PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 12, December 2018, Pages 5169–5179
https://doi.org/10.1090/proc/14144
Article electronically published on September 10, 2018

D-MINIMAL EXPANSIONS OF THE REAL FIELD HAVE THE ZERO SET PROPERTY

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(Communicated by Ken Ono)

ABSTRACT. If $E \subseteq \mathbb{R}^n$ is closed and the structure $(\mathbb{R}, +, \cdot, E)$ is d-minimal (that is, in every structure elementarily equivalent to $(\mathbb{R}, +, \cdot, E)$, every unary definable set is a disjoint union of open intervals and finitely many discrete sets), then for each $p \in \mathbb{N}$, there exist C^p functions $f: \mathbb{R}^n \to \mathbb{R}$ definable in $(\mathbb{R}, +, \cdot, E)$ such that E is the zero set of f.

Throughout, E denotes a closed subset of some $\mathbb{R}^n$.

We recall a result attributed to H. Whitney: There is a C^∞ function $f: \mathbb{R}^n \to \mathbb{R}$ such that $E = Z(f) := \{ x \in \mathbb{R}^n : f(x) = 0 \}$; see, e.g., Krantz and Parks [14, 3.3.6] for a proof. The construction can produce f that is rather far removed from how E arose. To illustrate: If $E = \{0\} \subseteq \mathbb{R}$, then E is the zero set of the squaring function, but the zero set of the derivative of the function produced by Whitney's method has infinitely many connected components. The loose question arises: If E is well behaved in some prescribed sense, can f be chosen to be similarly well behaved? In order to make this question precise we employ a notion from mathematical logic, namely, definability in expansions of $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot, (r)_{r \in \mathbb{R}})$, the real field with constants for all real numbers; readers not familiar with this notion may consult van den Dries and Miller [3, Sections 2 and 4] for an introduction. Let $\mathfrak{R}$ denote $(\overline{\mathbb{R}}, E)$, the structure on $\overline{\mathbb{R}}$ generated by E. Unless indicated otherwise, "definable" means "definable in $\mathfrak{R}$". The question arises: Is there a definable C^∞ function $f: \mathbb{R}^n \to \mathbb{R}$ such that $E = Z(f)$? While visibly true for some E (say, if E is finite), it is known to be false for some very simple cases such as $E = [0, 1] \subseteq \mathbb{R}$. But often what is needed for applications is only that, for each $k \in \mathbb{N}$, there is a C^k function $f_k: \mathbb{R}^n \to \mathbb{R}$ such that $E = Z(f_k)$. Hence, we shall modify the question.

From now on, let p denote a positive integer. Given open $U \subseteq \mathbb{R}^n$, let $\mathcal{C}(U)$ denote the collection of all definable C^p functions $U \to \mathbb{R}$. Our main question: *Are there $f \in \mathcal{C}(\mathbb{R}^n)$ such that $E = Z(f)$?* If $\mathfrak{R}$ defines the set $\mathbb{Z}$ of all integers, then yes, because $(\overline{\mathbb{R}}, \mathbb{Z})$ defines all closed subsets of each $\mathbb{R}^m$ (see, e.g., van den Dries [2, 2.6] or Kechris [13, 37.6]), hence also all continuous functions $\mathbb{R}^n \to \mathbb{R}$, and so the result of Whitney applies. (Thus, the case that $\mathfrak{R}$ defines $\mathbb{Z}$ is of no further interest.) If $\mathfrak{R}$ is **o-minimal** (that is, every definable subset of $\mathbb{R}$ either has interior or is finite),

Received by the editors January 17, 2017, and, in revised form, December 26, 2017.
2010 *Mathematics Subject Classification.* Primary 26B05; Secondary 03C64.
Key words and phrases. Expansion of the real field, definable sets, d-minimal, C^p functions, zero sets.
The research of the second author was conducted while he was a Zassenhaus Assistant Professor at the Department of Mathematics of The Ohio State University.

then again yes, by [3, 4.22]. There are some situations where we know the answer under further assumptions on E alone. The following are two such situations.

Proposition A. *If $E \subseteq \mathbb{R}$, then there exist $f \in \mathcal{C}(\mathbb{R})$ such that $E = Z(f)$.*

(The proof is straightforward; see 1.1 below.)

Proposition B (see 1.7)**.** *If E is a finite union of discrete sets (equivalently, countable and of finite Cantor–Bendixson rank), then there exist $f \in \mathcal{C}(\mathbb{R}^n)$ such that $E = Z(f)$.*

Taken in conjunction with the o-minimal case, these results suggest that our question might have a positive answer if every definable subset of $\mathbb{R}$ either has interior or is a finite union of discrete sets. We do not know if this is true, but we show that it is under a further assumption of uniformity. Following [17], we say that $\mathfrak{R}$ is **d-minimal** (short for "discrete minimal") if for every m and definable $A \subseteq \mathbb{R}^{m+1}$ there exists $N \in \mathbb{N}$ such that for every $x \in \mathbb{R}^m$ the set $\{ y \in \mathbb{R} : (x, y) \in A \}$ either has interior or is a union of at most N discrete sets (equivalently, by model-theoretic compactness, every unary set definable in any structure elementarily equivalent to $\mathfrak{R}$ is a disjoint union of open intervals and finitely many discrete sets). For context, history, and examples of d-minimal structures that are not o-minimal, see Friedman and Miller [8, 9], Miller and Tyne [20], and [17, 18]. (See also 2.2 below.)

Here is the main result of this paper.

Theorem A (see also 2.7 and 2.9)**.** *If $\mathfrak{R}$ is d-minimal, then there exist $f \in \mathcal{C}(\mathbb{R}^n)$ such that $E = Z(f)$.*

Corollary (see also 2.8)**.** *If $\mathfrak{R}$ is d-minimal and $A \subseteq \mathbb{R}^m$ is definable, then A is a finite union of sets of the form $\{ x \in \mathbb{R}^m : f(x) = 0, g_1(x) > 0, \dots, g_N(x) > 0 \}$, where $N \in \mathbb{N}$ and $f, g_1, \dots, g_N \in \mathcal{C}(\mathbb{R}^m)$.*

Proof. By [17, Theorem 3.2] and Dougherty and Miller [1], A is a boolean combination of closed definable subsets of $\mathbb{R}^m$. Apply Theorem A. $\square$

We defer further discussion of corollaries, variants and optimality.

