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CORRECTION TO "ON SOME SUBSPACES OF BANACH SPACES WHOSE DUALS ARE L_1 SPACES"

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Abstract. E. Casini, E. Miglierina, L. Piasecki, and L. Veselý have recently constructed an example of an L_1-predual hyperplane W of c which does not contain a subspace isometric to c, in spite of the fact that the closed unit ball of W contains an extreme point. This example shows that Remark A of Section 4 of [Proc. Amer. Math. Soc. 23 (1969), pp. 378-385], titled as above, is false. The purpose of this note is to present two correct versions of that Remark A and a short proof of our 1969 main result.

1. Introduction

Remark A of Section 4 of [Z] has been proved to be false in [CMPV]. We apologize for the mistake and present here two correct versions of Remark A. We do that by using only the information which was known in 1969, when [Z] was published. With one exception, all tools used below can be found in [Z]. This exception (the consequence of (2.2) below) is an observation which enables us to considerably simplify the (correct, but complicated) proof of our main Theorem 1 of [Z]. We do this in Remark 3 at the end of this paper.

We say that the Banach spaces U and V are $(1+\varepsilon)$ isomorphic if there is a one-to-one surjective operator $T : U \to V$ satisfying, for every $u \in U$, the inequality $(1+\varepsilon)^{-1}\|u\| \le \|T(u)\| \le (1+\varepsilon)\|u\|$. A subspace U of V is said to be $(1+\varepsilon)$-complemented if there is a projection P from V onto U with $\|P\| \le (1+\varepsilon)$. Let A be a subset of a space X; then $[A]$ denotes the closed linear span of A in X; $B(X)$ is the closed unit ball of X.

Note that there are several misprints in [Z]. (1). In the bottom line of page 379, u_i^{m+i} should be replaced by u_i^{m+1}. (2). In the top line of page 380, $1 \le i \le n$ should be replaced by $1 \le i \le m$.

If X is a separable space and X^* is an $L_1(\mu)$ space, then, as is well known (see, e.g., [LL1]), there exists a normalized monotone basis $\{x_i\}_{i=1}^\infty$ of X such that, if $X_n = [\{x_i\}_{i=1}^n]$, then, for every $n \ge 1$, X_n is isometric to ℓ_∞^n and contains a basis $\{e_i^n\}_{i=1}^n$ with $e_n^n = x_n$ which satisfies the following two conditions:

$$\left\| \sum_{i=1}^n c_i e_i^n \right\| = \max_{1\le i\le n} |c_i| \text{ for all } \{c_i\}_{i=1}^n \subset R \tag{1.1}$$

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and

$$\text{for every } 1 \le i \le n, e_i^n = e_i^{n+1} + a_i^n e_{n+1}^{n+1} \text{ where } \sum_{i=1}^n |a_i^n| \le 1. \tag{1.2}$$

It follows from (1.2) that the functional ϕ_j defined on $\bigcup_{n=1}^\infty X_n$ by

$$\phi_j(\sum_{i=1}^n c_i e_i^n) = c_j \quad \text{for all} \quad n \ge j \quad \text{and} \quad \{c_i\}_{i=1}^n \subset R \tag{1.3}$$

is a well-defined linear functional that can be uniquely extended by continuity to a linear functional on $X = [\bigcup_{n=1}^\infty X_n]$ which, as is proved in Lemma 1 of [Z], is an extreme point of $B(X^*)$. Clearly $\{\phi_j\}_{j=1}^\infty$ is isometrically equivalent to the standard basis of ℓ_1.

Remark 1. It follows from [MP] that the first element x_1 of the monotone basis can be chosen to be an arbitrary normalized vector of X.

The following result is a special case ($i(k) = n(k)$) of Lemma 2 of [Z]. It is a tool to construct subspaces of X which are also L_1 preduals. Because the proof of this special case is much simpler than that of Lemma 2 of [Z], we will present it here for completeness.

