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KLEIN'S FORMULAS AND ARITHMETIC OF TEICHMÜLLER MODULAR FORMS

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ABSTRACT. We apply the arithmetic theory of Teichmüller modular forms to calculating constants in relations, which are connected with Klein's (amazing) formulas, between certain invariants of canonical curves of genus $g = 3, 4$.

1. Introduction

Klein's formulas connect certain invariants of canonical curves of genus $g = 3, 4$ with the product θ_g of even theta constants for their Jacobian varieties. In this paper, motivated by results of Lachaud, Ritzenthaler, and Zykin [8, 9], we apply the arithmetic theory of Teichmüller modular forms [3, 4] to the calculation of the constants in Klein's formulas. Further, we describe θ_4 by derivatives of Schottky's modular form J, which is a Siegel modular form of degree 4 with zero divisor as the Jacobian locus. We obtain identities (up to a sign) between modular forms and invariants for canonical curves of genus 3, 4 by using the fact that these identities hold up to a nonzero constant (cf. [7, 10]) and then showing that both sides are integral and primitive.

First, we consider a smooth complex curve $C_P \subset \mathbb{P}^2_{\mathbb{C}}$ of genus 3 defined as $P = 0$, where P is a homogeneous quartic polynomial. Let (Ω_1, Ω_2) be the period 3×6 matrix of canonical regular 1-forms on C_P such that $Z = \Omega_1^{-1}\Omega_2$ belongs to the Siegel upper half space of degree 3. Then in [7], Klein proved that the discriminant $\mathrm{Disc}(P)$ of P satisfies

$$\mathrm{Disc}(P)^2 = c_1 \frac{\theta_3(Z)}{\det(\Omega_1)^{18}}$$

for a certain constant c_1 independent of C_P. In [8, 9], Lachaud, Ritzenthaler, and Zykin proved that

$$c_1 = \frac{(2\pi)^{54}}{2^{28}}$$

by big calculations (using the software MAGMA) of theta constants for the Jacobian variety of a special C_P which is isogenous to the product of three elliptic curves. Their work is also closely related to a conjecture of Serre [13] on a refinement of the Torelli theorem. In this paper, we give another proof of this result by showing that the invariant Disc equals (up to a sign) the integral and primitive Teichmüller modular form proportional to $\sqrt{\theta_3}$.

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Second, we show that this method can also be applied to determine the constant c_2 in the formula by Matone and Volpato [10, Theorem 3.4]

$$\det\left((1+\delta_{ij})\frac{\partial J(Z)}{\partial z_{ij}}\right)^2_{1\le i,j\le 4} = c_2\cdot\theta_4(Z),$$

where J denotes Schottky's modular form and $Z=(z_{ij})$ are the period matrices of canonical curves of genus 4. By using the arithmetic property of the Teichmüller modular form $\sqrt{\theta_4}$ and results of Brinkmann and Gerritzen [1,2] on the lowest term of J, we determine c_2. From this result, we can obtain a complete form of Klein's amazing formula (cf. [7, p. 462]) such as

$$\Delta(C)^2\cdot T(C)^8 = \frac{(2\pi)^{272}\theta_4(Z)}{2^{120}\det(\Omega_1)^{68}},$$

where $C\subset\mathbb{P}^3_{\mathbb{C}}$ is a canonical curve, and $\Delta(C)$ and $T(C)$ denote its (integral, primitive) discriminant and tact invariant respectively.

2. Preliminaries

2.1. A canonical curve is a proper smooth curve embedded into the projective space by the linear system associated with its canonical divisor. If a proper smooth curve C is not hyperelliptic, then a basis of the space $H^0(C,\Omega_C)$ of regular 1-forms on C gives a closed immersion $C\hookrightarrow\mathbb{P}^{g-1}$ which defines a canonical curve. Each canonical curve of genus 3 is given by a quartic curve in $\mathbb{P}^2$, and that of genus 4 is given by the intersection of a quadric surface and a cubic surface in $\mathbb{P}^3$.