1. Proofs

We begin with some global notation and conventions.

Given $m \in \mathbb{N}$ and $x, y \in \mathbb{R}^m$, we let $\mathrm{d}(x, y)$ denote the Euclidean distance between x and y, along with the usual variants such as $\mathrm{d}(x, B)$, $\mathrm{d}(A, y)$, and $\mathrm{d}(A, B)$ for $A, B \subseteq \mathbb{R}^m$.

Given $A \subseteq \mathbb{R}^m$, let $\operatorname{int} A$ denote the interior of A and $\operatorname{cl} A$ the closure of A. (We tend to omit parentheses in circumstances where they might proliferate so long as any resulting ambiguity is resolved by context. In particular, expressions such as $\operatorname{int} A \cup B$ always mean $(\operatorname{int} A) \cup B$, not $\operatorname{int}(A \cup B)$.) Put $\operatorname{fr} A = \operatorname{cl} A \setminus A$, the **frontier** of A, and $\operatorname{lc} A = A \setminus \operatorname{cl} \operatorname{fr} A$, the **locally closed points** of A (that is, the relative interior of A in $\operatorname{cl} A$). Note that $\operatorname{fr} A = \emptyset$ if and only if A is closed, and $\operatorname{fr} A$ is closed if and only if $A = \operatorname{lc} A$ if and only if A is locally closed (that is, open in its closure). We tend to write $\sim A$ instead of $\mathbb{R}^m \setminus A$ whenever m is clear from context. If A is regarded as a subset of some cartesian product $X \times Y$ and $x \in X$, then A_x denotes the **fiber** of A over x, that is, $A_x = \{ y \in Y : (x, y) \in A \}$.

Put $\mathbb{I} = [0, 1]$. Given $U \subseteq \mathbb{R}^m$, put $\mathcal{C}_{\mathbb{I}}(U) = \{ f \in \mathcal{C} : f(U) \subseteq \mathbb{I} \}$. Note that if $f \in \mathcal{C}(U)$, then there exists $g \in \mathcal{C}_{\mathbb{I}}(U)$ such that $Z(g) = Z(f)$ (say, $g = f^2/(1+f^2)$).

We define an auxiliary function $\hbar \in \mathcal{C}_{\mathbb{I}}(\mathbb{R})$ for use in several places. First, fix any polynomial $P\colon \mathbb{R} \to \mathbb{R}$ such that: P is strictly increasing on $\mathbb{I}$; P is p-flat (that is, all derivatives of order at most p vanish) at 0; and $1 - P$ is p-flat at 1 (say, $P = x^{2p}(x-2)^{2p}$, for concreteness). Now define $\hbar$ by $\hbar{\restriction}\mathbb{I} = P{\restriction}\mathbb{I}$, $\hbar{\restriction}(-\infty, 0) = 0$, and $\hbar{\restriction}(1, \infty) = 1$. (We use $\restriction$ to denote restriction of functions.) Note that $\hbar$ is definable in $\overline{\mathbb{R}}$.

1.1. **Proof of Proposition** A. Suppose that $\emptyset \neq E \subsetneq \mathbb{R}$. We find $f \in \mathcal{C}(\mathbb{R})$ such that $E = Z(f)$. Define $\alpha, \beta\colon \mathbb{R} \setminus E \to \mathbb{R} \cup \{\pm\infty\}$ by $\alpha(x) = \sup\bigl(E \cap (-\infty, x)\bigr)$ and $\beta(x) = \inf\bigl(E \cap (x, +\infty)\bigr)$. Define $g \in \mathcal{C}_{\mathbb{I}}(\mathbb{R})$ by $g{\restriction}\mathbb{I} = x^{2p}(x-1)^{2p}$ and $g{\restriction}{\sim}\mathbb{I} = 0$, and $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} 0, & \text{if } t \in E, \\ \hbar(t - \max E), & \text{if } E \cap [t, +\infty) = \emptyset, \\ \hbar(\min E - t), & \text{if } E \cap (-\infty, t] = \emptyset, \\ (\beta(t) - \alpha(t)) \cdot g\left(\dfrac{t - \alpha(t)}{\beta(t) - \alpha(t)}\right), & \text{otherwise.} \end{cases}$$

It is routine to check that f is as required. □

1.2. *Any expansion of* $\overline{\mathbb{R}}$ *that defines an infinite discrete subset of* $\mathbb{R}$ *also defines a closed infinite discrete subset of* $\mathbb{R}$.

This result (addressing an issue raised, but left open, in Miller and Speissegger [19]) was communicated in personal conversation to the first author by M. Tychonievich while the latter was a PhD student of the former. But it was never published in this form, as a more abstract (and more difficult) version was announced shortly thereafter (and independently) by Fornasiero [4, Remark 4.16], and was soon followed by another variant by Hieronymi [10, Theorem B]. For the reader's convenience, we give the following proof.

Proof of 1.2. Let $A \subseteq \mathbb{R}$ be definable, infinite, and discrete. If A is closed, then we are done, so assume that A is not closed; then there exists $b \in \operatorname{fr} A$. By replacing A with $\{\, 1/\,|b - a| : a \in A \,\}$, we reduce to the case that A is unbounded above. Put

$$S = \{\, (t, a) \in \mathbb{R}^{>0} \times A : \mathrm{d}(a, A \setminus \{a\}) \geq t \,\}.$$

Note that S is definable. If some fiber S_t is infinite, then we are done, so assume otherwise. As A is discrete, for each $a \in A$ there exists $t(a) > 0$ such that $a \in S_t$ for all $t \in (0, t(a)]$. As A is nonempty, there exists $t_0 > 0$ such that: (i) if $0 < t \leq t_0$, then $\max S_t$ exists; (ii) for all $0 < t < t' \leq t_0$, if $\max S_t \neq \max S_{t'}$, then $\max S_t \geq t + \max S_{t'}$; and (iii) $\lim_{t \downarrow 0} \max S_t = +\infty$. Hence, $\{\, \max S_t : 0 < t \leq t_0 \,\}$ is definable, closed, infinite, and discrete (as desired). □

Our current reason for interest in 1.2 is the following.