Lemma 1. *Let $\{n(k)\}_{k=1}^\infty$ be a strictly increasing subsequence of the integers with $n(1) = 1$. Then the sequence $\left\{e_{n(k)}^{n(k)}\right\}_{k=1}^\infty$ forms a monotone basis for the subspace U it spans in X. Also, in each subspace $U_m = \left[\left\{e_{n(k)}^{n(k)}\right\}_{k=1}^m\right]$ there is a basis $\{u_i^m\}_{i=1}^m$ satisfying the following three conditions:*

$$u_m^m = e_{n(m)}^{n(m)}, \tag{1.4}$$

$$\text{for } 1 \le i \le m, u_i^m = u_i^{m+1} + b_i^m u_{m+1}^{m+1} \text{ where} \sum_{i=1}^m |b_i^m| \le 1 \tag{1.5}$$

and

$$\left\| \sum_{i=1}^m c_i u_i^m \right\| = \max_{1 \le i \le m} |c_i| \text{ for any } \{c_i\}_{i=1}^m \subset R. \tag{1.6}$$

Finally, let $\psi_j = \phi_{n(j)}\big|_U$; then $\psi_j(u_i^m) = \delta_{i,j}$ for all $1 \le i, j \le m$ and, for every $u \in U$, we have

$$\|u\| = \sup_{j \ge 1} |\psi_j(u)| = \sup_{j \ge 1} \left|\phi_{n(j)}(u)\right|. \tag{1.7}$$

Proof. Define $u_k^k = e_{n(k)}^{n(k)}$ for all $k \ge 1$; then $\{u_k^k\}_{k=1}^\infty$ is a monotone basis for its closed linear span U because it is a subsequence of a monotone basis $\{e_n^n\}_{n=1}^\infty$. We now define u_h^k for $1 \le h \le k-1$ by induction on k: If $u_h^{k-1} = \sum_{j=1}^{n(k)} b_{h,j}^k e_j^{n(k)}$, then $u_h^k = \sum_{j=1}^{n(k)-1} b_{h,j}^k e_j^{n(k)}$. It follows that $u_h^{k-1} = u_h^k + b_h^{k-1} u_k^k$ where $b_h^{k-1} = b_{h,n(k)}^k$. We will show, by induction on k, that, for every choice of signs, $\left\|\sum_{h=1}^k \pm u_h^k\right\| = 1$. Indeed, $\left\|\pm u_1^1\right\| = 1$ and, assuming that $\left\|\sum_{h=1}^{k-1} \pm u_h^{k-1}\right\| = 1$, by (1.1) we get that, for every $1 \le j \le n(k)$, $\sum_{h=1}^{k-1} \left|b_{h,j}^k\right| \le 1$. It follows that $\left\|\sum_{h=1}^{k-1} \pm u_h^k\right\| \le 1$ and, since each u_h^k with $1 \le h \le k-1$ is supported on $\left\{e_j^{n(k)}\right\}_{j=1}^{n(k)-1}$ while $u_k^k = e_{n(k)}^{n(k)}$, we have $\left\|\sum_{h=1}^k \pm u_h^k\right\| = 1$. This property clearly implies (1.5) because $\sum_{j=1}^k \left|b_j^k\right| = \sum_{j=1}^k \left|\phi_{n(k+1)}(u_j^k)\right| \le \left\|\sum_{j=1}^k \pm u_j^k\right\| \le 1$ for some choice of the signs. We will now prove by induction on k that $\phi_{n(h)}(u_j^k) = \delta_{h,j}$ for all $1 \le h,j \le k$. By (1.3), $\phi_{n(1)}(u_1^1) = \phi_{n(1)}(e_{n(1)}^{n(1)}) = 1$.

