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TENSOR PRODUCTS AND SUMS OF p-HARMONIC FUNCTIONS, QUASIMINIMIZERS AND p-ADMISSIBLE WEIGHTS

ANDERS BJÖRN AND JANA BJÖRN

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Abstract. The tensor product of two p-harmonic functions is in general not p-harmonic, but we show that it is a quasiminimizer. More generally, we show that the tensor product of two quasiminimizers is a quasiminimizer. Similar results are also obtained for quasisuperminimizers and for tensor sums. This is done in weighted $\mathbf{R}^n$ with p-admissible weights. It is also shown that the tensor product of two p-admissible measures is p-admissible. This last result is generalized to metric spaces.

1. Introduction

It is well known (and easy to prove) that the tensor product and tensor sum of two harmonic functions are harmonic, i.e., if u_j is harmonic in $\Omega_j \subset \mathbf{R}^{n_j}$, $j = 1, 2$, then $u_1 \otimes u_2$ and $u_1 \oplus u_2$ are harmonic in $\Omega_1 \times \Omega_2 \subset \mathbf{R}^{n_1+n_2}$. Here

$$(u_1 \otimes u_2)(x, y) := u_1(x)u_2(y) \quad \text{and} \quad (u_1 \oplus u_2)(x, y) := u_1(x) + u_2(y).$$

It is also well known that the corresponding property for p-harmonic functions fails. However, as we show in this note, the tensor product of two p-harmonic functions is a quasiminimizer.

Here $u \in W^{1,p}_{\rm loc}(\Omega)$ is *p-harmonic* in the open set $\Omega \subset \mathbf{R}^n$ if it is a continuous weak solution of the p-Laplace equation

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty.$$

Moreover, $u \in W^{1,p}_{\rm loc}(\Omega)$ is a *Q-quasiminimizer* if

$$\int_{\varphi \neq 0} |\nabla u|^p \, dx \le Q \int_{\varphi \neq 0} |\nabla (u + \varphi)|^p \, dx$$

for all boundedly supported Lipschitz functions φ vanishing outside Ω. A quasiminimizer always has a continuous representative, and if $Q = 1$ this representative is a p-harmonic function.

In this note we show the following result.

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Theorem 1. *Let $1 < p < \infty$, and let u_j be a Q_j-quasiminimizer in $\Omega_j \subset \mathbf{R}^{n_j}$ with respect to a p-admissible weight w_j, $j = 1, 2$. Then $u = u_1 \otimes u_2$ and $v = u_1 \oplus u_2$ are Q-quasiminimizers in $\Omega_1 \times \Omega_2$ with respect to the p-admissible weight $w = w_1 \otimes w_2$, where*

$$Q = \begin{cases} \left(Q_1^{2/|p-2|} + Q_2^{2/|p-2|}\right)^{|p-2|/2} & \text{if } p \neq 2, \\ \max\{Q_1, Q_2\} & \text{if } p = 2. \end{cases} \tag{1}$$

In particular, if u_1 and u_2 are p-harmonic, then u and v are Q-quasiminimizers with $Q = 2^{|p-2|/2}$.

We also obtain a corresponding result for quasisuperminimizers. We pursue our studies on weighted $\mathbf{R}^n$ with respect to so-called p-admissible weights. To do so, we first show in Section 2 that the product of two p-admissible measures is p-admissible. This generalizes some earlier special cases from Lu–Wheeden [15, Lemma 2], Chua–Wheeden [7, Theorem 3.1], Kilpeläinen–Koskela–Masaoka [13, Lemma 2.2], and Björn [4, Lemma 11], but we have not seen it proved in this form in the literature. In fact, our result holds in the generality of metric spaces, see Remark 4.

Usually, $Q \geq 1$ in the definition of Q-quasiminimizers but here it is convenient to also allow for $Q = 0$ (which happens exactly when u is a.e. constant in every component of Ω). For example, if $Q_2 = 0$, then $Q = Q_1$ in Theorem 1. Even this special case of Theorem 1 seems to have gone unnoticed in the literature.

