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THE NUMBER OF GRIDPOINTS ON HYPERPLANE SECTIONS OF THE d-DIMENSIONAL CUBE

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In loving memory of my dear wife Anke (1967–2018)

Abstract. We deduce a formula for the exact number of gridpoints (i.e., elements of $\mathbb{Z}^d$) in the extended d-dimensional cube $nC_d = [-n, +n]^d$ on intersecting hyperplanes. In the special case of the hyperplanes $\{x \in \mathbb{R}^d \mid x_1 + \cdots + x_d = b\}$, $b \in \mathbb{Z}$, these numbers can be written as a finite sum involving products of certain binomial coefficients. Furthermore, we consider the limit as n tends to infinity which can be expressed in terms of Euler-Frobenius numbers. Finally, we state a conjecture on the asymptotic behaviour of this limit as the dimension d tends to infinity.

1. Introduction and main results

For $d \in \mathbb{N}$, let $C_d = [-1, +1]^d$ denote the d-dimensional cube in $\mathbb{R}^d$. Intersection of C_d with a hyperplane

$$H_{a,b} = \left\{x \in \mathbb{R}^d \mid a \cdot x = b\right\} \qquad \text{with } a = (a_1, \ldots, a_d) \in (\mathbb{R} \setminus \{0\})^d,\ b \in \mathbb{R}, \tag{1.1}$$

provides seemingly complicated sets $C_d \cap H_{a,b}$. Evers [7] showed that already in dimension $d = 4$ at least 30 different combinatorial types of intersection polyhedrons occur. In 1986 K. Ball [2] found the $(d-1)$-dimensional volume. After rescaling his remarkable formula reads

$$\mathrm{Vol}\left(C_d \cap H_{a,b}\right) = \frac{\|a\|}{\pi} 2^{d-1} \int_{-\infty}^{+\infty} \left(\prod_{k=1}^{d} \frac{\sin\left(a_k x\right)}{a_k x} \right) \cos\left(bx\right) dx, \tag{1.2}$$

where $\|a\| = \left(\sum_{k=1}^d a_k^2\right)^{1/2}$. The special case $a = (1, \ldots, 1)$ already appears in Pólya's 1913 paper [12]. Ball used formula (1.2) to prove the upper bound $\mathrm{Vol}(C_d \cap H_{a,b}) \leq 2^{d-1}\sqrt{2}$ which is best possible for each d. It was conjectured by Hensley [11] in 1979.

Pólya [12, pages 208–209] presented a geometric argument that the formula

$$\int_{-\infty}^{+\infty} x^{-d} \left(\prod_{k=1}^{d} \sin\left(a_k x\right) \right) dx$$

$$= \frac{\pi}{2^d (d-1)!} \sum_{\nu \in \{-1,+1\}^d} \left(\prod_{k=1}^{d} \nu_k \right) (\nu \cdot a)^{d-1} \operatorname{sgn}(\nu \cdot a). \tag{1.3}$$

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is valid. By application of elementary trigonometric identities the more general integral in equation (1.2) can be reduced to an integral of the type in equation (1.3) (see [1]). Pólya [12, page 204] remarked that the integral in (1.3) is a special case of the integral

$$\int_{-\infty}^{+\infty} e^{-px} x^{-d} \left(\prod_{k=1}^{d} \sin(a_k x) \right) dx \qquad (p>0)$$

in the integral table [6, Table 371, Nr. 5] by David Bierens de Haan, wherein the 1862 book [5, pages 344–346] is cited. D. Borwein and J. M. Borwein [3, Theorem 2 (ii)] gave a very elegant proof of equation (1.2). For further recent attempts to calculate the integral in equation (1.2) see, e.g., the work by R. Frank und H. Riede [8], [9] and the paper [1].

In this note we consider the discrete analogue of the above problem. We count the gridpoints (that are the elements of $\mathbb{Z}^d$) in the extended d-dimensional cube $nC_d = [-n, +n]^d$ on intersecting hyperplanes $H_{a,b}$. That means we consider the set $nC_d \cap H_{a,b} \cap \mathbb{Z}^d$. We denote its cardinality by

$$S_d(a,b,n) := \sharp \left(nC_d \cap H_{a,b} \cap \mathbb{Z}^d \right).$$

In the discrete case it is quite natural to assume that $a = (a_1, \ldots, a_d) \in (\mathbb{Z} \setminus \{0\})^d$, and $b \in \mathbb{Z}$.