1.3. *If* $\mathfrak{S}$ *is an expansion of* $\overline{\mathbb{R}}$ *by boolean combinations of open sets, then* $\mathfrak{S}$ *either is o-minimal or defines a closed infinite discrete subset of* $\mathbb{R}$.

(Thus, if $\mathfrak{R}$ is not o-minimal, then it defines a closed infinite discrete subset of $\mathbb{R}$.)

Proof. By [1], $\mathfrak{S}$ is (interdefinable with) an expansion of $\overline{\mathbb{R}}$ by open sets. Assume that $\mathfrak{S}$ is not o-minimal. By [19, Theorem (b)], $\mathfrak{S}$ defines an infinite discrete subset of $\mathbb{R}$, hence also a closed infinite discrete subset of $\mathbb{R}$ by 1.2. □

Following [3] (though our notation is a bit different), let Φ denote the set of all $\phi \in \mathcal{C}(\mathbb{R})$ that are odd, strictly increasing, surjective, and p-flat at 0. Note the following easy facts:

- Φ contains every odd power function of exponent $> p$.
- If $\phi \in \Phi$ and $1 < N \in \mathbb{N}$ is odd, then $\phi^N \in \Phi$, $\phi^N < \phi$ on $(0,1)$, and $\lim_{t\to 0} \phi^N(t)/\phi(t) = 0$.
- For all $\phi_1 \in \Phi$ there exist $\phi_2 \in \Phi$ and $c > 0$ such that $\phi_2(1) = 1$ and $\phi_2(t) \leq \phi_1(t)$ for all $t \in (0, c]$.

1.4. *Let $f\colon (0,1] \to (0,\infty)$ be definable such that $\liminf_{t\to c} f(t) > 0$ for all $c \in (0,1]$. Then there exist $\phi \in \Phi$ such that $\phi(t) \leq f(t)$ for all $t \in (0,1]$.*

(Cf. [3, C.5], but also see 2.3 below.)

Proof. The conclusion is known if $\mathfrak{R}$ is o-minimal (see [3, C.5] and its proof), so assume that $\mathfrak{R}$ is not o-minimal. (We could even reduce to the case that $\mathfrak{R}$ defines no functions $g\colon (0,b) \to (0,\infty)$ such that $0 < b < 1$, $g \leq f{\restriction}(0,b)$, and $(\overline{\mathbb{R}}, g)$ is o-minimal, but this would not change our proof.) By 1.3, there is a sequence $(s_k)_{k\in\mathbb{N}}$ of positive real numbers strictly decreasing to 0 such that $s_0 = 1$ and $S := \{\, s_k : k \in \mathbb{N} \,\}$ is definable. The idea now is essentially just differential calculus. We first produce a C^p function $\psi\colon (0,1] \to \mathbb{R}$, having the desired properties of ϕ on $(0,1]$, by pasting together appropriately translated and scaled copies of $\hbar{\restriction}\mathbb{I}$ defined on the intervals $[s_{k+1}, s_k]$. We can do this definably because the range S of the sequence $(s_k)_{k\in\mathbb{N}}$ is definable; doing the scaling definably is the only part that requires any finesse. It is then routine to extend ψ to some ϕ, as desired. In more detail: For $t \in (0,1]$, put

$$\begin{aligned}
\alpha(t) &= \max(S \cap (0,t)),\\
\beta(t) &= \min(S \cap [t,1]),\\
g(t) &= t \min\bigl(\inf f{\restriction}[t,1], \inf(\beta-\alpha)^p{\restriction}[t,1]\bigr),\\
\psi(t) &= g(\alpha(\alpha(t))) + [g(\alpha(t)) - g(\alpha(\alpha(t)))]\hbar\left(\frac{t-\alpha(t)}{\beta(t)-\alpha(t)}\right).
\end{aligned}$$

Observe that ψ is definable, and g is positive and strictly increasing. If $s_{k+1} < t \leq s_k$, then

$$\psi(t) = g(s_{k+2}) + \bigl(g(s_{k+1}) - g(s_{k+2})\bigr)\hbar\left(\frac{t - s_{k+1}}{s_k - s_{k+1}}\right).$$

By inspection, ψ is C^p, positive, strictly increasing and bounded above by f, and $\psi - \psi(s_k)$ is p-flat at each s_k. Let $j \in \{1,\dots,p\}$. If $s_{k+1} < t < s_k$, then

$$\psi^{(j)}(t) = \frac{g(s_{k+1}) - g(s_{k+2})}{(s_k - s_{k+1})^j}\hbar^{(j)}\left(\frac{t-s_{k+1}}{s_k - s_{k+1}}\right).$$

As $\hbar^{(j)}$ is bounded and

$$g(s_{k+1}) - g(s_{k+2}) < g(s_{k+1}) < g(s_k) \leq s_k(\beta(s_k) - \alpha(s_k))^p = s_k(s_k - s_{k+1})^p,$$

we have $\lim_{t\downarrow 0} \psi^{(j)}(t) = 0$ for $j = 0,\dots,p$. Extend ψ to $[0,\infty)$ by putting $\psi(0) = 0$ and $\psi(t) = \psi(1) + (t-1)^{p+1}$ for $t > 1$. Finally, define $\phi \in \Phi$ by $\psi(t)$ for $t \geq 0$, and $\phi(t) = -\psi(-t)$ for $t < 0$. By L'Hôpital's rule (iterated), ϕ is C^p and p-flat at 0. Hence, $\phi \in \Phi$. $\square$

The next result is essentially the same as [3, C.8], but we must provide a proof that is free of tameness assumptions on $\mathfrak{R}$. (As written, [3, C.8] relies indirectly on [3, C.4].)