Quasiminimizers were introduced by Giaquinta and Giusti [8], [9] in the early 1980s as a tool for a unified treatment of variational integrals, elliptic equations, and quasiregular mappings on $\mathbf{R}^n$. In those papers, De Giorgi's method was extended to quasiminimizers, yielding in particular their local Hölder continuity. Quasiminimizers have since then been studied in a large number of papers, first on unweighted $\mathbf{R}^n$ and later on metric spaces; see Appendix C in Björn–Björn [3] and the introduction in Björn [5] for further discussion and references.

Quasiminimizers form a much more flexible class than p-harmonic functions. For example, Martio–Sbordone [16] showed that quasiminimizers have an interesting and nontrivial theory also in one dimension, and Kinnunen–Martio [14] developed an interesting nonlinear potential theory for quasiminimizers, including quasisuperharmonic functions. Unlike p-harmonic functions and solutions of elliptic PDEs, quasiminimizers can have singularities of any order, as shown in Björn–Björn [2].

2. Tensor products of p-admissible measures

Let w be a weight function on $\mathbf{R}^n$, i.e., a nonnegative locally integrable function, and let $d\mu = w\,dx$. In this section we also let $1 \leq p < \infty$ be fixed. For a ball $B = B(x_0, r) := \{x : |x - x_0| < r\}$ in $\mathbf{R}^n$ we use the notation $\lambda B = B(x_0, \lambda r)$.

Definition 2. The measure μ (or the weight w) is *p-admissible* if the following two conditions hold:

- It is *doubling*, i.e., there exists a *doubling constant* $C > 0$ such that for all balls B,
$$0 < \mu(2B) \leq C\mu(B) < \infty.$$

- It supports a *p-Poincaré inequality*, i.e., there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls B and all bounded locally Lipschitz functions u on λB,

$$\fint_B |u - u_B| \, d\mu \leq C \operatorname{diam}(B) \biggl(\fint_{\lambda B} |\nabla u|^p \, d\mu \biggr)^{1/p},$$

where ∇u is the a.e. defined gradient of u and

$$u_B := \fint_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.$$

This is one of many equivalent definitions of p-admissible weights in the literature; see, e.g., Corollary 20.9 in Heinonen–Kilpeläinen–Martio [11] (which is not in the first edition) and Proposition A.17 in Björn–Björn [3]. It can be shown that on $\mathbf{R}^n$, the dilation λ in the Poincaré inequality can be taken equal to 1; see Jerison [12], Hajłasz–Koskela [10], and the discussion in [11, Chapter 20].

It is not known whether there exist any admissible measures on $\mathbf{R}^n$, $n \geq 2$, which are not absolutely continuous with respect to the Lebesgue measure (and thus given by admissible weights). (On $\mathbf{R}$ all p-admissible measures are absolutely continuous, and even A_p weights; see Björn–Buckley–Keith [6].) We therefore formulate our next result in terms of p-admissible measures.

Theorem 3. *Let μ_1 and μ_2 be p-admissible measures on $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$, respectively. Then the product measure $\mu = \mu_1 \times \mu_2$ is p-admissible on $\mathbf{R}^{n_1+n_2}$.*

For a function u on an open subset $\Omega \subset \mathbf{R}^{n_1+n_2}$ we will denote the gradient by ∇u. The gradients with respect to the first n_1, resp., the last n_2 variables will be denoted by $\nabla_x u$ and $\nabla_y u$. In this section we will only consider gradients of locally Lipschitz functions, which are thus defined a.e. and coincide with the Sobolev gradients determined by the admissible measures; see Heinonen–Kilpeläinen–Martio [11, Lemma 1.11].

Proof. Let $z = (z_1, z_2) \in \mathbf{R}^{n_1+n_2}$ and $r > 0$. We denote balls in $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, and $\mathbf{R}^{n_1+n_2}$, by B', B'', and B, respectively. Let

$$Q(z,r) = B'(z_1, r) \times B''(z_2, r)$$

and note that

$$B(z,r) \subset Q(z,r) \subset B(z, \sqrt{2}r). \tag{2}$$

It follows that for $B = B(z,r)$ we have

$$\begin{aligned} \mu(2B) &\leq \mu(Q(z,2r)) = \mu_1(B'(z_1,2r))\mu_2(B''(z_2,2r)) \\ &\leq C\mu_1\bigl(B'\bigl(z_1, \tfrac{1}{2}r\bigr)\bigr)\mu_2\bigl(B''\bigl(z_2, \tfrac{1}{2}r\bigr)\bigr) = C\mu\bigl(Q\bigl(z, \tfrac{1}{2}r\bigr)\bigr) \leq C\mu(B), \end{aligned}$$

and hence μ is doubling. Here and below, the letter C denotes various positive constants whose values may vary even within a line.