In the next section we derive a formula for $S_d(a,b,n)$ in terms of a finite sum over all subsets of $\{1, \ldots, d\}$. The subsequent section considers the special case $\mathbf{a} = \mathbf{1} := (1, \ldots, 1) \in \mathbb{Z}^d$. The main result is a representation of $S_d(\mathbf{1}, b, n)$ as a combinatorial sum. Finally, we consider the limit

$$\sqrt{d} \lim_{n \to \infty} (2n+1)^{-(d-1)} S_d(\mathbf{1}, nx, n)$$

and close with a conjecture on its asymptotic behaviour as the dimension d tends to infinity.

2. A formula for $S_d(a,b,n)$

We use the following notation: For $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{Z}^d$ let $|\nu| := \nu_1 + \cdots + \nu_r$. Furthermore, define the function $\delta : \mathbb{R} \to \{0,1\}$ by $\delta(t) = 1$, for $t = 0$, and $\delta(t) = 0$ otherwise. Taking advantage of the equation $\int_{-\pi}^{\pi} e^{int} dt = 2\pi\delta(n)$, for $n \in \mathbb{Z}$, we obtain

$$\begin{aligned} S_d(a,b,n) &= \sum_{\nu \in nC_d} \delta(a \cdot \nu - b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\nu \in nC_d} e^{i(a \cdot \nu - b)t} \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\prod_{k=1}^{d} \frac{\sin\left(a_k \frac{2n+1}{2} t\right)}{\sin\left(a_k \frac{t}{2}\right)} \right) e^{-ibt} dt, \end{aligned}$$

where we used the formula for geometric series. Geometric evidence as well as the latter integral reveal the symmetry $S_d(a,b,n) = S_d(a,-b,n)$ for all $b \in \mathbb{Z}$. Hence, we arrive at the well-known formula

$$S_d(a,b,n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\prod_{k=1}^{d} \frac{\sin\left(a_k \frac{2n+1}{2} t\right)}{\sin\left(a_k \frac{t}{2}\right)} \right) \cos(bx)\, dt.$$

This procedure can easily be extended to the more general condition $A\nu^T = \mathbf{b}^T$, where A is an (r,d) −matrix, for a certain integer $r \leq d$, and $\mathbf{b} \in \mathbb{Z}^r$. It seems to be natural to suppose that rank$(A) = r$. The number of gridpoints in the extended d-dimensional cube $nC_d = [-n, +n]^d$ satisfying $A\nu^T = \mathbf{b}^T$, can be expressed by an r-fold integral

$$(2\pi)^{-r} \int_{[-\pi,\pi]^r} \left(\sum_{\nu \in nC_d} e^{i\left(\mathbf{t}A\nu^T\right)} \right) e^{-i(\mathbf{bt})} d\mathbf{t},$$

where $\mathbf{t} = (t_{1,\dots,}t_r)$. Application of the geometric series as above leads to the following theorem.

Theorem 2.1. *For integers r,d satisfying $1 \leq r \leq d$, let $A = (a_{jk})$ be an (r,d)-matrix and $\mathbf{b} \in \mathbb{Z}^r$. The number of gridpoints in the extended d-dimensional cube $nC_d = [-n, +n]^d$ satisfying $A\nu^T = \mathbf{b}^T$, is given by*

$$\frac{1}{(2\pi)^r} \int_{[-\pi,\pi]^r} \left(\prod_{k=1}^{d} \frac{\sin\left(\frac{2n+1}{2} \sum_{j=1}^{r} a_{jk} t_j\right)}{\sin\left(\frac{1}{2} \sum_{j=1}^{r} a_{jk} t_j\right)} \right) e^{-i(\mathbf{bt})} d\mathbf{t}.$$

Let us return to the special case of a crossing hyperplane, i.e., the case $r = 1$. The change of variable $z = e^{it}$ yields the representation as a contour integral

$$S_d(a,b,n) = \frac{1}{2\pi i} \int_{|z|=\rho} \left(\prod_{k=1}^{d} \frac{z^{a_k(2n+1)} - 1}{z^{a_k} - 1} \right) z^{-|\mathbf{a}|n-b-1} dz \tag{2.1}$$

for all positive $\rho < 1$. Another possibility, to obtain this formula is the observation that $S_d(a,b,n)$ is the coefficient of z^b in the power series expansion around $z = 0$ of the function

$$\sum_{\nu \in nC_d} z^{a\cdot\nu} = z^{-|\mathbf{a}|n} \prod_{k=1}^{d} \frac{z^{a_k(2n+1)} - 1}{z^{a_k} - 1}$$