2.2. For an integer $g>1$, let $\mathcal{M}_g$ and $\mathcal{A}_g$ be the moduli stacks of proper smooth curves of genus g and of principally polarized abelian varieties of dimension g respectively. Then there exists the Hodge line bundle λ on these stacks whose fiber is spanned by the exterior product of a basis of regular 1-forms. Let R be a commutative ring with unity, and let h be an integer. Then Siegel (resp. Teichmüller) modular forms of degree g and weight h over R are defined as global sections of $\lambda^{\otimes h}$ on $\mathcal{A}_g\otimes R$ (resp. $\mathcal{M}_g\otimes R$). The pullback by the Torelli map

$$\tau:\mathcal{M}_g\to\mathcal{A}_g$$

gives an R-linear map τ^* between the R-modules of Siegel modular forms and of Teichmüller modular forms over R which preserves their degree and weight. These forms are called integral if $R=\mathbb{Z}$ and given as elements of $\mathcal{O}_S$ if λ is trivialized on a scheme S over $\mathcal{A}_g\otimes R$ (resp. $\mathcal{M}_g\otimes R$).

By taking regular 1-forms $\omega_i = d\zeta_i/\zeta_i$ $(1\le i\le g)$ on Mumford's abelian scheme of dimension g [11] formally represented as

$$\{(\zeta_1,\dots,\zeta_g)\in\mathbb{G}_{\mathrm{m}}^g\}/\left\langle (q_{ij})_{1\le i\le g}\ \middle|\ 1\le j\le g\right\rangle,$$

the trivialization of λ by $\omega_1\wedge\cdots\wedge\omega_g$ gives rise to the Fourier (q_{ij}-) expansion for Siegel modular forms of degree g. For variables $\alpha_{\pm1},\dots,\alpha_{\pm g}$, put

$$A_0=\mathbb{Z}\left[\alpha_k,\frac{1}{\alpha_l-\alpha_m}\ \ (k,l,m\in\{\pm1,\dots,\pm g\},\ l\ne m)\right],$$

and let

$$A=A_0[[\beta_1,\dots,\beta_g]]$$

be the ring of formal power series over A_0 of variables $\beta_1, \dots, \beta_g$. Then it is shown in [4, Theorem 3.5] that there exists a generalized Tate curve $\mathcal{C}_g$ which is a stable curve of genus g over A and has the following properties:

- The curve $\mathcal{C}_g$ is a universal deformation over A of the singular projective line obtained as $\alpha_i = \alpha_{-i}$ $(1 \le i \le g)$ in $\mathbb{P}^1_{A_0}$ and is Mumford uniformized by
$$\left\langle \begin{pmatrix} \alpha_i & \alpha_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta_i \end{pmatrix} \begin{pmatrix} \alpha_i & \alpha_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \middle| \ 1 \le i \le g \right\rangle.$$
- The Jacobian of $\mathcal{C}_g$ becomes Mumford's abelian scheme
$$\mathbb{G}_{\mathrm{m}}^g / \langle (p_{ij})_{1 \le i \le g} \mid 1 \le j \le g \rangle,$$
where $p_{ij} \in A$ are called the universal periods whose lowest terms become β_i (if $i = j$) or $\dfrac{(\alpha_i - \alpha_j)(\alpha_{-i} - \alpha_{-j})}{(\alpha_i - \alpha_{-j})(\alpha_{-i} - \alpha_j)}$ (if $i \ne j$).

Then the evaluation of Teichmüller modular forms of degree g over R on $\mathcal{C}_g$ gives elements of $A \otimes R$, and this evaluation of the images by τ^* of Siegel modular forms is obtained by putting $q_{ij} = p_{ij}$ on their Fourier expansions.

