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INSTABILITY AND SINGULARITY OF PROJECTIVE HYPERSURFACES

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ABSTRACT. In this paper, we will show that the Hesselink stratification of a Hilbert scheme of hypersurfaces is independent of the choice of Plücker coordinate and there is a positive relation between the length of Hesselink's worst virtual 1-parameter subgroup and multiplicity of a projective hypersurface.

1. INTRODUCTION

Let k be an algebraically closed field. The table in [8, p. 80] says that a hypersurface over k is unstable if and only if it has a singular point of some special tangent cone for some possibly small degree and dimension.

If there is a singular point p of a hypersurface X, we can measure its *magnitude* of singularity by its multiplicity $n_{p,X}$, which is defined to be the degree of the tangent cone TC_pX as a subscheme of the tangent space T_pX([2, p. 258]). In particular, $n_{p,X} \geq 2$ if and only if p is a singular point of X. The number $n_X = \max_{p\in X} n_{p,X}$ is a geometric invariant.

On the other hand, we can measure how much a hypersurface is unstable. A hypersurface of a projective space is represented by a k point of a Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^r_k)$, when P is of the form

$$P(t) = \binom{r+t}{r} - \binom{r+t-d}{r}$$

for some r and d. There is a canonical action of $\mathrm{SL}_{r+1}(k)$ on $\mathrm{Hilb}^P(\mathbb{P}^r_k)$, possibly with a choice of Plücker coordinate

$$\mathrm{Hilb}^P(\mathbb{P}^r_k) \to \mathbb{P}\Bigg(\bigwedge^{Q(d+t)} H^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(d+t))\Bigg)$$

for some $t \geq 0$. For each choice of t, we have the unstable locus of a Hilbert scheme and Hesselink stratification of the unstable locus

$$\mathrm{Hilb}^P(\mathbb{P}^r_k)^{\mathrm{us}}_{d+t} = \coprod_{[\lambda],\delta>0} E_{d+t,[\lambda],\delta}, \tag{1}$$

where each $E_{d+t,[\lambda],\delta}$ is a constructible subset of the Hilbert scheme ([3]).

In this paper, we will show that every Hesselink stratification in (1) is actually independent of the choice of the natural number t (Theorem 3.1). We will also show

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that there is a positive relation between n_H and the pair $([\lambda], \delta)$, where $x \in E_{d,[\lambda],\delta}$ for *the* Hilbert point x which represents H, for every projective hypersurface H over k (Theorem 4.3). This is given by (6), which is a sharp inequality.

2. Preliminaries and notation

A sequence of closed immersions which map a Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^r_k)$ into projective spaces is given in [1, (3.4)]. If g_P is the Gotzmann number corresponding to P, then for any $d \geq g_P$, there is a closed immersion

$$\phi_d : \mathrm{Hilb}^P(\mathbb{P}^r_k) \to \mathbb{P}\Big(\bigwedge^{Q(d)} H^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(d))\Big).$$

Here,

$$Q(d) = h^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(d)) - P(d).$$

Let $S = k[x_0, \dots, x_r]$ be graded by the degree of polynomial. Then we have the natural isomorphism $S \cong \bigoplus_{i \geq 0} H^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(i))$ as graded rings over k.

For any closed point $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)$ which represents a saturated homogeneous ideal $I \subset S$, $\phi_d(x) = [I^{\wedge Q(d)}]$. We can describe the set of closed points in the image of ϕ_d explicitly:

$$\phi_d\left(\mathrm{Hilb}^P(\mathbb{P}^r_k)\right) = \{V \in \mathrm{Gr}(S_d, Q(d)) |\, \dim_k S_1 V \leq \dim_k S_1 U, \forall U \in \mathrm{Gr}(S_d, Q(d))\}.$$

Here we consider $\mathrm{Gr}(S_d, Q(d))$ as a closed subscheme of $\mathbb{P}\left(\bigwedge^{Q(d)} S_d\right)$ via Plücker embedding.

Now $\mathbb{P}\left(\bigwedge^{Q(d)} S_d\right)$ admits a canonical $G := \mathrm{GL}_{r+1}(k)$-action which stabilizes the image of ϕ_d so that we can discuss the unstable locus of a Hilbert scheme via induced $H = \mathrm{SL}_{r+1}(k)$ action [8, Theorem 1.19 and Theorem 2.1, Chapter 2]. We call $\phi_d(x)$ a dth Hilbert point for any $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)$ and $d \geq g_P$.