or the coefficient of $z^{|\mathbf{a}|n+b}$ in the power series expansion of the function

$$\prod_{k=1}^{d} \frac{z^{a_k(2n+1)} - 1}{z^{a_k} - 1}.$$

By the Cauchy integral formula, both quantities are equal. With the notation $[d] := \{1, \dots, d\}$, the numerator

$$\prod_{k=1}^{d} \left(z^{a_k(2n+1)} - 1\right) = \sum_{j=0}^{d} (-1)^{d-j} \sum_{\substack{K \subseteq [d] \\ \sharp K = j}} z^{(2n+1)\sum_{k\in K} a_k}$$

has the derivatives

$$\left[\left(\frac{d}{dz}\right)^{\mu} \prod_{k=1}^{d} \left(z^{a_k(2n+1)} - 1\right) \right]_{z=0}$$
$$= \mu! \sum_{K \subseteq [d]} (-1)^{d-\sharp K} \delta\left((2n+1) \left(\sum_{k\in K} a_k \right) - \mu \right)$$

for $\mu \in \mathbb{Z}_{\geq 0}$. On the other hand, we have

$$\left[\left(\frac{d}{dz}\right)^{\mu} (z^{a_k}-1)^{-1}\right]_{z=0} = -\left[\left(\frac{d}{dz}\right)^{\mu} \sum_{j=0}^{\infty} z^{ja_k}\right]_{z=0}$$
$$= \begin{cases} -\mu! & \text{if } \mu = ja_k \text{ for some } j \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise,} \end{cases}$$

and thus

$$\left[\left(\frac{d}{dz}\right)^{\mu} \prod_{k=1}^{d} (z^{a_k}-1)^{-1}\right]_{z=0} = (-1)^d \mu! \cdot \sharp\left\{\nu \in \mathbb{Z}_{\geq 0}^d \mid \nu \cdot a = \mu\right\}.$$

Hence, the Leibniz rule for differentiation yields

$$S_d(a,b,n) = (-1)^d \sum_{s=0}^{|\mathbf{a}|n+b} \binom{|\mathbf{a}|\, n+b}{s} s! \cdot \sharp\left\{\nu \in \mathbb{Z}_{\geq 0}^d \mid \nu \cdot a = s\right\}$$
$$\times \sum_{K \subseteq [d]} (-1)^{d-\sharp K} \cdot (|\mathbf{a}|\, n+b-s)!\,\delta\left((2n+1)\left(\sum_{k\in K} a_k\right) - |\mathbf{a}|\, n - b + s\right)$$

and we arrive at the following theorem.

Theorem 2.2. *Let r, n be positive integers. For $a = (a_1, \ldots, a_d) \in (\mathbb{Z} \setminus \{0\})^d$ and $b \in \mathbb{Z}$,*

$$S_d(a,b,n) = (|\mathbf{a}|\, n+b)! \sum_{s=0}^{|\mathbf{a}|n+b} \sharp\left\{\nu \in \mathbb{Z}_{\geq 0}^d \mid \nu \cdot a = s\right\}$$
$$\times \sum_{K \subseteq [d]} (-1)^{\sharp K} \cdot \delta\left((2n+1)\left(\sum_{k\in K} a_k\right) - |\mathbf{a}|\, n - b + s\right).$$

3. The special case $\mathbf{a} = (1, \ldots, 1) \in \mathbb{Z}^d$

Now we consider the vector $a = \mathbf{1} := (1, \ldots, 1) \in \mathbb{Z}^d$. In this special case the formula (2.1) reduces to

$$S_d(\mathbf{1},b,n) = \frac{1}{2\pi i} \int_{|z|=\rho} \left(\frac{z^{2n+1}-1}{z-1}\right)^d z^{-dn-b-1} dz.$$

By the Cauchy integral formula and using the symmetry $S_d(\mathbf{1},b,n) = S_d(\mathbf{1},-b,n)$, we have

$$S_d(\mathbf{1},b,n) = \frac{1}{(dn+b)!} \left[\left(\frac{z^{2n+1}-1}{z-1}\right)^d\right]^{(dn+b)}\Bigg|_{z=0}.$$