Let
$$\theta_g(Z) = \prod_{\substack{a,\, b \,\in\, \{0, 1/2\}^g \\ 4a^t b\,:\,\text{even}}} \sum_{n \in \mathbb{Z}^g} \exp\left(2\pi\sqrt{-1} \left[\frac{1}{2}(n+a)Z^t(n+a) + (n+a)^t b \right] \right)$$
be the product of even theta constants (i.e., theta null-values) of degree g, where Z belongs to the Siegel upper half space H_g of degree g. For $g \ge 3$, θ_g is an integral Siegel modular form of degree g and weight $2^{g-2}(2^g+1)$. Then in [14, Theorem 1], Tsuyumine proved that $\tau^*(\theta_g)$ has a square root as a Teichmüller modular form over $\mathbb{C}$. Put
$$N_g = \begin{cases} -2^{28} & (g = 3), \\ 2^{2^{g-1}(2^g - 1)} & (g \ge 4). \end{cases}$$
Then using the above expansion theory of Teichmüller modular forms, it is shown in [3, Theorem 4.4] and [4, Corollary 4.6] that $\mu_g = \sqrt{\tau^*(\theta_g)/N_g}$ is an integral Teichmüller modular form which is also primitive, i.e., not congruent to 0 modulo any prime. Actually, [3, Proposition 4.3] implies that the lowest term of the expansion of μ_g as an element of A is primitive.

3. Genus 3 case

3.1. For a homogeneous polynomial P of x_1, x_2, x_3 of degree 4, define the discriminant $\mathrm{Disc}(P)$ of P as
$$\mathrm{Disc}(P) = 2^{-14} \cdot \mathrm{Res}\left(\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \frac{\partial P}{\partial x_3} \right),$$
where Res denotes the multivariate resultant. It is known that $\mathrm{Disc}(P)$ is an integral and primitive polynomial of the coefficients of P and that $\mathrm{Disc}(P) = 0$ if and only if $\partial P/\partial x_i$ $(i = 1, 2, 3)$ have a common nonzero root (cf. [8, 2.2] and [9, 2.1]). If $\mathrm{Disc}(P) \ne 0$, then the curve $C_P \subset \mathbb{P}^2$ defined as $P = 0$ is a smooth curve of genus 3. For each $f = x_i$, let ω_f be a 1-form on C_P defined as
$$dP \wedge \omega_f = f\left(-x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 - x_3 dx_1 \wedge dx_2\right),$$

namely,

$$\omega_f = \frac{f \cdot (x_j dx_k - x_k dx_j)}{\partial P / \partial x_i} \quad \text{if } \partial P/\partial x_i \neq 0 \text{ and } (ijk) \text{ is even.}$$

From this expression, one can see that ω_{x_i} $(1 \leq i \leq 3)$ are well defined and give a basis of $H^0(C_P, \Omega_{C_P})$. We call ω_{x_i} the canonical regular 1-forms on C_P.

Let X_4 be the affine space over $\mathbb{Z}$ consisting of homogeneous polynomials P of x_1, x_2, x_3 of degree 4, and let X_4^{o} be its Zariski open subspace on which P is irreducible and $\mathrm{Disc}(P)$ is invertible. Then the correspondence $P \mapsto C_P$ gives rise to a morphism

$$\phi : X_4^{\mathrm{o}} \to \mathcal{M}_3$$

whose image is the locus of nonhyperelliptic curves. Using invariant theory, Klein proved in [7] that the pullback $(\tau \circ \phi)^*(\theta_3)$ of θ_3 by $\tau \circ \phi : X_4^{\mathrm{o}} \to \mathcal{A}_3$ is a multiple by a nonzero constant of the square Disc^2 of Disc. Further, Lachaud, Ritzenthaler, and Zykin [8, 9] calculated this constant and noticed that their result is equivalent to the following fact, which we will show using the arithmetic property of μ_3:

Theorem 3.1 (cf. [9, Remark 2.2.4])**.** *We have*

$$\phi^*(\mu_3) = \pm \mathrm{Disc}$$

under the trivialization of λ by $\omega_{x_1} \wedge \omega_{x_2} \wedge \omega_{x_3}$.

Proof. Note that $\phi^*(\mu_3)$ and Disc are integral polynomials of x_i and that they are seen to be proportional. Further, Disc is primitive, and hence there exists an integer c such that

$$\phi^*(\mu_3) = c \cdot \mathrm{Disc} \cdot (\omega_{x_1} \wedge \omega_{x_2} \wedge \omega_{x_3})^{\otimes 9}.$$

Since μ_3 is primitive on $\mathcal{M}_3$ and ϕ is dominant over all the special fibers of $\mathrm{Spec}(\mathbb{Z})$, $\phi^*(\mu_3)$ is not congruent to 0 modulo any prime. Therefore, $\phi^*(\mu_3)$ is also primitive, and hence $c = \pm 1$. $\square$