1.5 (cf. [3, C.8]). *Let $A \subseteq \mathbb{R}^n$ be locally closed, let $g\colon A \to \mathbb{R}$ be definable and continuous, and let $\mathcal{H}$ be a finite set of definable locally bounded functions $A \setminus Z(g) \to \mathbb{R}$. Then there exist $\phi \in \Phi$ such that $\phi(1) = 1$ and $\lim_{x\to y} \phi(g(x))h(x) = 0$ for each $y \in Z(g)$ and $h \in \mathcal{H}$.*

Proof. It suffices to consider the case that $A \setminus Z(g) \neq \emptyset$ and $\mathcal{H}$ is a singleton $\{h\}$ with $h \geq 0$. Define $S \subseteq \mathbb{R}^{>0} \times A$ fiberwise by

$$S_t = \{\, x \in A : |x| \leq 1/t \;\;\&\;\; \mathrm{d}(x, \operatorname{fr} A) \geq t \;\;\&\;\; |g(x)| \geq t \,\}.$$

Observe that S is definable and each S_t is compact. Define $f\colon (0,1] \to \mathbb{R}$ by $f(t) = 1/(1 + \sup h{\restriction}S_t)$ if $S_t \neq \emptyset$, and $f(t) = 1$ if $S_t = \emptyset$. By 1.4 and its preceding paragraph, there exists $\phi \in \Phi$ such that $\phi(1) = 1$ and $\lim_{t\downarrow 0} \phi(t) \sup h{\restriction}S_t = 0$. Let $y \in Z(g)$, and let $(x_k)_{k\in\mathbb{N}}$ be a sequence of points in $A \setminus Z(g)$ such that $k \mapsto \mathrm{d}(x_k, y)$ is strictly decreasing to 0. Since $\mathrm{d}(y, \operatorname{fr} A) > 0$ and $g(x_k) \to 0$, we may take each $x_k \in S_{|g(x_k)|}$; then

$$|\phi(g(x_k))h(x_k)| = \phi(|g(x_k)|)h(x_k) \leq \phi(|g(x_k)|) \sup h{\restriction}S_{|g(x_k)|}.$$

Thus, $\lim_{k\to\infty} \phi(g(x_k))h(x_k) = 0$, hence also $\lim_{x\to y} \phi(g(x))h(x) = 0$. ∎

Some routine, but very useful, consequences follow.

1.6. *Let $U \subseteq \mathbb{R}^n$ be open, and let $g\colon U \to \mathbb{R}$ be definable.*

(1) *If g is continuous, and C^p on $U \setminus Z(g)$, then there exist $\phi \in \Phi$ such that $\phi \circ g$ is C^p and $Z(1 - \phi \circ g) = Z(1 - g)$.*
(2) *If g is C^p, then there exist $f \in \mathcal{C}(U)$ such that $Z(f) = Z(g)$ and f is p-flat on $Z(f)$.*
(3) *If g is C^p and $Z(g) = (E \setminus \operatorname{int} E) \cap U$, then there exist $f \in \mathcal{C}(U)$ such that $Z(f) = E \cap U$.*
(4) *If g is C^p and $h \in \mathcal{C}(U \setminus Z(g))$, then there exist $f \in \mathcal{C}(U)$ such that $Z(f) = Z(g) \cup Z(h)$ and $Z(1 - f) \supseteq Z(1 - g) \cap Z(1 - h)$.*

(Results (2), (3), and (4) are standard consequences of (1). All of the main ideas for the proof of (1) occur when $p = 1$: Let ϕ_0 be as in 1.5 with $\mathcal{H}$ equal to the set of all first partials of $g{\restriction}(U \setminus Z(g))$; then ϕ_0^3 $(= \phi_0^{2p+1})$ belongs to Φ and

$$\nabla(\phi_0^3 \circ g)(x) = [(\phi_0 \circ g)\nabla g](x) \cdot [3(\phi_0\phi_0') \circ g](x)$$

for all $x \in U \setminus Z(g)$.)

Note. We shall refer to property 1.6(4) as **gluing**.

We are now ready to establish Proposition B.

1.7. *Let $A \subseteq U \subseteq \mathbb{R}^n$ be definable such that A is a finite union of discrete sets and U is open. Then there exist $f \in \mathcal{C}_{\mathbb{I}}({\sim}\operatorname{cl}\operatorname{fr} A)$ such that $Z(f) = \operatorname{lc} A$ and $Z(1 - f) \supseteq {\sim}U \setminus \operatorname{cl}\operatorname{fr} A$.*

(Proposition B is the case that A is closed and $U = \mathbb{R}^n$. The more technical statement arises from inductive needs.)

Proof. We proceed by induction on the minimal number of discrete sets that comprise A (that is, on the Cantor–Bendixson rank of A regarded as a subspace of $\mathbb{R}^n$).

Suppose that A is discrete. Define $\rho\colon A \to \mathbb{R}$ by

$$\rho(a) = \min\bigl(1, \mathrm{d}(a, A \setminus \{a\}), \mathrm{d}(a, \operatorname{fr} U)\bigr).$$

Put $V = \bigcup_{a\in A}\{\, v \in \mathbb{R}^n : 3\,\mathrm{d}(v,a) < \rho(a)\,\}$. Define $\sigma\colon V \to \mathbb{R}^n$ by letting $\sigma(v)$ be the center of the ball containing v; observe that $\sigma(v)$ is the unique $a \in A$ such that $3\,\mathrm{d}(v,a) < \rho(a)$. For $v \in V$, put $f(v) = \hbar(10\,\mathrm{d}^2(v,\sigma(v))/\rho^2(\sigma(v)))$. For $v \notin V$, put $f(v) = 1$. As A is locally closed, $\operatorname{fr} A$ is closed. It is routine to check that the restriction of f to $\sim \operatorname{fr} A$ is as desired.