Application of the Leibniz rule yields

$$\begin{aligned}
&\left[\left(\frac{z^{2n+1}-1}{z-1}\right)^{d}\right]^{(dn+b)}\Bigg|_{z=0} \\
&= \left(\sum_{k=0}^{dn+b}\binom{dn+b}{k}\left[\left(\frac{d}{dz}\right)^{k}\sum_{j=0}^{d}\binom{d}{j}(-1)^{d-j}\,z^{j(2n+1)}\right]\right. \\
&\qquad \left.\times\left[\left(\frac{d}{dz}\right)^{dn+b-k}(z-1)^{-d}\right]\right)\Bigg|_{z=0} \\
&= \sum_{j=0}^{d}\binom{d}{j}(-1)^{d-j}\sum_{k=0}^{dn+b}\binom{dn+b}{k} \\
&\qquad \times (j\,(2n+1))^{\underline{k}}\,\delta\,(j\,(2n+1)-k)\,(-d)^{\underline{dn+b-k}}\,(-1)^{-d-(dn+b-k)}\,.
\end{aligned}$$

Because

$$(-1)^{dn+b-k}\,(-d)^{\underline{dn+b-k}} = (d-1+dn+b-k)^{\underline{dn+b-k}}$$

we obtain

$$\begin{aligned}
S_d\,(\mathbf{1},b,n) &= \frac{1}{(dn+b)!}\sum_{0\le j\le (dn+b)/(2n+1)}(-1)^{j}\binom{d}{j}(dn+b)^{\underline{j(2n+1)}} \\
&\qquad \times (d-1+dn+b-j\,(2n+1))^{\underline{dn+b-j(2n+1)}}\,.
\end{aligned}$$

Thus we have proved the following representation of $S_d\,(\mathbf{1},b,n)$.

Theorem 3.1. *Let d,n be positive integers. For $\mathbf{1}=(1,\ldots,1)\in\mathbb{Z}^d$ and $b\in\mathbb{Z}$,*

$$S_d\,(\mathbf{1},b,n) = \sum_{0\le j\le (dn+b)/(2n+1)}(-1)^{j}\binom{d}{j}\binom{d-1+dn+b-j\,(2n+1)}{d-1}.$$

For scaling, we put $b=nx$ with $-d\le x\le d$. Fix d and x. We have

$$\begin{aligned}
&\lim_{n\to\infty}(2n+1)^{-(d-1)}\,S_d\,(\mathbf{1},nx,n) \\
&= \frac{2^{-(d-1)}}{(d-1)!}\sum_{0\le j\le (d+x)/2}(-1)^{j}\binom{d}{j}(d+x-2j)^{d-1}\,. \qquad (3.1)
\end{aligned}$$

Now we study the asymptotic behaviour of

$$L_d\,(x) := \sqrt{d}\lim_{n\to\infty}(2n+1)^{-(d-1)}\,S_d\,(\mathbf{1},nx,n) \qquad (3.2)$$

as the dimension d tends to infinity.

The right-hand side of equation (3.1) is intimately connected with the Euler-Frobenius numbers $A_{m,\ell}\,(\lambda)$. These numbers are defined as the coefficients of the Euler-Frobenius polynomials $P_{m,\lambda}\,(z)$ which can be introduced via the rational function expansion

$$\sum_{\nu=0}^{\infty}(\nu+\lambda)^{m}\,z^{\nu} = \frac{P_{m,\lambda}\,(z)}{(1-z)^{m+1}} \qquad (m=0,1,2,\ldots)\,,$$

where $0 \leq \lambda < 1$. They satisfy the relation

$$(z+1-\lambda)^m = \sum_{\ell=0}^{m} A_{m,\ell}(\lambda) \binom{z+\ell}{m}$$

and have the explicit representation

$$A_{m,\ell}(\lambda) = \sum_{j=0}^{\ell} (-1)^j \binom{m+1}{j} (\ell+\lambda-j)^m \tag{3.3}$$

(see [10, Lemma 2.2 (v) and (iii)]). Denoting by $\lfloor z \rfloor$ the largest integer less than or equal to z and by $\{z\}$ the integer part of $z \in \mathbb{R}$ (such that $z = \lfloor z \rfloor + \{z\}$) we have

$$L_d(x) = \frac{\sqrt{d}}{(d-1)!} \sum_{j=0}^{\lfloor (d+x)/2 \rfloor} (-1)^j \binom{d}{j} \left(\frac{d+x}{2} - j\right)^{d-1}.$$

By equation (3.3) we obtain the representation

$$L_d(x) = \frac{\sqrt{d}}{\Gamma(d)} A_{d-1,\lfloor (d+x)/2 \rfloor}(\{(d+x)/2\}).$$

