(1 - \frac{1}{(z-1)^2}\Big)\right]\bar{F}\\ &+ \overline{\left[-\frac{\kappa}{2} z^2 \partial^2_{zz} F + z\Big(\frac{z+1}{z-1} - \frac{\kappa}{2}\Big)\partial_z F + 2\Big(1 - \frac{1}{(z-1)^2}\Big)\right]} F.\end{aligned} \tag{2.45}$$

By (2.1), the two last lines vanish. The quantity (2.45) is thus simplified as

$$\mathcal{P}_{prin}(D)(R) + \kappa z\bar{z}\partial_z F \partial_{\bar{z}} \bar{F}. \tag{2.46}$$

That leads us to the following proposition.

Proposition 2.3.

$$\mathcal{P}_{prin}(D)(R) = -\kappa z\bar{z}\partial_z F \partial_{\bar{z}} \bar{F}. \tag{2.47}$$

2.3. **Asymptotic variance of** SLE_2**.** The function R is analytic in the bidisk $\mathbb{D}\times\mathbb{D}$; we thus write it in a series form

$$R(z,\bar{z}) = \sum_{n\geq 0} a_{n,m} z^n \bar{z}^m, \tag{2.48}$$

since the normalization of the SLE map f, $a_{0,0}=0$.

The equation (2.47) gives us recurrent relations of the coefficients $a_{n,m}$. In the case of $\kappa = 2$, these recurrent relations make us able to easily obtain an explicit formula of R and thus of G (see [9] for a similar argument). Let us set $\kappa = 2$ and use the corresponding expression (2.21) of F to derive the following equation from (2.47):

$$\begin{aligned}&\frac{z+1}{z-1} z\partial_z R + \frac{\bar{z}+1}{\bar{z}-1}\bar{z}\partial_{\bar{z}} R - \frac{\kappa}{2}(z\partial_z - \bar{z}\partial_{\bar{z}})^2 R\\ &\quad = -2z\bar{z}\Big(-\frac{4}{3}z + \frac{2}{3}\log(1-z)\Big)\Big(-\frac{4}{3}\bar{z} + \frac{2}{3}\log(1-\bar{z})\Big).\end{aligned} \tag{2.49}$$

After putting the series form (2.48) of R into (2.49) and identifying the two sides of the equation, we obtain

$$\begin{aligned}&a_{1,0} = a_{0,1} = a_{2,0} = a_{0,2} = 0, a_{1,1} = 4, a_{2,2} = \frac{14}{9},\\ &a_{n,m} = \frac{1}{(n-m)^2 + n + m}\Bigg[\Bigg((n-m-1)^2 - n + m + 1\Bigg)a_{n-1,m}\\ &+ \Bigg((n-m+1)^2 + n - m + 1\Bigg)a_{n,m-1} + \Bigg(-(n-m)^2 + n + m - 2\Bigg)a_{n-1,m-1}\Bigg].\end{aligned}$$

By using the inductive method, we can prove that

$$a_{1,1} = 4, \tag{2.50}$$

$$a_{n,n} = \frac{4}{3}\left(\frac{4}{n^2} + \frac{1}{3n}\right) \qquad \forall n \geq 2, \tag{2.51}$$

$$a_{n,n+1} = a_{n+1,n} = \frac{-8}{3n(n+1)} \qquad \forall n \geq 1, \tag{2.52}$$

$$a_{n,m} = 0 \quad \text{otherwise}. \tag{2.53}$$

These identities are equivalent to

$$R(z,\bar z) = \frac{4}{3}\left[-\frac{4}{3}z\bar z + \frac{2(z+\bar z)}{z\bar z}\int_0^{z\bar z}\log(1-u)du - 4\int_0^{z\bar z}\frac{\log(1-u)}{u}du - \frac{1}{3}\log(1-z\bar z)\right], \tag{2.54}$$

and hence

$$\begin{aligned} G(z,\bar z) =& \left(-\frac{4}{3}z + \frac{2}{3}\log(1-z)\right)\left(-\frac{4}{3}\bar z + \frac{2}{3}\log(1-\bar z)\right) \\ &+ \frac{4}{3}\left[-\frac{4}{3}z\bar z + \frac{2(z+\bar z)}{z\bar z}\int_0^{z\bar z}\log(1-u)du - 4\int_0^{z\bar z}\frac{\log(1-u)}{u}du - \frac{1}{3}\log(1-z\bar z)\right]. \end{aligned} \tag{2.55}$$

Equation (2.55) directly implies the main result of this section.