3.2. We apply Theorem 3.1 to re-prove the following main result of [8].

Corollary 3.2 (cf. [8, Theorem 4.3.1 and Corollary 4.3.3] and [9, Theorem 2.2.3])**.** *Let $C_P \subset \mathbb{P}^2_{\mathbb{C}}$ be a canonical complex curve as above, and let $\gamma_1, \ldots, \gamma_6$ be a symplectic basis of $H_1(C_P, \mathbb{Z})$ for the intersection pairing. Put*

$$\Omega = (\Omega_1, \Omega_2) = \left(\int_{\gamma_j} \omega_{x_i} \right)_{i,j}$$

and $Z = \Omega_1^{-1}\Omega_2 \in H_3$. Then

$$\mathrm{Disc}(P)^2 = \frac{(2\pi)^{54} \cdot \theta_3(Z)}{2^{28} \cdot \det(\Omega_1)^{18}}.$$

Proof. As is mentioned in subsection 2.2, $\mu_3 = \sqrt{-\tau^*(\theta_3)/2^{28}}$, and hence the assertion follows from Theorem 3.1 and that

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = \left(2\pi\sqrt{-1}\right)^3 dz_1 \wedge dz_2 \wedge dz_3$$

under the identification by $\zeta_i = \exp\left(2\pi\sqrt{-1} z_i\right)$ of Mumford's abelian scheme of dimension 3 over $\mathbb{C}$ with the complex torus $\mathbb{C}^3 / \left(\mathbb{Z}^3 + \mathbb{Z}^3 \cdot Z\right)$, where (z_1, z_2, z_3) are the natural coordinates on $\mathbb{C}^3$. $\square$

4. Genus 4 case

4.1. For a multiple n of 4, let

$$\Theta_n(Z) = \sum_{(\lambda_1,\dots,\lambda_4)\in L_{2n}^4} \exp\left(\pi\sqrt{-1}\sum_{i,j=1}^{4}\langle\lambda_i,\lambda_j\rangle z_{ij}\right) \qquad (Z=(z_{ij})\in H_4)$$

be the theta series for the lattice

$$L_{2n} = \left\{(a_1,\dots,a_{2n})\in\mathbb{R}^{2n}\;\middle|\; 2a_i,\ a_i-a_j,\ \frac{1}{2}\sum_i a_i\in\mathbb{Z}\right\}$$

with standard inner product $\langle\ ,\ \rangle$. Then Schottky's J defined as

$$J(Z) = \frac{2^2}{3^2\cdot 5\cdot 7}\left(\Theta_4(Z)^2-\Theta_8(Z)\right)$$

is a Siegel modular form of degree 4 and weight 8 which is represented as an integral polynomial of theta constants (cf. [5,6]) and hence is integral. Further, J is characterized (up to a constant) as a nonzero Siegel modular form of degree 4 of minimal weight vanishing on the Jacobian locus. Then Brinkmann and Gerritzen [1,2] proved that the lowest term of (the Fourier expansion of) J for $q_{11},q_{22},q_{33},q_{44}$ becomes

$$-2^{16}\left(\prod_{1\le i<j\le 4} q_{ij}^{-1}\right) F\cdot q_{11}q_{22}q_{33}q_{44}.$$

Here F is an integral polynomial of q_{ij} $(1\le i<j\le 4)$ which is a (unique up to a sign) generator of the kernel of the algebra homomorphism

$$\varphi:\mathbb{Z}\left[q_{ij}\ (1\le i<j\le 4)\right]\to\mathbb{Z}\left[\alpha_k,\frac{1}{\alpha_l-\alpha_m}\ \ (k,l,m\in\{\pm1,\dots,\pm4\},\ l\neq m)\right]$$

given by

$$\varphi(q_{ij}) = \frac{(\alpha_i-\alpha_j)(\alpha_{-i}-\alpha_{-j})}{(\alpha_i-\alpha_{-j})(\alpha_{-i}-\alpha_j)},$$

where $\alpha_{\pm1},\dots,\alpha_{\pm4}$ are variables. Actually, F is written as $\Delta H-G$ by the following:

$$\Delta = \prod_{1\le i<j\le 4}(q_{ij}-1),$$

$$H = \prod_{1\le i<j\le 4} q_{ij} - \left(\sum_{1\le i\le 4}\ \prod_{\substack{1\le k<l\le 4\\ k,l\neq i}} q_{kl}\right) + q_{12}q_{34}+q_{13}q_{24}+q_{14}q_{23},$$

$$G = q_{12}q_{34}\prod_{\substack{1\le i<j\le 4\\ (i,j)\neq(1,2),(3,4)}}(q_{ij}-1)^2 + q_{13}q_{24}\prod_{\substack{1\le i<j\le 4\\ (i,j)\neq(1,3),(2,4)}}(q_{ij}-1)^2$$

$$+\ q_{14}q_{23}\prod_{\substack{1\le i<j\le 4\\ (i,j)\neq(1,4),(2,3)}}(q_{ij}-1)^2.$$

Put

$$S_{ij} = \frac{-1}{2^{16}\cdot 2\pi\sqrt{-1}}\frac{\partial J(Z)}{\partial z_{ij}} = -\frac{q_{ij}}{2^{16}}\frac{\partial J}{\partial q_{ij}} \qquad (1\le i,j\le 4),$$

and denote by S the 4×4 symmetric matrix whose (i,j)-components are $(1+\delta_{ij})S_{ij}$, where δ_{ij} denotes Kronecker's delta.

Theorem 4.1. *The pullback* $\tau^*(\det(S))$ *of the determinant* $\det(S)$ *of* S *by the Torelli map* τ *is equal to* $\pm\mu_4$*, and hence it is an integral and primitive Teichmüller modular form of degree* 4.

Proof. As mentioned in subsection 2.2, [3, Proposition 4.3] implies that the lowest term of the expansion of μ_g as an element of A is integral and primitive, and by [10, Theorem 3.4], $\tau^*(\det(S))$ is proportional to μ_4 as a Teichmüller modular form. In the following, we will prove that the lowest term of $\tau^*(\det(S))$ as a formal power series of β_i $(1\le i\le 4)$ is a primitive element of A_0. Then one can see that $\tau^*(\det(S))=\pm\mu_4$, and hence it is integral and primitive.

Since the expansion $\tau^*(\det(S))$ by β_i is obtained by putting $q_{ij}=p_{ij}$ on $\det(S)$, the lowest term of $\tau^*(\det(S))$ for β_i belongs to A_0. Put $D=\prod_{k=1}^4 q_{kk}$, and for each $1\le i,j\le 4$, denote by L_{ij} the constant term of $(1+\delta_{ij})S_{ij}/D$ for q_{kk} $(1\le k\le 4)$ which belongs to $\mathbb{Z}\left[q_{lm}^{\pm 1}\ (1\le l<m\le 4)\right]$. Then we prove that the lowest term of $\tau^*(\det(S))$ is primitive by showing that L_{ij} $(1\le i,j\le 4)$ form a regular matrix modulo the above F over any field. Put $N=\prod_{1\le l<m\le 4} q_{lm}$. Then this assertion follows from the fact that $\det(L_{ij}\cdot N)$ is not divided by F as a polynomial of q_{lm} $(1\le l<m\le 4)$ over any field since F/q_{lm} is not a polynomial for any $l<m$. By putting $q_{lm}=0$ for $(l,m)\ne(1,2),(3,4)$, we have

$$F\mapsto F'\overset{\text{def}}{=}(q_{12}-1)(q_{34}-1)q_{12}q_{34}-q_{12}q_{34}=q_{12}^2q_{34}^2-q_{12}^2q_{34}-q_{12}q_{34}^2$$

and

$$L_{ij}\cdot N\mapsto\begin{cases}2F' & (i=j),\\ q_{12}^2q_{34}^2-q_{12}^2q_{34} & ((i,j)=(1,2)),\\ q_{12}^2q_{34}^2-q_{12}q_{34}^2 & ((i,j)=(3,4)),\\ -F' & (\text{otherwise}).\end{cases}$$