2.1. **Hesselink stratification of a Hilbert scheme and state polytope.**

We can define a real-valued norm $\|\cdot\|$ on the group of every 1-parameter subgroup $\Gamma(G)$ of G which satisfies conditions in [5, p. 305]. Let T_0 be the maximal torus of G which consists of all diagonal matrices. Then $\Gamma(T_0) \cong \mathbb{Z}^{r+1}$, so there is the Euclidean norm on $\Gamma(T_0) \otimes \mathbb{R}$, associated with the standard basis given by $\{\lambda_i | 0 \leq i \leq r\}$ satisfying

$$\lambda_i(t)_{ab} = \begin{cases} t & a = b = i, \\ 1 & a = b \neq i, \\ 0 & a \neq b. \end{cases}$$

For any $g \in G$ and $\lambda \in \Gamma(G)$, let $g \star \lambda \in \Gamma(G)$ be the 1-parameter subgroup which maps $t \in \mathbb{G}_m$ to $g\lambda(t)g^{-1}$. The norm $\|\cdot\|$ is invariant under the action of the Weyl group of T_0 in G. Thus this norm can be extended to $\Gamma(G)$ via conjugation by [4, Corollary A, p. 135]. Let $T = H \cap T_0$, which is a maximal torus of H.

Suppose $\lambda' \in \Gamma(G)$ and V is a G-representation. There is a decomposition

$$V = \bigoplus_{i \in \mathbb{Z}} V_i$$

where $V_i = \{w \in V | \lambda'(t).w = t^i w\}$ ([7, Proposition 4.7]), and there is a decomposition $v = \sum_{i\in\mathbb{Z}} v_i$ with $v_i \in V_i$ so that we can explain the value $\mu(v, \lambda')$ as follows:

$$\mu(v, \lambda') = \min\{i \in \mathbb{Z} | v_i \neq 0\}.$$

For any unstable dth Hilbert point $\phi_d(x)$, any two indivisible worst 1-parameter subgroups of $\phi_d(x)$ are conjugate so that we can measure the magnitude of instability of $\phi_d(x)$ by the conjugacy classes of indivisible 1-parameter subgroups [5, Theorem 3.4]. Let $\Lambda_{x,d}$ be the set of all indivisible worst 1-parameter subgroups of $\phi_d(x)$. That is,

$$\Lambda_{x,d} = \left\{ \lambda \in \Gamma(H) \middle| \lambda \text{ is indivisible and } \frac{\mu(\phi_d(x), \lambda)}{\|\lambda\|} = \max_{\lambda' \in \Gamma(H)} \frac{\mu(\phi_d(x), \lambda')}{\|\lambda'\|} \right\}.$$

The set $\Lambda_{x,d}$ is non-empty. Moreover,

$$E_{d,[\lambda],\delta} = \left\{ x \in \mathrm{Hilb}^P(\mathbb{P}^r_k) \middle| \Lambda_{x,d} \cap [\lambda] \neq \emptyset, \quad \frac{\mu(\phi_d(x), \lambda_{\max})}{\|\lambda_{\max}\|} = \delta, \ \forall \lambda_{\max} \in \Lambda_{x,d} \right\}$$

is a constructible subset of $\mathrm{Hilb}^P(\mathbb{P}^r_k)$ for a conjugacy class $[\lambda]$ containing an indivisible 1-parameter subgroup λ of T [3]. Consequently, there is a stratification

$$\mathrm{Hilb}^P(\mathbb{P}^r_k)^{\mathrm{us}}_d = \coprod_{[\lambda],\delta>0} E_{d,[\lambda],\delta}$$

for each integer $d \geq g_P$.