Assume the assertion for $k-1$ and proceed by induction. Since $\phi_{n(h)}(e_{n(k)}^{n(k)}) = 0$ for all $1 \le h \le k-1$ we get that $\phi_{n(h)}(u_j^k) = \phi_{n(h)}(u_j^{k-1} - b_j^{k-1}e_{n(k)}^{n(k)}) = \phi_{n(h)}(u_j^{k-1}) = \delta_{h,j}$ for all $1 \le j,h \le k-1$. By definition, $\phi_{n(k)}(u_j^k) = 0$ if $1 \le j \le k-1$ while $\phi_{n(k)}(u_k^k) = 1$. This proves the assertion $\phi_{n(h)}(u_j^k) = \delta_{h,j}$ for all $k \ge 1$ and $1 \le h,j \le k$. It follows that, for every $k \ge 1$ and real $\{c_j\}_{j=1}^k$, $\max\limits_{1\le i\le k}|c_i| = \max\limits_{1\le i\le k}\left|\phi_{n(i)}(\sum_{j=1}^k c_j u_j^k)\right| \le \left\|\sum_{j=1}^k c_j u_j^k\right\| \le (\max\limits_{1\le i\le k}|c_i|)\left\|\sum_{j=1}^k \pm u_j^k\right\| = \max\limits_{1\le i\le k}|c_i|$ for some choice of signs. This settles (1.6) and shows that (1.7) holds for every $u \in \bigcup_{i=1}^\infty U_i$. Since $\bigcup_{i=1}^\infty U_i$ is dense in U the proof of Lemma 1 is complete. $\square$

Let c denote the space of convergent sequences $s = (s(1), s(2), s(3), \ldots)$ with

$$\|s\| = \sup_{j\ge 1}|s(j)|.$$

We use the usual notation with respect to the standard bases of c and c^*: we let $w_0 = (1,1,1,\ldots)$ and, for $i \ge 1$, we define $w_i \in c$ by $w_i = (w_i(1), w_i(2), w_i(3), \ldots)$ where $w_i(j) = \delta_{i,j}$. We define the functionals $\{\varphi_i\}_{i=0}^\infty \subset c^*$ by

$$\varphi_0(s) = \lim_{j\to\infty} s(j)$$

and $\varphi_i(s) = s(i)$ if $i \ge 1$ for every $s = (s(1), s(2), s(3), \ldots)$. We are now ready to state the following two versions of Remark A of [Z].

Theorem 1. *Let X be a separable $L_1(\mu)$ predual space. Then there exist a subspace $U \subset X$ and an isometry $T : U \to c$ such that the range $Y = T(U)$ is either the whole space c or a closed hyperplane of c. Moreover, if $B(X)$ has an extreme point, then Y contains the element $(1,1,1,\ldots)$ of c.*

Theorem 2. *Let X be a separable $L_1(\mu)$ predual space and assume that $B(X)$ has an extreme point. Then, for every $\varepsilon > 0$, X contains a $(1+\varepsilon)$-complemented subspace Z which is $(1+\varepsilon)$ isomorphic to c.*

Note that Theorem 1 is "linearly close" while Theorem 2 is "norm close" to Remark A of [Z].

2. Proofs of the theorems

Proof of Theorem 1. We use the structure of X described in the introduction. Since $B(X^*)$ is w^* compact we can select a subsequence $\left\{\phi_{n(k)}\right\}_{k=1}^\infty$ with $n(1) = 1$ which w^*-converges to a functional $\phi \in B(X^*)$. Then we construct the subspace U and the functionals $\{\psi_j\}_{j=1}^\infty$ as in Lemma 1. Because $\psi_j = \phi_{n(j)}\big|_U$, we get that

$$\lim_{j\to\infty}\psi_j(u) = \phi(u)$$

for every $u \in U$. Put $\psi_0 = \phi|_U$and assume, first, that $\psi_0 \neq 0$. Use (1.7) to obtain