We now turn to the Poincaré inequality. As mentioned above we can assume that the p-Poincaré inequalities for μ_1 and μ_2 hold with dilation $\lambda = 1$. Let $B = B(z,r)$ and $Q = Q(z,r) = B' \times B''$. Also let u be an arbitrary bounded locally Lipschitz function on $2B$ and set

$$c = \fint_Q u \, d\mu = \fint_{B''} \fint_{B'} u(s,t) \, d\mu_1(s) \, d\mu_2(t).$$

Then by the Fubini theorem,

(3)

$$\begin{aligned}\fint_Q |u-c|\,d\mu &\le \fint_{B''}\biggl(\fint_{B'}\biggl|u(x,y)-\fint_{B'}u(s,y)\,d\mu_1(s)\biggr|\,d\mu_1(x)\biggr)\,d\mu_2(y)\\ &\quad+\fint_{B''}\biggl|\fint_{B'}u(s,y)\,d\mu_1(s)-\fint_{B'}\fint_{B''}u(s,t)\,d\mu_2(t)\,d\mu_1(s)\biggr|\,d\mu_2(y)\\ &=: I_1+I_2.\end{aligned}$$

The first integral I_1 can be estimated using the p-Poincaré inequality for μ_1 and $u(\,\cdot\,,y)$ on B', and then the Hölder inequality with respect to μ_2, as follows:

$$\begin{aligned}I_1 &\le \fint_{B''} Cr\biggl(\fint_{B'}|\nabla_x u(x,y)|^p\,d\mu_1(x)\biggr)^{1/p}\,d\mu_2(y)\\ &\le Cr\biggl(\fint_{B''}\fint_{B'}|\nabla_x u(x,y)|^p\,d\mu_1(x)\,d\mu_2(y)\biggr)^{1/p} \le Cr\biggl(\fint_Q|\nabla u|^p\,d\mu\biggr)^{1/p}.\end{aligned}$$

As for the second integral I_2 in (3) we have by the Fubini theorem,

$$\begin{aligned}I_2 &\le \fint_{B''}\fint_{B'}\biggl|u(s,y)-\fint_{B''}u(s,t)\,d\mu_2(t)\biggr|\,d\mu_1(s)\,d\mu_2(y)\\ &= \fint_{B'}\fint_{B''}\biggl|u(s,y)-\fint_{B''}u(s,t)\,d\mu_2(t)\biggr|\,d\mu_2(y)\,d\mu_1(s),\end{aligned}$$

which can be estimated in the same way as I_1, by switching the roles of the variables. Thus

$$I_2 \le Cr\biggl(\fint_Q|\nabla u|^p\,d\mu\biggr)^{1/p}.$$

Summing the estimates for I_1 and I_2 and using the doubling property for μ we see that

$$\fint_B|u-c|\,d\mu \le C\fint_Q|u-c|\,d\mu \le Cr\biggl(\fint_Q|\nabla u|^p\,d\mu\biggr)^{1/p} \le Cr\biggl(\fint_{2B}|\nabla u|^p\,d\mu\biggr)^{1/p}.$$

Finally, a standard argument allows us to replace c on the left-hand side by u_B at the cost of an extra factor 2 on the right-hand side; cf. [3, Lemma 4.17]. We conclude that μ supports a p-Poincaré inequality on $\mathbf{R}^{n_1+n_2}$, and thus that μ is p-admissible. □

Remark 4. The proof of Theorem 3 easily generalizes to metric spaces. More precisely, if (X_j,d_j), $j=1,2$, are (not necessarily complete) metric spaces equipped with doubling measures μ_j supporting p-Poincaré inequalities with dilation constant λ, then $X=X_1\times X_2$, equipped with the product measure $\mu=\mu_1\times\mu_2$, supports a p-Poincaré inequality with dilation constant 2λ and μ is a doubling measure. See, e.g., Björn–Björn [3] for the precise definitions of these notions in metric spaces.