Proposition 2.4. *Let $f := f_0$ be the interior whole-plane* SLE_2 *map at time* 0. *Then*

$$\lim_{r\to 1^-}\frac{1}{2\pi|\log(1-r)|}\int_0^{2\pi}\mathbb{E}(|\log f'(re^{i\theta})|^2)d\theta = \frac{4}{9}. \tag{2.56}$$

Proof. By using the Maclaurin expansion of the logarithmic function $\log(1-z)$,

$$\log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n}, \quad |z|<1,$$

one rewrites the function $G(z,\bar z)$ (2.55) as

$$\begin{aligned} G(z,\bar z) =& \frac{8}{9}\left(z\sum_{n=1}^{\infty}\frac{\bar z^n}{n} + \bar z\sum_{n=1}^{\infty}\frac{z^n}{n}\right) + \frac{4}{9}\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{n=1}^{\infty}\frac{\bar z^n}{n} \\ &- \frac{8}{3}(z+\bar z)\sum_{n=1}^{\infty}\frac{(z\bar z)^n}{n(n+1)} + \frac{16}{3}\sum_{n=1}^{\infty}\frac{(z\bar z)^n}{n^2} - \frac{4}{9}\log(1-z\bar z). \end{aligned} \tag{2.57}$$

Plancherel's theorem then yields

$$\frac{1}{2\pi}\int_0^{2\pi}\mathbb{E}(|\log f'(re^{i\theta})|^2)d\theta = \frac{16}{9}r^2 + \frac{52}{9}\sum_{n=1}^{\infty}\frac{r^{2n}}{n^2} - \frac{4}{9}\log(1-r^2), \quad r<1. \tag{2.58}$$

It follows that

$$\begin{aligned}
&\lim_{r\to 1^-}\frac{1}{2\pi|\log(1-r)|}\int_0^{2\pi}\mathbb{E}(|\log f'(re^{i\theta})|^2)d\theta\\
&=\lim_{r\to 1^-}\frac{\frac{16}{9}r^2+\frac{52}{9}\sum_{n=1}^{\infty}\frac{r^{2n}}{n^2}-\frac{4}{9}\log(1-r^2)}{-\log(1-r)}\\
&=\frac{4}{9}.
\end{aligned}\tag{2.59}$$

□

We believe that the relation (1.13) is true for SLE_κ with an abitrary $\kappa > 0$. However, this is still an open problem.

Acknowledgment

We thank Michel Zinsmeister for introducing us to the question of McMullen and for fruitful discussions.

References

[1] Dmitry Beliaev, Bertrand Duplantier, and Michel Zinsmeister, *Integral means spectrum of whole-plane SLE*, Comm. Math. Phys. **353** (2017), no. 1, 119–133. MR3638311

[2] D. Beliaev and S. Smirnov, *Harmonic measure and SLE*, Comm. Math. Phys. **290** (2009), no. 2, 577–595. MR2525631

[3] Bertrand Duplantier, Xuan Hieu Ho, Thanh Binh Le, and Michel Zinsmeister, *Logarithmic coefficients and multifractality of whole-plane sle*, arXiv preprint arXiv:1504.05570 (2015).

[4] Bertrand Duplantier, Xuan Hieu Ho, *Logarithmic coefficients and generalized multifractality of whole-plane sle*, Communications in Mathematical Physics (2017).

[5] Bertrand Duplantier, Chi Nguyen, Nga Nguyen, and Michel Zinsmeister, *The coefficient problem and multifractality of whole-plane SLE & LLE*, Ann. Henri Poincaré **16** (2015), no. 6, 1311–1395. MR3345643

[6] Kenneth Falconer, *Fractal geometry,* Mathematical foundations and applications, 2nd ed., John Wiley & Sons, Inc., Hoboken, NJ, 2003. MR2118797

[7] Nhat Le Thanh Hoang and Michel Zinsmeister, *On Minkowski dimension of Jordan curves*, Ann. Acad. Sci. Fenn. Math. **39** (2014), no. 2, 787–810. MR3237051

[8] Gregory F. Lawler, *Conformally invariant processes in the plane*, Mathematical Surveys and Monographs, vol. 114, American Mathematical Society, Providence, RI, 2005. MR2129588

[9] Igor Loutsenko, SLE_κ*: correlation functions in the coefficient problem*, J. Phys. A **45** (2012), no. 27, 275001, 10. MR2947222

[10] Curtis T. McMullen, *Thermodynamics, dimension and the Weil-Petersson metric*, Invent. Math. **173** (2008), no. 2, 365–425. MR2415311

[11] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 299, Springer-Verlag, Berlin, 1992. MR1217706

[12] Steffen Rohde and Oded Schramm, *Basic properties of SLE [MR2153402]*, Selected works of Oded Schramm. Volume 1, 2, Sel. Works Probab. Stat., Springer, New York, 2011, pp. 989–1030. MR2883396

[13] Oded Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288. MR1776084

[14] Michel Zinsmeister, *Deterministic and random growth processes*, lectures held at the morningside center of mathematics ed., Chinese Academy of Sciences, Beijing, 2010.

Bâtiment de Mathématiques MAPMO, UFR Sciences, Université d'Orléans, rue de Chartres, B.P.6759, Orleans Cedex 2 45067, France
Email address: xuan-hieu.ho@univ-orleans.fr

Department of Mathematics, University of Quy Nhon, 170 An Duong Vuong Street, Quy Nhon, Binh Dinh, Vietnam
Email address: mr.lethanhbinh@gmail.com