$$\|u\| = \max_{j\geq 0} |\psi_j(u)| \text{ for every } u \in U. \tag{2.1}$$

It follows that the operator $T : U \to c$ defined by $T(u)=(\psi_1(u), \psi_2(u), \psi_3(u), \dots)$ is an isometric isomorphism from U onto a subspace Y of c. If this isometry is not onto c, then there is a functional $\varphi \in c^*$ which annihilates $T(U) = Y$. Using the notation introduced in the introduction, we get a representation $\varphi = \sum_{i=0}^{\infty} a_i\varphi_i$ and, since for every $u \in U$, $T^*\varphi_j(u) = \varphi_j(Tu) = \varphi_j(\psi_1(u), \psi_2(u), \psi_3(u), \dots) = \psi_j(u)$, we get that $T^*(\varphi_j) = \psi_j$ for all $j \geq 0$ hence $T^*(\varphi) = \sum_{j=0}^{\infty} a_j\psi_j$. By using the structure of U we will show that, up to a numerical multiple, the functional φ is unique. Indeed, by Lemma 1, we know that, for every $m \geq 1$ and $1 \leq i \leq m$, $\psi_i(u_j^m) = \delta_{i,j}$. Therefore,

$$0 = T^*(\varphi)(u_j^m) = \sum_{i=0}^{\infty} a_i\psi_i(u_j^m) = a_0\psi_0(u_j^m) + a_j + \sum_{i=m+1}^{\infty} a_i\psi_i(u_j^m). \tag{2.2}$$

Since $\sum_{i=0}^{\infty} |a_i|$ converges, we get from (2.2) that

$$\left|a_0\psi_0(u_j^m) + a_j\right| = \left|\sum_{j=m+1}^{\infty} a_i\psi_i(u_j^m)\right| \underset{m\to\infty}{\to} 0.$$

Therefore, if $a_0 \neq 0$, then $\lim_{m\to\infty} \psi_0(u_j^m)$ exists and $a_0^{-1}a_j = -\lim_{m\to\infty} \psi_0(u_j^m)$ for all $j \geq 1$. It follows that all annihilators φ of $T(U)$ with $a_0 \neq 0$ are numerical multiples of each other hence $T(U)$ is a hyperplane of c. On the other hand, if $a_0 = 0$ for every φ annihilating Y, then, by (2.2), $a_j = 0$ for all $j \geq 1$, hence $\varphi = 0$ and thus $T(U) = c$. Suppose that $\psi_0 = 0$; then T, defined above, becomes an operator $T : U \to c_0$. If there exists a fuctional $\varphi \in c_0^*$ which annihilates $T(U)$, then $\varphi = \sum_{i=1}^{\infty} a_i\varphi_i$. It follows from (2.2) that $a_i = 0$ for all $i \geq 1$ hence $\varphi = 0$ and T is an isometry from U onto c_0.

It remains to consider the case where $B(X)$ has an extreme point. In this case, by [S], X is isometric to a space $A(S)$ of affine continuous functions on a Choquet simplex S. It is proved in Theorem 5.2 of [LL2] that, under these circumstances, the space X has a structure $X = [\bigcup_{n=1}^{\infty} X_n]$ of finite dimensional subspaces $X_n = \ell_\infty^n$, each with a basis $\{e_i^n\}_{i=1}^n$ satisfying (1.1), (1.2), and the following additional property:

$$a_i^n \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} a_i^n = 1 \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad 1 \leq i \leq n. \tag{2.3}$$

It follows that $e_1^1 = \sum_{i=1}^n e_i^n$ for all $n \geq 1$ and, by (1.3), $\phi_i(e_1^1) = 1$ for all $i \geq 1$. Since $\psi_j = \phi_{n(j)}\big|_U$ and $n(1) = 1$, we get that $\psi_j(u_1^1) = \phi_{n(j)}(e_1^1) = 1$ for every $j \geq 1$ and therefore, by the definition of the isometry T, $T(u_1^1) = (1, 1, 1, \dots) \in Y$. This proves Theorem 1. □

Remark 2. This consequence of [LL2] that we use above was discussed in J. Lindenstrauss' Functional Analysis Seminar at The Hebrew University of Jerusalem in 1967. Actually, it can be deduced from Remark 1 above. Indeed, if we choose $x_1 = e_1^1$ to be the unit function of $A(S)$, then, by multiplying x_n by -1, if necessary, we can build, for every n, the basis $\{e_i^n\}_{i=1}^n$ to satisfy (2.3).