Poincaré inequalities in metric spaces are defined using so-called upper gradients, and the main property needed for the proof of Theorem 3 in the metric setting is that whenever $g(\,\cdot\,,\cdot\,)$ is an upper gradient of $u(\,\cdot\,,\cdot\,)$ in X and $y\in X_2$, then $g(\,\cdot\,,y)$ is an upper gradient of $u(\,\cdot\,,y)$ with respect to X_1, and similarly for $g(x,\cdot\,)$ and $u(x,\cdot\,)$ with $x\in X_1$. For this to hold, the metric on $X_1\times X_2$ can actually be defined using

$$d((x_1,y_1),(x_2,y_2)) = \|(d_1(x_1,x_2),d_2(y_1,y_2))\|$$

with an arbitrary norm $\|\cdot\|$ on $\mathbf{R}^2$. In this generality we cannot assume that $\lambda=1$, and therefore λ also needs to be inserted at suitable places in the proof. (If the norm does not satisfy $\|(x,0)\|\le\|(x,y)\|$ and $\|(0,y)\|\le\|(x,y)\|$, then the inclusions (2) need to be modified, necessitating similar changes also later in the proof.) We refrain from this generalization in this note. Also Theorem 5 below can be similarly generalized to metric spaces.

We conclude this section by showing that Theorem 3 admits a converse.

Theorem 5. *Assume that $\mu=\mu_1\times\mu_2$ is a p-admissible measure on $\mathbf{R}^{n_1+n_2}$. Then μ_1 and μ_2 are p-admissible measures on $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$, respectively.*

Proof. It suffices to show the p-admissibility of μ_1. Let $B'=(z',r)\subset\mathbf{R}^{n_1}$ be a ball and let $B'':=B(0,r)\subset\mathbf{R}^{n_2}$. Let u be an arbitrary bounded locally Lipschitz function on B' and for $(x,y)\in B'\times B''$ define $v(x,y)=u(x)$. Then

$$v_{B'\times B''}=\fint_{B'}\fint_{B''}v(x,y)\,d\mu_1(x)\,d\mu_2(y)=u_{B'}.$$

Note that for $z=(z',0)\in\mathbf{R}^{n_1+n_2}$,

$$B(z,r)\subset B'\times B''\subset B(z,\sqrt{2}r)=:\widehat{B}\subset 2B'\times 2B''\subset B(z,2\sqrt{2}r). \tag{4}$$

It then follows from the doubling property of μ that

$$\mu_1(2B')\mu_2(2B'')\le\mu(B(z,2\sqrt{2}r))\le C\mu(B(z,r))\le C\mu_1(B')\mu_2(B'')$$

and division by $\mu_2(2B'')\ge\mu_2(B'')$ yields $\mu_1(2B')\le C\mu_1(B')$, i.e., μ_1 is doubling.

As for the Poincaré inequality, we have by (4), the doubling property of μ and [3, Lemma 4.17] that

$$\begin{aligned}\fint_{B'}|u-u_{B'}|\,d\mu_1&=\fint_{B'\times B''}|v-v_{B'\times B''}|\,d\mu\le 2\fint_{B'\times B''}|v-v_{\widehat{B}}|\,d\mu\\&\le C\fint_{\widehat{B}}|v-v_{\widehat{B}}|\,d\mu.\end{aligned}$$

The last integral is estimated using the p-Poincaré inequality for μ and the fact that $\nabla v(x,y)=\nabla u(x)$ as follows:

$$\begin{aligned}\fint_{\widehat{B}}|v-v_{\widehat{B}}|\,d\mu&\le Cr\biggl(\fint_{\widehat{B}}|\nabla v|^p\,d\mu\biggr)^{1/p}\le Cr\biggl(\fint_{2B'\times 2B''}|\nabla v|^p\,d\mu\biggr)^{1/p}\\&\le Cr\biggl(\fint_{2B'}|\nabla u|^p\,d\mu_1\biggr)^{1/p}.\end{aligned}$$ □

3. Tensor products and sums of quasiminimizers

Throughout this section, $1<p<\infty$ and $\mathbf{R}^{n_j}$ is equipped with a p-admissible weight w_j, $j=1,2$. It follows from Theorem 3 that $w=w_1\otimes w_2$ is p-admissible on $\mathbf{R}^{n_1+n_2}$. We let $d\mu_j=w_j\,dx$, $j=1,2$, and $d\mu=w\,dx$.