More generally, let A_1 be the set of isolated points of A. By the preceding paragraph, there exists $f_1 \in \mathcal{C}_{\mathbb{I}}(\sim \operatorname{fr} A_1)$ such that $Z(f_1) = A_1$ and $Z(1 - f_1) \supseteq {\sim U} \setminus \operatorname{fr} A_1$. Put $A_2 = A \setminus A_1$. Inductively, there exists $f_2 \in \mathcal{C}_{\mathbb{I}}(\sim \operatorname{cl}\operatorname{fr} A_2)$ such that $Z(f_2) = A_2 \setminus \operatorname{cl}\operatorname{fr} A_2$ and $Z(1 - f_2) \supseteq {\sim U} \setminus \operatorname{cl}\operatorname{fr} A_2$. The result now follows by gluing. $\square$

It is natural to next consider the case that $0 < m < n$ and every fiber of E over $\mathbb{R}^m$ is a finite union of discrete sets, but we do not yet know how to deal with this level of generality, even if every fiber of E over $\mathbb{R}^m$ is discrete. Thus, we shall assume both a uniform bound on the Cantor–Bendixson rank of the fibers and a tameness condition on $\mathfrak{R}$.

As noted earlier, we are done if $\mathfrak{R}$ defines $\mathbb{Z}$. It is suspected that if $\mathfrak{R}$ does not define $\mathbb{Z}$, then every definable set either has interior or is nowhere dense (see Hieronymi and Miller [11] for some evidence), a condition that holds if $\mathfrak{R}$ is d-minimal (by [17], it is enough to show that every definable subset of $\mathbb{R}$ has interior or is nowhere dense). Next is a key technical lemma.

1.8. *Suppose that every definable set has interior or is nowhere dense. Let $m \in \{0,\dots,n\}$, and let π denote a projection on the first m coordinates. Let $N \in \mathbb{N}$, and let $A \subseteq U \subseteq \mathbb{R}^n$ be definable such that U is open and every A_x is a union of at most N discrete sets ($x \in \mathbb{R}^m$). Then there exist a definable open and dense $W \subseteq \mathbb{R}^m$ and $f \in \mathcal{C}_{\mathbb{I}}(\pi^{-1}W \setminus \operatorname{cl}\operatorname{fr} A)$ such that $Z(f) = \pi^{-1}W \cap \operatorname{lc} A$ and $Z(1-f) \supseteq {\sim U} \cap \pi^{-1}W \setminus \operatorname{cl}\operatorname{fr} A$. If every A_x is discrete, then W can be taken such that $f \in \mathcal{C}_{\mathbb{I}}(\pi^{-1}W \setminus \operatorname{fr} A)$, $Z(f) = \pi^{-1}W \cap A$, and $Z(1-f) \supseteq {\sim U} \cap \pi^{-1}W \setminus \operatorname{fr} A$.*

Proof. The case $m = 0$ is just 1.7, so we take $m > 0$. As $\pi A \setminus \operatorname{int}\pi A$ is nowhere dense, we reduce to the case that πA is nonempty and open. For each $d \in \mathbb{N}$, the set of $x \in \mathbb{R}^m$ such that A_x has Cantor–Bendixson rank d is definable. Thus, we reduce to the case that $N \geq 1$ and every fiber of A over πA has Cantor–Bendixson rank N. We now proceed by induction on N. In order to reduce clutter, we sometimes identify maps with their graphs.

Let $N = 1$, that is, A_x is discrete for each $x \in \pi A$. Define $\rho\colon A \to \mathbb{R}$ by

$$\rho(x,y) = \min\bigl(1, \mathrm{d}(y, A_x \setminus \{y\}), \mathrm{d}(y, \operatorname{fr} U_x)\bigr).$$

Let $\tau\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a projection on the first n variables. Put $\tilde\pi = \pi \circ \tau$ (that is, $\tilde\pi$ is a projection of $\mathbb{R}^{n+1}$ on the first m variables). Let C be the set of all $a \in A$ for which there exists an open box $B \subseteq \mathbb{R}^{n+1}$ centered at $(a, \rho(a))$ such that $A \cap \tau B$ is a C^p map $\tilde\pi B \to \mathbb{R}^{n-m}$ and $\rho\restriction(A \cap \tau B)$ is also C^p. It is an exercise to see that C is definable (see, e.g., [3, Appendices A and B]). We now reduce to the case that $A = C$ by showing that $\pi(A \setminus C)$ has no interior (and thus is nowhere dense). Suppose to the contrary that $\pi(A \setminus C)$ has interior; then we can reduce to the case that $C = \emptyset$. As every fiber of A over $\mathbb{R}^m$ is discrete, so is every fiber of ρ over $\mathbb{R}^m$. By the Baire Category Theorem, there is an open box $B \subseteq \mathbb{R}^{n+1}$ such that the set $\{ x \in \mathbb{R}^m : \operatorname{card}((B \cap \rho)_x) = 1 \}$ is somewhere dense, and thus has interior. By shrinking B, we reduce to the case that there is a definable map $\varphi\colon \tilde{\pi}B \to \rho$. By [17, Theorem 3.3], we may shrink B so that φ is C^p. But then the projection of φ on the first n coordinates is contained in C, contradicting that $C = \emptyset$.