Proof of Theorem 2. We use Theorem 1 and the notation as above. Let $\varphi = \sum_{i=0}^{\infty} a_i \varphi_i$ be a nontrivial functional which annihilates Y. Since $\lim\limits_{i\to\infty} a_i = 0$, given $0 < \delta < \frac{1}{2}(1+\varepsilon)^{-1}\varepsilon$ and an integer $h > 0$, there is a $k > h$ and a number d with $|d| < \delta$ such that $\varphi(w_k - dw_h) = a_k - da_h = 0$ and, consequently, $w_k - dw_h \in Y$. Therefore, there exist an infinite sequence $\{p(j)\}_{j=1}^{\infty}$ of positive integers and numbers $\{d_j\}_{j=1}^{\infty}$ with $\max\limits_{j\geq 1} |d_j| < \delta$ such that, for all $j \geq 1$, $w_{p(2j)} + d_j w_{p(2j-1)} \in Y$. Put $p(0) = p(-1) = d_0 = 0$ and let $V = \left[\left\{w_{p(2j)} + d_j w_{p(2j-1)}\right\}_{j=0}^{\infty}\right]$. Clearly, the subspace $W = \left[\left\{w_{p(2j)}\right\}_{j=0}^{\infty}\right]$ is isometric to the whole space c. Let $v_j = w_{p(2j)} + d_j w_{p(2j-1)}$; then $v_0 = w_0$ and, for any null sequence $\{b_j\}_{j=0}^{\infty}$ of real numbers,

$$\left\|\sum_{j=0}^{\infty} b_j (v_j - w_{p(2j)})\right\| = \left\|\sum_{j=1}^{\infty} b_j d_j w_{p(2j-1)}\right\| = \max_{j\geq 1} |b_j d_j| \leq \delta \max_{j\geq 1} |b_j|$$

$$\leq 2\delta \max\left\{|b_0|, \max\left\{|b_0 + b_j| : j \geq 1\right\}\right\} = 2\delta \left\|\sum_{j=0}^{\infty} b_j w_{p(2j)}\right\|.$$

Let $S : W \to V$ be the operator defined by $S(\sum_{j=0}^{\infty} b_j w_{p(2j)}) = \sum_{j=0}^{\infty} b_j v_j$; then, as we have just proved, $\|Sw - w\| \leq 2\delta \|w\|$ for all $w \in W$ and hence V is $(1+\varepsilon)$ -isomorphic to c. Since T is an isometry, the space $Z = T^{-1}(V)$ is $(1+\varepsilon)$ -isomorphic to c. It remains to show that there is a good projection from X onto Z.

Let $z_i = T^{-1} v_i$ for all $i \geq 0$ and recall that $e_1^1 = u_1^1 = T^{-1}(w_0) = T^{-1}(v_0) = z_0$ and, for all $i \geq 1$, $\phi_{n(i)}(e_1^1) = 1$ and also $\phi(z_0) = \phi(e_1^1) = \lim\limits_{i\to\infty} \phi_{n(i)}(e_1^1) = 1$. Define the operator $P : X \to Z$ by $P(x) = \phi(x) z_0 + \sum_{i=1}^{\infty} (\phi_{n(p(2i))} - \phi)(x) z_i$. Then $P(z_0) = z_0$ and, since $\phi_{n(p(2j))}\big|_U = \psi_{p(2j)} = T^*(\varphi_{p(2j)})$, we obtain, for $j, i \geq 1$, the relation $(\phi_{n(p(2j))} - \phi)(z_i) = T^*(\varphi_{p(2j)} - \varphi_0)(z_i) = (\varphi_{p(2j)} - \varphi_0)(T(z_i)) = (\varphi_{p(2j)} - \varphi_0)(w_{p(2i)} - d_j w_{p(2i-1)}) = \delta_{i,j}$ while $\phi(z_i) = T^*(\varphi_0)(z_i) = \varphi_0(T(z_i)) = \lim\limits_{k\to\infty} \varphi_k(w_{p(2i)} - d_i w_{p(2i-1)}) = 0$ and therefore $P(z_i) = z_i$ for all $i \geq 0$. Hence P is a projection from X onto Z.