Our aim is to prove Theorem 1. We will also obtain similar results for quasisuperminimizers, which we now define. Let $\Omega\subset\mathbf{R}^n$ be an open set. By $\operatorname{Lip}_0(\Omega)$ we denote the space of boundedly supported Lipschitz functions vanishing outside Ω.

Definition 6. A function $u : \Omega \to [-\infty, \infty]$ is a *Q-quasi(sub/super)minimizer* with respect to a p-admissible weight w in a nonempty open set $\Omega \subset \mathbf{R}^n$ if $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mu)$ and

$$\int_{\varphi \neq 0} |\nabla u|^p \, d\mu \leq Q \int_{\varphi \neq 0} |\nabla (u + \varphi)|^p \, d\mu$$

for all (nonpositive/nonnegative) $\varphi \in \mathrm{Lip}_0(\Omega)$.

By splitting φ into its positive and negative parts, it is easily seen that a function is a Q-quasiminimizer if and only if it is both a Q-quasisubminimizer and a Q-quasi-superminimizer.

The Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega; \mu)$ is defined as in Heinonen–Kilpeläinen–Martio [11] (although they use the letter H instead of W). See [11, Section 1.9] and [3, Proposition A.17] for the definition of the gradient ∇u for $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mu)$, which need not be the distributional gradient of u.

Definition 6 is one of several equivalent definitions of quasi(sub/super)minimizers; see Björn [1, Proposition 3.2], where this was shown on metric spaces. It follows from Propositions A.11 and A.17 in [3] that the metric space definitions coincide with the usual ones on weighted $\mathbf{R}^n$ (with a p-admissible weight).

For quasisuperminimizers, an analogue of Theorem 1 takes the following form.

Theorem 7. *Let u_j be a Q_j-quasisuperminimizer in $\Omega_j \subset \mathbf{R}^{n_j}$ with respect to p-admissible weights w_j, $j = 1, 2$, and Q be given by* (1). *Then $u_1 \oplus u_2$ is a Q-quasisuperminimizer in $\Omega = \Omega_1 \times \Omega_2$ with respect to $w = w_1 \otimes w_2$.*

In addition, if both u_1 and u_2 are nonnegative/nonpositive, then $u_1 \otimes u_2$ is a Q-quasisuper/subminimizer in Ω with respect to w.

By considering $-u_1$ and $-u_2$, we easily obtain a corresponding result for quasisubminimizers. Usually, $Q_j \geq 1$ but we also allow for $Q_j = 0$. This can only happen when u_j is constant (a.e. in each component of Ω_j), but when this is fulfilled in Theorem 1 or 7 it immediately implies the following conclusion.

Corollary 8. *If u is a Q-quasi(super)minimizer in $\Omega \subset \mathbf{R}^{n_1}$ with respect to a p-admissible weight w_1, and we let $v(x, y) = u(x)$ for $(x, y) \in \Omega \times \mathbf{R}^{n_2}$, then v is a Q-quasi(super)minimizer in $\Omega \times \mathbf{R}^{n_2}$ with respect to $w = w_1 \otimes w_2$, whenever w_2 is a p-admissible weight on $\mathbf{R}^{n_2}$.*

Proof. As $v = u \oplus \mathbf{0}$, where $\mathbf{0}$ is the zero function, this follows directly from Theorems 1 and 7. □

Proof of Theorem 1. Since u_1 and u_2 are finite a.e., and the quasiminimizing property is the same for all representatives of an equivalence class in the local Sobolev space, we may assume that u_1 and u_2 are finite everywhere.

First, we show that $u := u_1 \otimes u_2$ is a Q-quasiminimizer. Note that

$$|\nabla u(x, y)|^p = (|\nabla_x u(x, y)|^2 + |\nabla_y u(x, y)|^2)^{p/2},$$

where $\nabla_x u(x, y) = u_2(y) \nabla u_1(x)$ and $\nabla_y u(x, y) = u_1(x) \nabla u_2(y)$.