Therefore,

$$\begin{aligned}&\det(L_{ij}\cdot N)\\ &\mapsto\det\begin{pmatrix}2F' & q_{12}^2q_{34}^2-q_{12}^2q_{34} & -F' & -F'\\ q_{12}^2q_{34}^2-q_{12}^2q_{34} & 2F' & -F' & -F'\\ -F' & -F' & 2F' & q_{12}^2q_{34}^2-q_{12}q_{34}^2\\ -F' & -F' & q_{12}^2q_{34}^2-q_{12}q_{34}^2 & 2F'\end{pmatrix}\\ &\equiv q_{12}^6q_{34}^6\not\equiv 0 \bmod(F'),\end{aligned}$$

and hence $\det(L_{ij}\cdot N)$ is not congruent to 0 modulo F over any field. This implies that the lowest term of $\tau^*(\det(S))$ is primitive, and hence Theorem 4.1 follows from the above argument. □

Corollary 4.2. *Let* $C\subset\mathbb{P}^3_{\mathbb{C}}$ *be a canonical complex curve of genus* 4 *whose Jacobian variety is represented as a complex torus* $\mathbb{C}^4/\left(\mathbb{Z}^4+\mathbb{Z}^4\cdot Z\right)$*, where* $Z=(z_{ij})\in H_4$*. Denote by* $S(C)$ *the* 4×4 *matrix obtained by putting* $q_{ij}=\exp\left(2\pi\sqrt{-1}z_{ij}\right)$ *in* S*. Then*

$$\det(S(C))^2=\frac{(2\pi)^{272}\cdot\theta_4(Z)}{2^{120}}.$$

Proof. This assertion follows from Theorem 4.1 and that $\mu_4=\sqrt{\tau^*(\theta_4)/2^{120}}$. □

4.2. The above results are also applied to calculating the unknown constant in Klein's amazing formula [7, p. 462]. Let $C \subset \mathbb{P}^3 = \{(x_1 : x_2 : x_3 : x_4)\}$ be (possibly singular) curves which are obtained as the intersections of quadric surfaces and cubic surfaces given by $Q = 0$ and $E = 0$ respectively. Then the discriminant $\Delta(C)$ of C is defined as $\det\left((1+\delta_{ij})Q_{ij}\right)$, where Q is given by $\sum_{1 \le i \le j \le 4} Q_{ij} x_i x_j$. Further, the tact invariant $T(C)$ of C is defined in [12, p. 122] as a polynomial of the coefficients of Q, E corresponding to the locus over which C are singular. Especially, we take $T(C)$ as an integral and primitive polynomial. If C is smooth, then its genus is 4 and there exists a basis of $H^0(C, \Omega_C)$ which consists of canonical regular 1-forms ω_{x_i} $(1 \le i \le 4)$ satisfying

$$dQ \wedge dE \wedge \omega_{x_i} = x_i \sum_{j=1}^{4} (-1)^j x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_4$$

on $C \subset \mathbb{P}^3$. Then Klein [7, p. 462] (see also [10, Remark 3.6]) states that there exists a constant d independent of C such that

$$\Delta(C)^2 \cdot T(C)^8 = d \frac{\theta_4(Z)}{\det(\Omega_1)^{68}},$$

where (Ω_1, Ω_2) is the period matrix of ω_{x_i} $(1 \le i \le 4)$ for a symplectic basis of $H_1(C, \mathbb{Z})$ such that $Z = \Omega_1^{-1}\Omega_2 \in H_4$.

For $d = 2, 3$, let Y_d be the affine space over $\mathbb{Z}$ consisting of homogeneous polynomials of x_i $(1 \le i \le 4)$ with degree d. Then the above discriminant and tact invariant give integral and primitive polynomials defined on $Y_2 \times Y_3$, which we denote by Δ and T respectively. Let $Y_{2,3}^{\mathrm{o}}$ be the Zariski open subspace of $Y_2 \times Y_3$ which consists of (Q, E) satisfying that $\{Q = 0\} \cap \{E = 0\}$ are smooth curves in $\mathbb{P}^3$. Then we have a family of canonical curves of genus 4 over $Y_{2,3}^{\mathrm{o}}$ with basis of canonical regular 1-forms ω_{x_i}. Therefore, there exists a natural morphism