We have now reduced to the case that $A = C$. Put

$$V = \bigcup_{(x,z)\in A} \{ (x,y) \in \mathbb{R}^n : 3\,\mathrm{d}(y,z) < \rho(x,z) \}.$$

Define $\xi\colon V \to \mathbb{R}^{n-m}$ by letting $\xi(x,y)$ be the unique $z \in A_x$ such that $3\,\mathrm{d}(y,z) < \rho(x,z)$. Since V is definable, it now suffices (by arguing similarly as in 1.7) to show that V is open and

$$(x,y) \mapsto \hbar(10\,\mathrm{d}^2(y,\xi(x,y))/\rho^2(x,\xi(x,y)))\colon V \to \mathbb{R}$$

is C^p. Let $(x_0,y_0) \in V$; then $(x_0,\xi(x_0,y_0)) \in C$, so there exist an open box $B \subseteq \mathbb{R}^{n+1}$ centered at the point $\big((x_0,\xi(x_0,y_0)),\rho(x_0,\xi(x_0,y_0))\big)$ and a C^p map $\gamma\colon \tilde{\pi}B \to \mathbb{R}^{n-m}$ such that $\gamma = A \cap \tau B$ and $\rho{\restriction}\gamma$ is C^p. Put

$$S = \{ (x,y) \in \tilde{\pi}B \times \mathbb{R}^{n-m} : 3\,\mathrm{d}(y,\gamma(x)) < \rho(x,\gamma(x)) \};$$

then S is open, $(x_0,y_0) \in S \subseteq V$, and $\xi(x,y) = \gamma(x)$ for all $(x,y) \in S$. Hence, S is open, $\xi{\restriction}S$ is C^p, and the function

$$(x,y) \mapsto \hbar(10\,\mathrm{d}^2(y,\xi(x,y))/\rho^2(x,\xi(x,y)))\colon S \to \mathbb{R}$$

is C^p. As $(x_0,y_0) \in V$ was arbitrary, this also holds with V in place of S, as was to be shown. (This concludes the proof for the case $N = 1$.)

As the union of the set of isolated points of the fibers of A over πA is definable, the rest of the induction is a routine modification of the corresponding part of the argument for 1.7. □

Let $\Pi(n,m)$ denote the collection of all coordinate projection maps

$$(x_1,\ldots,x_n) \mapsto \big(x_{\lambda(1)},\ldots,x_{\lambda(m)}\big) : \mathbb{R}^n \to \mathbb{R}^m,$$

where $\lambda\colon \{1,\ldots,m\} \to \{1,\ldots,n\}$ is strictly increasing. For $A \subseteq \mathbb{R}^n$, let $\dim A$ be the supremum of all $m \in \mathbb{N}$ such that πA has interior for some $\pi \in \Pi(n,m)$. By [17, §7], if every subset of $\mathbb{R}$ definable in $(\overline{\mathbb{R}}, A)$ has interior or is nowhere dense (in particular, if $(\overline{\mathbb{R}}, A)$ is d-minimal), then $\dim \operatorname{cl} A = \dim A$, hence also $\dim \operatorname{cl}\operatorname{fr} A \le \dim A$.

Proof of Theorem A. Assume that $\mathfrak{R}$ is d-minimal. We must find $f \in \mathcal{C}(\mathbb{R}^n)$ such that $Z(f) = E$. It suffices to show that if $A \subseteq U \subseteq \mathbb{R}^n$ is definable and U is open, then there exist $f \in \mathcal{C}_{\mathbb{I}}(\sim \operatorname{cl}\operatorname{fr} A)$ and $g \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $Z(f) = \operatorname{lc} A \subseteq {\sim} Z(g) \subseteq U$ (consider $A = E$ and $U = \mathbb{R}^n$). The result is trivial if $A = \emptyset$. Assume that $A \neq \emptyset$. We proceed by induction on $d = \dim A \ge 0$ and $n \ge 1$.

Suppose that $d = 0$. By d-minimality, A is a finite union of discrete sets, hence also $\operatorname{cl}\operatorname{fr} A$ is a finite union of discrete sets. By 1.7, there exist $f \in \mathcal{C}_{\mathbb{I}}(\sim \operatorname{cl}\operatorname{fr} A)$ and $h \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $Z(f) = \operatorname{lc} A$, $Z(1-f) \supseteq {\sim} U \setminus \operatorname{cl}\operatorname{fr} A$, and $Z(h) = \operatorname{cl}\operatorname{fr} A$. Let g be the result of gluing h and $1-f$. Then f and g are as desired.

It now suffices by 1.6(3) to consider the case that $0 < d < n$ and the result holds for all lesser values of n or d.

We first show that there exist $f \in \mathcal{C}_{\mathbb{I}}(\sim \operatorname{cl} \operatorname{fr} A)$ with $Z(f) = \operatorname{lc} A$. (Note that this statement is independent of U, and if A is closed, then $f \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ and $Z(f) = A$.) Let $\pi \in \Pi(n,d)$. By [17, Lemma 8.5], there exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$, either $\dim(A \cap \pi^{-1}x) > 0$ or $A \cap \pi^{-1}x$ is a union of N discrete sets. As $d = \dim A$, the set $\{\, x \in \mathbb{R}^d : \dim(A \cap \pi^{-1}x) > 0 \,\}$ has no interior, and thus is nowhere dense. By 1.8, there exist dense open definable $W_\pi \subseteq \mathbb{R}^d$ and $\alpha_\pi \in \mathcal{C}_{\mathbb{I}}(\pi^{-1}W_\pi \setminus \operatorname{cl} \operatorname{fr} A)$ such that $Z(\alpha_\pi) = \operatorname{lc} A \cap \pi^{-1}W_\pi$. Inductively, there exist $\beta_\pi \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^d)$ such that $Z(\beta_\pi) = {\sim}W_\pi$, hence also $\gamma_\pi \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $Z(\gamma_\pi) = {\sim}\pi^{-1}W_\pi$. By gluing α_π and γ_π there exist $f_\pi \in \mathcal{C}_{\mathbb{I}}(\sim \operatorname{cl} \operatorname{fr} A)$ such that $Z(f_\pi) = \operatorname{lc} A \cup {\sim}\pi^{-1}W_\pi$. Put $X = \bigcap_{\pi \in \Pi(n,d)} {\sim}\pi^{-1}W_\pi$. Note that X is closed and has $\dim < d$. Put $Y = (\operatorname{lc} A \cup X) \setminus \operatorname{cl} A$; then $\operatorname{cl} Y \subseteq X$ (so $\dim Y < d$), $Y \subseteq \sim \operatorname{cl} A$, and Y is locally closed. Inductively, there exist $g_1 \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $Y \subseteq {\sim}Z(g_1) \subseteq \sim \operatorname{cl} A$; then

$$f = \frac{(g_1 \restriction \sim \operatorname{cl} \operatorname{fr} A) + \sum_{\pi \in \Pi(n,d)} f_\pi}{1 + \operatorname{card} \Pi(n,d)}$$

is as desired.