Using the fact that $T : U \to c$ is an isometry we get, for every $x \in B(X)$, the inequality

$$\begin{aligned}
\|P(x)\| = \|TP(x)\| &= \left\|\phi(x) v_0 + \sum_{j=1}^{\infty} (\phi_{n(p(2j))} - \phi)(x) v_j\right\| \\
&= \left\|\phi(x) w_0 + \sum_{j=1}^{\infty} (\phi_{n(p(2j))} - \phi)(x)(w_{p(2j)} + d_j w_{p(2j-1)})\right\| \\
&\leq \left\|\phi(x) w_0 + \sum_{j=1}^{\infty} (\phi_{n(p(2j))} - \phi)(x) w_{p(2j)}\right\| \\
&\qquad + \left\|\sum_{j=1}^{\infty} (\phi_{n(p(2j))} - \phi)(x) d_j w_{p(2j-1)}\right\|
\end{aligned}$$

$$= \max\left\{|\phi(x)|, \max\left\{\left|\phi_{n(p(2j))}(x)\right| : j \geq 1\right\}\right\} + \left\|\sum_{j=1}^{\infty}(\phi_{n(p(2j))} - \phi)(x) d_j w_{p(2j-1)}\right\|$$
$$\leq 1 + 2\max\left\{|d_j| : j \geq 1\right\} \leq 1 + 2\delta < 1 + \varepsilon.$$

It follows that $\|P\| \leq 1 + \varepsilon$ and this completes the proof of Theorem 2. □

Remark 3. Theorem 1 of [Z] states that every separable L_1 predual X contains a 1-complemented subspace Z which is isometric to the space c_0. The following is a simple proof of Theorem 1 of [Z] which is based on Theorem 1 above.

Let U be the subspace of X constructed in Theorem 1 above and let $\varphi = \sum_{i=0}^{\infty} a_i \varphi_i$ be the annihilator of $T(U)$ in c^*. For some small $\varepsilon > 0$ pick a subsequence of integers $\{p(j)\}_{j=1}^{\infty}$ and vectors $v_j = w_{p(2j)} + d_j w_{p(2j-1)}$ with $|d_j| \leq \varepsilon$ which belong to $T(U)$. Let $z_j = T^{-1}(v_j)$ and put $Z = \left[\{z_j\}_{j=1}^{\infty}\right]$. The operator T is an isometry and $\left[\{v_j\}_{j=1}^{\infty}\right]$ is isometric to c_0 hence Z is isometric to c_0. We know that $T^*(\varphi_{p(2j)}) = \psi_{p(2j)} = \phi_{n(p(2j))}\big|_U$ and $\varphi_{p(2j)}(v_i) = \delta_{i,j}$. The subspace $Z_0 = \left[\{z_{2j} - z_{2j-1}\}_{j=1}^{\infty}\right]$ is isometric to c_0 and $\omega^* \lim(\phi_{n(p(4j))} - \phi_{n(p(4j-2))}) = 0$. It follows that the operator P defined by

$$P(x) = (1/2)\sum_{j=1}^{\infty}(\phi_{n(p(4j))} - \phi_{n(p(4j-2))})(x)(z_{2j} - z_{2j-1})$$

for every $x \in X$ is a projection of norm 1 from X onto Z_0. This proves Theorem 1 of [Z].

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