The T_0 action induced by the G action on $V^P_d = \bigwedge^{Q(d)} S_d$ has decomposition

$$V^P_d = \bigoplus_{m \in M_{dQ(d)}} V^P_{m,d},$$

where $V^P_{m,d}$ is a k-subspace of $\bigwedge^{Q(d)} S_d$ which is spanned by

$$\left\{ \bigwedge_{i=1}^{Q(d)} m_i \middle| m_i \in M_d \text{ for all } 1 \leq i \leq Q(d) \quad \text{and} \quad \prod_{i=1}^{Q(d)} m_i = m \right\}.$$

Here M_d is the set of monomials of degree d in S. This decomposition coincides with weight decomposition of the T_0 action because wedges of monomials are exactly the eigenvectors of the action. A multiplicative semi-group of monomials in S is isomorphic to $\mathbb{N}^{r+1}$ so that it is naturally embedded in $X(T_0) \cong \mathbb{Z}^{r+1}$ via morphism, sending an eigenvector to its weight, which is given by a character of T_0. There is also a natural isomorphism $\eta : \Gamma(T_0) \to X(T_0)$ defined by a basis $\{\lambda_i\}_{i=0}^r$ of $\Gamma(T_0)$ and its dual basis with respect to the pairing in [5, p. 304]. For arbitrary dth Hilbert point $\phi_d(x)$,

$$\phi_d(x) = \left[\sum_{m \in M_{dQ(d)}} \phi_d(x)_m \right]$$

for a unique choice of sequence (up to scalar multiplication) $\{\phi_d(x)_m\}_{m \in M_{dQ(d)}}$ satisfying $\phi_d(x)_m \in V^P_{m,d}$. Let

$$\Xi_{x,d} = \{m \in M_{dQ(d)} | \phi_d(x)_m \neq 0\}.$$

This set is called the state of the Hilbert point $\phi_d(x)$ ([5] and [6]). For $f \in S_d$ let $\{f_m\}_{m\in M_d}$ be the sequence satisfying

$$f = \sum_{m\in M_d} f_m m.$$

Let $\Delta_{x,d}$ be the convex hull of $\Xi_{x,d} \otimes \mathbb{R}$ in $X(T_0) \otimes \mathbb{R} \cong \mathbb{R}^{r+1}$ and let $|\Delta_{x,d}|$ be the distance between $\Delta_{x,d}$ and $\xi_d := \frac{dQ(d)}{r}\mathbb{1} \in X(T_0) \otimes \mathbb{R}$. $\mathbb{1} \in \mathbb{R}^{r+1}$ is the vector whose coefficients are 1. There is also a unique point $h_{x,d}$ in $\Delta_{x,d}$ satisfying $\|h_{x,d}-\xi_d\| = |\Delta_{x,d}|$ and unique indivisible $\lambda_{x,d} \in \Gamma(T)$ satisfying $\eta(\lambda_{x,d})\otimes\mathbb{R} = qh_{x.d}$ for some $q \in \mathbb{R}^+$. Now we are ready to state

Theorem 2.1. *Suppose $\lambda \in \Gamma(H)$ is an indivisible worst 1-parameter subgroup of $\phi_d(x)$ for some $d \geq g_P$ and $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)^{\mathrm{us}}_d$. Then,*

$$\phi_d(x) \in E_{d,[\lambda_{g.x,d}],|\Delta_{g.x,d}|}$$

for every $g \in G$ such that the image of $g\star\lambda$ is a subset of T. Such a $g \in G$ exists for any choice of x and d. In particular, $\lambda \in [\lambda_{g.x,d}]$ and $|\Delta_{g.x,d}| = \max_{h\in G} |\Delta_{h.x,d}|$.

Proof. See [5, Lemma 3.2 and Theorem 3.4] and [6, Criterion 3.3]. □

Theorem 2.1 means that finding $E_{d,[\lambda],\delta}$ containing $\phi_d(x)$ is an optimization problem. That is, it is equivalent to finding $g \in G$ maximizing $|\Delta_{g.x,d}|$.

2.2. **The multiplicity of a projective hypersurface at a point.** Suppose $p = [1 : 0 : \ldots : 0] \in \mathbb{P}^r_k$ where X is a projective hypersurface corresponding to the homogeneous ideal generated by the single generator $f \in S_d$, without loss of generality. Multiplicity of X at a point p is

$$n_{p,X} = \min\left\{t \in \mathbb{N} \middle| x_0^{d-t} | f\right\}. \tag{2}$$

This is an extrinsic definition. However, we also have an intrinsic definition. $n_{p,X}$ is the degree of the tangent cone of X at p as a subscheme of the tangent space $T_p(X)$ of X at p ([2, p. 258]).