$$\psi : Y_{2,3}^{\mathrm{o}} \to \mathcal{M}_4$$

whose image corresponds to the locus of nonhyperelliptic curves of genus 4. Then Klein's statement implies that under the trivialization of λ by $\omega_{x_1} \wedge \omega_{x_2} \wedge \omega_{x_3} \wedge \omega_{x_4}$, $\psi^*(\mu_4)$ is a multiple of $\Delta \cdot T^4$ by a nonzero constant. Since ψ is dominant over all the special fibers of $\mathrm{Spec}(\mathbb{Z})$, $\psi^*(\mu_4)$ is integral and primitive, and hence we have

$$\psi^*(\mu_4) = \pm \Delta \cdot T^4.$$

Therefore, as in the proof of Corollaries 3.2 and 4.2, the above constant d can be determined as

$$d = \frac{(2\pi)^{272}}{2^{120}}.$$

References

[1] B. Brinkmann and L. Gerritzen, *The lowest term of the Schottky modular form*, Math. Ann. **292** (1992), no. 2, 329–335, DOI 10.1007/BF01444624. MR1149038

[2] L. Gerritzen, *Equations defining the periods of totally degenerate curves*, Israel J. Math. **77** (1992), no. 1-2, 187–210, DOI 10.1007/BF02808017. MR1194792

[3] Takashi Ichikawa, *Theta constants and Teichmüller modular forms*, J. Number Theory **61** (1996), no. 2, 409–419, DOI 10.1006/jnth.1996.0156. MR1423061

[4] Takashi Ichikawa, *Generalized Tate curve and integral Teichmüller modular forms*, Amer. J. Math. **122** (2000), no. 6, 1139–1174. MR1797659

[5] Jun-ichi Igusa, *Schottky's invariant and quadratic forms*, E. B. Christoffel (Aachen/Monschau, 1979), Birkhäuser, Basel-Boston, Mass., 1981, pp. 352–362. MR661078

[6] Jun-ichi Igusa, *Problems on theta functions*, Theta functions—Bowdoin 1987, Part 2 (Brunswick, ME, 1987), Proc. Sympos. Pure Math., vol. 49, Amer. Math. Soc., Providence, RI, 1989, pp. 101–110. MR1013168

[7] Felix Klein, *Zur Theorie der Abel'schen Functionen* (German), Math. Ann. **36** (1890), no. 1, 1–83, DOI 10.1007/BF01199432. MR1510611

[8] Gilles Lachaud and Christophe Ritzenthaler, *On some questions of Serre on abelian threefolds*, Algebraic geometry and its applications, Ser. Number Theory Appl., vol. 5, World Sci. Publ., Hackensack, NJ, 2008, pp. 88–115, DOI 10.1142/9789812793430_0005. MR2484050

[9] Gilles Lachaud, Christophe Ritzenthaler, and Alexey Zykin, *Jacobians among abelian threefolds: a formula of Klein and a question of Serre*, Math. Res. Lett. **17** (2010), no. 2, 323–333, DOI 10.4310/MRL.2010.v17.n2.a11. MR2644379

[10] Marco Matone and Roberto Volpato, *Vector-valued modular forms from the Mumford forms, Schottky-Igusa form, product of Thetanullwerte and the amazing Klein formula*, Proc. Amer. Math. Soc. **141** (2013), no. 8, 2575–2587, DOI 10.1090/S0002-9939-2012-11526-6. MR3056547

[11] David Mumford, *An analytic construction of degenerating abelian varieties over complete rings*, Compositio Math. **24** (1972), 239–272. MR0352106

[12] G. Salmon, *Traité de géométrie analytique à trois dimensions*, Troisième partie, Ouvrage traduit de l'anglais sur la quatrième édition, Paris, 1892.

[13] Jean-Pierre Serre, *Two letters to Jaap Top*, Algebraic geometry and its applications, Ser. Number Theory Appl., vol. 5, World Sci. Publ., Hackensack, NJ, 2008, pp. 84–87, DOI 10.1142/9789812793430_0004. MR2484049

[14] Shigeaki Tsuyumine, *Thetanullwerte on a moduli space of curves and hyperelliptic loci*, Math. Z. **207** (1991), no. 4, 539–568, DOI 10.1007/BF02571407. MR1119956

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