Gawronski and Neuschel [10, Theorem 4.3] proved, for $k \geq 3$, the asymptotic relation

$$\begin{aligned} &\sqrt{\frac{m+1}{12}} \frac{1}{m!} A_{m,\ell}(\lambda) \\ &\quad = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \left(1 + \sum_{\mu=1}^{\lfloor (k-2)/2 \rfloor} \frac{p_{4\mu}(t)}{(m+1)^\mu}\right) + o\left(\frac{1}{(m+1)^{(k-2)/2}}\right) \end{aligned} \tag{3.4}$$

as $m \to \infty$ uniformly in $\ell \in \mathbb{Z}$, with explicitly computable even polynomials $p_{4\mu}$ of the quantity

$$t = \left(\ell + \lambda - \frac{m+1}{2}\right) \sqrt{\frac{12}{m+1}}$$

the degrees of which are at most 4μ. We conjecture that (3.4) holds uniformly in the variable λ. If so the following conjecture would be true.

Conjecture 3.2. *For $d \in \mathbb{Z}$, $k \geq 3$, and $-d \leq x \leq d$, the limit* (3.1) *satisfies the asymptotic relation*

$$L_d(x) = \sqrt{\frac{6}{\pi}} e^{-3x^2/(2d)} \left(1 + \sum_{\mu=1}^{\lfloor (k-2)/2 \rfloor} \frac{p_{4\mu}\left(\sqrt{\frac{3}{d}} x\right)}{d^\mu}\right) + o\left(\frac{1}{d^{(k-2)/2}}\right)$$

as $d \to \infty$. In particular, for $x \in \mathbb{R}$,

$$\lim_{d \to \infty} L_d\left(\sqrt{d} x\right) = \sqrt{\frac{6}{\pi}} e^{-3x^2/2}.$$

References

[1] Ulrich Abel, *Integration von sinc-Produkten mit funktionentheoretischen Methoden* (German, with German summary), Math. Semesterber. **61** (2014), no. 2, 153–158, DOI 10.1007/s00591-013-0122-0. MR3265204

[2] Keith Ball, *Cube slicing in* $\mathbf{R}^n$, Proc. Amer. Math. Soc. **97** (1986), no. 3, 465–473, DOI 10.2307/2046239. MR840631

[3] David Borwein and Jonathan M. Borwein, *Some remarkable properties of sinc and related integrals*, Ramanujan J. **5** (2001), no. 1, 73–89, DOI 10.1023/A:1011497229317. MR1829810

[4] David Borwein, Jonathan M. Borwein, and Bernard A. Mares Jr., *Multi-variable sinc integrals and volumes of polyhedra*, Ramanujan J. **6** (2002), no. 2, 189–208, DOI 10.1023/A:1015727317007. MR1908197

[5] D. B. de Haan, *Exposé de la théorie, des propri'etés, des formules de transformation, et des méthodes d'évaluation des intégrales définies*, Amsterdam: C. G. Van der Post, 1862. (Online at http://quod.lib.umich.edu/u/umhistmath/ARL0113.0001.001)

[6] D. B. de Haan, *Nouvelles tables d'intégrales défines*, Leyden 1867.

[7] D. Evers, *Hyperebenenschnitte des vierdimensionalen Würfels*, Wissenschaftl. Prüfungsarbeit, Universität Koblenz-Landau, Koblenz, 2010.

[8] Rolfdieter Frank and Harald Riede, *Hyperplane sections of the n-dimensional cube*, Amer. Math. Monthly **119** (2012), no. 10, 868–872, DOI 10.4169/amer.math.monthly.119.10.868. MR2999590

[9] Rolfdieter Frank and Harald Riede, *Die Berechnung des Integral* $\int_{-\infty}^{\infty}(\prod_{k=1}^{n}\frac{\sin(a_k x)}{a_k x})\cdot\cos(bx)dx$ (German, with German summary), Math. Semesterber. **61** (2014), no. 2, 145–151, DOI 10.1007/s00591-012-0115-4. MR3265203

[10] Wolfgang Gawronski and Thorsten Neuschel, *Euler-Frobenius numbers*, Integral Transforms Spec. Funct. **24** (2013), no. 10, 817–830, DOI 10.1080/10652469.2012.762362. MR3171995

[11] Douglas Hensley, *Slicing the cube in* $\mathbf{R}^n$ *and probability (bounds for the measure of a central cube slice in* $\mathbf{R}^n$ *by probability methods)*, Proc. Amer. Math. Soc. **73** (1979), no. 1, 95–100, DOI 10.2307/2042889. MR512066

[12] Georg Polya, *Berechnung eines bestimmten Integrals* (German), Math. Ann. **74** (1913), no. 2, 204–212, DOI 10.1007/BF01456040. MR1511759

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