We now produce $g \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $\operatorname{lc} A \subseteq {\sim}Z(g) \subseteq U$. Since $\dim \operatorname{cl} \operatorname{fr} A \leq \dim A \leq d$, we may apply the result of the preceding paragraph to obtain $h \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $Z(h) = \operatorname{cl} \operatorname{fr} A$.

For $\pi \in \Pi(n,d)$, let W_π, α_π, and let δ_π be as in the preceding paragraph, but also taking U into account when applying 1.8; then we also have $Z(1-\alpha_\pi) \supseteq {\sim}U \cap \pi^{-1}W_\pi \setminus \operatorname{cl} \operatorname{fr} A$. By gluing $h\delta_\pi$ and $1 - \alpha_\pi$, there exists $g_\pi \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $\operatorname{lc} A \cap \pi^{-1}W_\pi \subseteq {\sim}Z(g_\pi) \subseteq U$. With X as before, $\operatorname{lc} A \cap X$ is locally closed and has $\dim < d$. Hence, inductively, there exists $g_2 \in \mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $\operatorname{lc} A \cap X \subseteq {\sim}Z(g_2) \subseteq U$. Put

$$g = \frac{g_2 + \sum_{\pi \in \Pi(n,d)} g_\pi}{1 + \operatorname{card} \Pi(n,d)}$$

to finish. $\square$

2. Remarks

2.1. Theorem A does not settle our main question. To illustrate, suppose that $E \subseteq \mathbb{R}^2$ is compact, has no isolated points, $\dim E = 0$, and $\mathfrak{R}$ does not define $\mathbb{Z}$. In this generality, we do not know if there exist $f \in \mathcal{C}(\mathbb{R}^2)$ such that $Z(f) = E$ (indeed, we do not even know if there exist C^p functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $E = Z(f)$ and $(\mathfrak{R}, f)$ does not define $\mathbb{Z}$). There are examples of Cantor subsets K of $\mathbb{R}$ such that the expansion of $\overline{\mathbb{R}}$ by all subsets of each K^m satisfies the “interior or nowhere dense” condition of 1.8; see Friedman et al. [7].

2.2. There are currently no verified examples of E such that $\mathfrak{R}$ is not d-minimal but every definable subset of $\mathbb{R}$ either has interior or is a finite union of discrete sets.

2.3. Here is a concrete example justifying our extra work for the proof of 1.4. Put $a_0 = 1$ and $a_k = \exp(\exp(k^2))$ for $0 < k \in \mathbb{N}$. Suppose that $E = \{\, a_k : k \in \mathbb{N} \,\}$. By [9], $\mathfrak{R}$ is d-minimal. There is an obvious choice for the set S in the proof of 1.4: Take $s_k = 1/a_k$ for $k \in \mathbb{N}$. Let $f\colon (0,1] \to \mathbb{R}$ be given by $f \restriction (s_{k+1}, s_k] = s_{k+2}$; then f is definable and $\lim_{t \downarrow 0} f(t)/t^d = 0$ for each $d \in \mathbb{N}$. If $\mathfrak{R}$ were to define some

$g\colon (0,b) \to (0,\infty)$ such that $0 < b < 1$, $g \leq f{\restriction}(0,b)$, and $(\overline{\mathbb{R}}, g)$ is o-minimal (thus obviating the need for our construction of ψ), then it would define the function exp (by growth dichotomy [15]), hence also $\mathbb{N}$, contradicting d-minimality.

More generally: By Fornasiero et al. [6], if $\mathfrak{R}$ does not define $\mathbb{N}$, then every nonempty bounded nowhere dense definable subset of $\mathbb{R}$ has Minkowksi dimension zero. (See [11] for an even more stringent result.) This forces the sequence (s_k) in the proof of 1.4 to approach 0 extremely rapidly if $\mathfrak{R}$ also defines exp.

Some of our proofs suggest more general results. We give one example, beginning with a result about o-minimality.

2.4. *Let $S \subseteq \mathbb{R}^{m+n}$ be such that $(\overline{\mathbb{R}}, S)$ is o-minimal and every fiber of S over $\mathbb{R}^m$ is closed. Then there exist $F\colon \mathbb{R}^m \times \mathbb{R}^{>0} \times \mathbb{R}^n \to \mathbb{R}$ definable in $(\overline{\mathbb{R}}, S)$ such that for all $u \in \mathbb{R}^m$ and $r > 0$:*

- *$F(u, r, \mathbb{R}^n) \subseteq \mathbb{I}$,*
- *$v \mapsto F(u, r, v)\colon \mathbb{R}^n \to \mathbb{R}$ is C^p,*
- *if $v \in \mathbb{R}^n$, then $F(u, r, v) = 0$ if and only if $v \in S_u$,*
- *if $v \in \mathbb{R}^n$ and $\mathrm{d}(v, S_u) \geq r$, then $F(u, r, v) = 1$.*

Proof. Let $u \in \mathbb{R}^m$ and $r > 0$. By [3, C.12], there is a C^p function $F_{u,r}\colon \mathbb{R}^n \to [0,1]$ definable in $(\overline{\mathbb{R}}, S)$ such that $Z(F_{u,r}) = S_u$ and $Z(1 - F_{u,r}) = \{\, v \in \mathbb{R}^n : \mathrm{d}(v, S_u) \geq r \,\}$. Put $F(u, r, v) = F_{u,r}(v)$ for $v \in \mathbb{R}^n$. An examination of the proof of [3, C.12] (including all supporting results) yields that F is definable in $(\overline{\mathbb{R}}, S)$. □