Let $\varphi \in \mathrm{Lip}_0(\Omega)$ be arbitrary. For a fixed $y \in \Omega_2$, let

$$\Omega_1^y = \{x \in \Omega_1 : \varphi(x, y) \neq 0\}.$$

As u_1 is a Q_1-quasiminimizer in Ω_1, so is $u(\,\cdot\,, y) = u_2(y)u_1(\,\cdot\,)$. Since $\varphi(\,\cdot\,, y) \in \operatorname{Lip}_0(\Omega_1^y)$, we get

$$\int_{\Omega_1^y} |\nabla_x u(x,y)|^p \, d\mu_1(x) \le Q_1 \int_{\Omega_1^y} |\nabla_x (u(x,y) + \varphi(x,y))|^p \, d\mu_1(x).$$

Integrating over all $y \in \Omega_2$ with nonempty Ω_1^y yields

$$\int_{\varphi\neq 0} |\nabla_x u|^p \, d\mu \le Q_1 \int_{\varphi\neq 0} |\nabla_x (u+\varphi)|^p \, d\mu. \tag{5}$$

Similarly,

$$\int_{\varphi\neq 0} |\nabla_y u|^p \, d\mu \le Q_2 \int_{\varphi\neq 0} |\nabla_y (u+\varphi)|^p \, d\mu. \tag{6}$$

Now we consider four cases.

Case 1 ($Q_1 = 0$). In this case, $\nabla u_1 \equiv 0$ a.e., and so $\nabla_x u \equiv 0$ a.e. Hence, by (6),

$$\begin{aligned}\int_{\varphi\neq 0} |\nabla u|^p \, d\mu = \int_{\varphi\neq 0} |\nabla_y u|^p \, d\mu &\le Q_2 \int_{\varphi\neq 0} |\nabla_y (u+\varphi)|^p \, d\mu \\ &\le Q_2 \int_{\varphi\neq 0} |\nabla (u+\varphi)|^p \, d\mu,\end{aligned}$$

and thus u is a Q_2-quasiminimizer.

Case 2 ($Q_2 = 0$). This is similar to Case 1.

Case 3 ($p \le 2$). In this case, summing (5) and (6) gives

$$\begin{aligned}\int_{\varphi\neq 0} |\nabla u|^p \, d\mu &\le \int_{\varphi\neq 0} (|\nabla_x u|^p + |\nabla_y u|^p) \, d\mu \\ &\le \int_{\varphi\neq 0} (Q_1 |\nabla_x (u+\varphi)|^p + Q_2 |\nabla_y (u+\varphi)|^p) \, d\mu.\end{aligned}$$

This proves the result for $p = 2$. For $p < 2$, the Hölder inequality applied to the sum $Q_1 a^p + Q_2 b^p$ in the last integrand shows that

$$\begin{aligned}\int_{\varphi\neq 0} |\nabla u|^p \, d\mu &\le \Big(Q_1^{2/(2-p)} + Q_2^{2/(2-p)} \Big)^{1-p/2} \\ &\qquad \times \int_{\varphi\neq 0} (|\nabla_x (u+\varphi)|^2 + |\nabla_y (u+\varphi)|^2)^{p/2} \, d\mu \\ &= \Big(Q_1^{2/(2-p)} + Q_2^{2/(2-p)} \Big)^{1-p/2} \int_{\varphi\neq 0} |\nabla (u+\varphi)|^p \, d\mu.\end{aligned}$$

Case 4 ($p \ge 2$ and $Q_1, Q_2 > 0$). Rewrite $|\nabla u|^p$ as

$$|\nabla u|^p = (|\nabla_x u|^2 + |\nabla_y u|^2)^{p/2} = \left(Q_1^{2/p} \left(\frac{1}{Q_1} \right)^{2/p} |\nabla_x u|^2 + Q_2^{2/p} \left(\frac{1}{Q_2} \right)^{2/p} |\nabla_y u|^2 \right)^{p/2}.$$

The Hölder inequality applied to the sum $Q_1^{2/p} a^2 + Q_2^{2/p} b^2$ implies

$$|\nabla u|^p \le \Big(Q_1^{2/(p-2)} + Q_2^{2/(p-2)} \Big)^{(p-2)/2} \left(\frac{1}{Q_1} |\nabla_x u|^p + \frac{1}{Q_2} |\nabla_y u|^p \right).$$