2.5. *With all data as in* 2.4, *assume moreover that S is definable (in $\mathfrak{R}$). Let $A \subseteq U \subseteq \mathbb{R}^n$ be definable such that U is open and there is a definable $B \subseteq \mathbb{R}^m$ such that $A = \bigcup_{b \in B} S_b$ and $\mathrm{d}(S_b, A \setminus S_b) > 0$ for all $b \in B$. Then $\operatorname{fr} A$ is closed and there exist $f \in \mathcal{C}_{\mathbb{I}}(\sim \operatorname{fr} A)$ such that $Z(f) = A$ and $Z(1 - f) \supseteq {\sim}U \setminus \operatorname{fr} A$.*

Proof. It is immediate from assumptions (and o-minimality) that A is locally closed, so $\operatorname{fr} A$ is closed. For $u \in \mathbb{R}^m$, put

$$\rho(u) = \min\bigl(1, \mathrm{d}(S_u, A \setminus S_u), \mathrm{d}(S_u, {\sim}U)\bigr).$$

Put $V = \bigcup_{b \in B} \{\, v \in \mathbb{R}^n : 3\,\mathrm{d}(v, S_b) < \rho(b) \,\}$. For $v \in V$, let $\sigma(v) = b$ where $S_b \in \mathbb{R}^n$ is the fiber of S such that $\mathrm{d}(v, S_b) < \rho(b)$, and put $f(v) = F(\sigma(v), \rho(\sigma(v)), v)$. For $v \notin V$, put $f(v) = 1$. The restriction of f to $\sim \operatorname{fr} A$ is as desired. □

If A is discrete, then we recover the conclusion of 1.7 from 2.5 by setting $S = \{\, (x, x) : x \in A \,\}$ and $B = A$. While this might sound promising, a moment's thought reveals that any straightforward attempt to extend 2.5 so as to recover the full conclusion of 1.7 would seem to require a rather tedious (albeit fairly obvious) hypothesis, because the assumptions of 2.5 are about *the form of A* rather than a tameness property of $\mathfrak{R}$. One can imagine more results along these lines, but again with assumptions that tend to become tedious and disconnected from tameness of $\mathfrak{R}$.

2.6. As mentioned in the introduction, the C^∞ version of Theorem A does not hold in general; indeed, it fails if $\mathfrak{R}$ is o-minimal and does not define exp (an easy consequence of results from [15, 16]). On the other hand, if exp is definable, then the C^∞ version holds for at least some o-minimal $\mathfrak{R}$ (indeed, for most of the expansions of $(\overline{\mathbb{R}}, \exp)$ that are currently known to be o-minimal); see Jones [12]. By [20], the expansion of $(\overline{\mathbb{R}}, \exp)$ by the set of "towers" $\{e, e^e, e^{e^e}, \dots\}$ is d-minimal; we suspect that the C^∞ version of Theorem A holds for any closed set definable in this structure (but, as yet, it is unclear to us whether it would be the effort to prove it).

Our remaining remarks are more of model-theoretic interest. First, Theorem A is independent of parameters, as defined below.

2.7. *If* $(\mathbb{R},+,\cdot,E)$ *is d-minimal, then there exist* $f\in C^p(\mathbb{R}^n,\mathbb{R})$ *such that* $Z(f)=E$ *and* f *is* $\emptyset$*-definable in* $(\mathbb{R},+,\cdot,E)$*.*

This can be established by tracking parameters throughout the proof (including all supporting results from elsewhere), but what follows is another approach of potentially independent interest.

Proof. First, observe that in any expansion of $(\mathbb{R},<,+,1)$, every rational number is $\emptyset$-definable. Hence, if every nonempty definable subset of $\mathbb{R}$ either has interior or an isolated point, then every nonempty definable subset of $\mathbb{R}$ has an $\emptyset$-definable point. An easy induction then yields that every nonempty $\emptyset$-definable set contains a $\emptyset$-definable point.

Assume that $(\mathbb{R},+,\cdot,E)$ is d-minimal; then every nonempty $\emptyset$-definable set contains a $\emptyset$-definable point. By Theorem A, there exist $m\in\mathbb{N}$, $a\in\mathbb{R}^m$, and $\emptyset$-definable $f\colon\mathbb{R}^{m+n}\to\mathbb{R}$ such that $x\mapsto f(a,x)$ is C^p with zero set equal to E. The set of all $u\in\mathbb{R}^m$ such that $x\mapsto f(u,x)$ is C^p with zero set equal to E is $\emptyset$-definable and nonempty, so there is a $\emptyset$-definable $b\in\mathbb{R}^m$ such that $x\mapsto f(b,x)$ is C^p with zero set equal to E. □

An example of consequences is as follows.

2.8. *If* $\mathfrak{S}$ *is a d-minimal expansion of* $(\mathbb{R},+,\cdot\,)$, *then the expansion* in the syntactic sense *of* $(\mathbb{R},<)$ *by all* C^p *functions that are* $\emptyset$*-definable in* $\mathfrak{S}$ *admits elimination of quantifiers.*

Proof. By [17, Theorem 3.2] and [1], every $\emptyset$-definable set is a boolean combination of $\emptyset$-definable closed sets. Apply 2.7. □

It can be shown that Theorem A holds over arbitrary ordered fields provided that an appropriate definition of d-minimality is given.

2.9. *Let* $\mathfrak{M}$ *be an expansion of an ordered field such that, for every* $\mathfrak{M}'\equiv\mathfrak{M}$, *every unary set definable in* $\mathfrak{M}'$ *is a disjoint union of open intervals and finitely many discrete sets. Then every closed set definable in* $\mathfrak{M}$ *is the zero set of some definable (total)* C^p *function.*

A proof can be obtained by modifying our proof of Theorem A via the emerging subject of "definably complete" expansions of ordered fields (see, e.g., Fornasiero and Hieronymi [5] and its bibliography). We leave details to the interested reader.

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