Integrating over the set $\{(x,y)\in\Omega:\varphi(x,y)\neq 0\}$ and using (5) and (6) we obtain

$$\int_{\varphi\neq 0}|\nabla u|^p\,d\mu \le \Big(Q_1^{2/(p-2)}+Q_2^{2/(p-2)}\Big)^{(p-2)/2} \times \int_{\varphi\neq 0}(|\nabla_x(u+\varphi)|^p+|\nabla_y(u+\varphi)|^p)\,d\mu.$$

As $p/2\ge 1$, the elementary inequality $a^p+b^p\le(a^2+b^2)^{p/2}$ concludes the proof for u.

We now turn to $v:=u_1\oplus u_2$. Let $\varphi\in \operatorname{Lip}_0(\Omega)$ be arbitrary. Note that

$$|\nabla v(x,y)|^p=(|\nabla_x v(x,y)|^2+|\nabla_y v(x,y)|^2)^{p/2}=(|\nabla u_1(x)|^2+|\nabla u_2(y)|^2)^{p/2}$$

and

$$|\nabla (v+\varphi)|^p=(|\nabla_x (v+\varphi)|^2+|\nabla_y (v+\varphi)|^2)^{p/2}.$$

For a fixed $y\in\Omega_2$, let

$$\Omega_1^y=\{x\in\Omega_1:\varphi(x,y)\neq 0\}.$$

As u_1 is a Q_1-quasiminimizer in Ω_1 and $\varphi(\,\cdot\,,y)\in\operatorname{Lip}_0(\Omega_1^y)$, we get

$$\int_{\Omega_1^y}|\nabla u_1(x)|^p\,d\mu_1(x)\le Q_1\int_{\Omega_1^y}|\nabla_x(u_1(x,y)+\varphi(x,y))|^p\,d\mu_1(x).$$

Integrating over all $y\in\Omega_2$ with nonempty Ω_1^y yields

$$\int_{\varphi\neq 0}|\nabla u_1|^p\,d\mu_1(x)\,d\mu_2(y)\le Q_1\int_{\varphi\neq 0}|\nabla_x(v+\varphi)|^p\,d\mu_1(x)\,d\mu_2(y),$$

i.e., (5) holds. Similarly, (6) holds and the rest of the proof is as for u. □

Proof of Theorem 7. This proof is very similar to the proof above. In this case we of course assume that $\varphi\in\operatorname{Lip}_0(\Omega)$ is nonnegative/nonpositive.

The only other difference in the proof is that since u_1 is a Q_1-quasisuperminimizer in Ω_1 and $u_2(y)$ is nonnegative/nonpositive, we can conclude that

$$u(\,\cdot\,,y)=u_2(y)u_1(\,\cdot\,)$$

is a Q_1-quasisuper/subminimizer in Ω_1. The rest of the proof is the same; in particular the proof for v needs no nontrivial changes, and is thus valid also when u_1 and u_2 change sign. □

For tensor sums one can use Theorem 7 to deduce (the corresponding part of) Theorem 1. For tensor products this is not possible as in this case the quasisuperminimizers in Theorem 7 need to be nonnegative. This nonnegativity is an essential assumption for quasisuperminimizers, which is not required for quasiminimizers. (To see this consider what happens when $u_2\equiv -1$.) We can, however, obtain the following result.

Theorem 9. *Let u_1 be a Q_1-quasisub/superminimizer in Ω_1 and $u_2\ge 0$ be a Q_2-quasiminimizer in Ω_2, with respect to p-admissible weights w_1 and w_2, respectively.*

Then $u_1\otimes u_2$ is a Q-quasisub/superminimizer in $\Omega=\Omega_1\times\Omega_2$ with respect to $w=w_1\otimes w_2$, where Q is given by (1).

Proof. This is proved using a similar modification of the proof of Theorem 1 as we did when proving Theorem 7. The key fact is that quasiminimizers are preserved under multiplication by real numbers, while the corresponding fact for quasisub/superminimizers is only true under multiplication by nonnegative real numbers. □

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Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
Email address: anders.bjorn@liu.se

Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
Email address: jana.bjorn@liu.se