3. Computation of worst state polytope

Let $\langle,\rangle$ be the standard inner-product on $X(T) \otimes \mathbb{R}$ with respect to the basis $\{\eta(\lambda_i)\}_{i=0}^r$. For $\lambda \in \Gamma(T)$, let's define a monomial order $<_\lambda$ as follows:

$$m <_\lambda m' \iff \left\langle \eta(\lambda) \otimes \frac{1}{\|\lambda\|}, m \right\rangle < \left\langle \eta(\lambda) \otimes \frac{1}{\|\lambda\|}, m' \right\rangle \quad \text{or}$$

$$\left[\left\langle \eta(\lambda) \otimes \frac{1}{\|\lambda\|}, m \right\rangle = \left\langle \eta(\lambda) \otimes \frac{1}{\|\lambda\|}, m \right\rangle \quad \text{and} \quad m <_{\mathrm{lex}} m' \right].$$

Here $<_{\mathrm{lex}}$ is a lexicographic order with respect to the term order $x_i < x_{i+1}$. We can describe $\Delta_{x,t}$ for $t \geq g_P$ with this notation:

$$|\Delta_{x,t}| = \max_{\lambda\in\Gamma(T)} \min_{m\in\Xi_{x,t}} \left\langle \eta(\lambda) \otimes \frac{1}{\|\lambda\|}, m \right\rangle. \tag{3}$$

The value

$$\min_{m\in\Xi_{x,t}} \left\langle \eta(\lambda) \otimes \frac{1}{\|\lambda\|}, m \right\rangle$$

is equal to

$$\frac{\mu(\phi_t(x), \lambda)}{\|\lambda\|}.$$

3.1. **Stability of Hesselink stratifications of a Hilbert scheme of hypersurfaces.** In this subsection, assume that

$$P(t) = \binom{r+t}{r} - \binom{r+t-d}{r}. \tag{4}$$

Considering a lex-segment homogeneous ideal whose Hilbert polynomial is P, we get $g_P = d\ ([1])$. This means that $\mathrm{Hilb}^P(\mathbb{P}^r_k)$ is a parameter space of every hypersurface of $\mathbb{P}^r_k$ defined by a homogeneous polynomial of degree d. Also, we have a Hesselink stratification

$$\mathrm{Hilb}^P(\mathbb{P}^r_k)^{\mathrm{us}}_t = \coprod_{[\lambda],\delta>0} E_{t,[\lambda],\delta}$$

for each integer $t \geq d$. Before we state a new theorem, let

$$\tau(\delta, D) = |M_D|\delta$$

for $D \in \mathbb{N}$ and $\delta \in \mathbb{R}^+$.

Theorem 3.1. *For the above stratifications, there are identities*

$$E_{d+D,[\lambda],\tau(\delta,D)} = E_{d,[\lambda],\delta}$$

between constructible subsets of $\mathrm{Hilb}^P(\mathbb{P}^r_k)$. *Consequently, all Hesselink stratifications of* $\mathrm{Hilb}^P(\mathbb{P}^r_k)^{\mathrm{us}}_t$ *corresponding to* $t \geq d$ *coincide. In particular,* $\mathrm{Hilb}^P(\mathbb{P}^r_k)^{\mathrm{us}}_{d+D} = \mathrm{Hilb}^P(\mathbb{P}^r_k)^{\mathrm{us}}_d$ *as sets.*

Proof. We will show that $E_{d+D,[\lambda],\tau(\delta,D)} = E_{d,[\lambda],\delta}$ as a subset of $\mathrm{Hilb}^P(\mathbb{P}^r_k)$. Suppose that $x \in E_{d,[\lambda],\delta} \cap E_{d+D,[\lambda'],\delta'}$ for some $D \in \mathbb{N} \setminus \{0\}$, $\lambda, \lambda' \in \Gamma(T)$, and $\delta, \delta' > 0$. Fix $\gamma \in \Gamma(T)$. For each $t \geq d$, we have a unique sequence $\{m_{i,t}\}_{i=1}^{|M_t|}$ satisfying $m_{i,t} \in M_t$ for all $1 \leq i \leq |M_t|$ and $m_{i,t} >_\gamma m_{i+1,t}$ for all $1 \leq i < |M_t|$. Let f be the unique generator (up to scalar) of the saturated ideal represented by x. Then we can write $[g.f] = g.\phi_d(x)$ for arbitrary $g \in G$ as follows:

$$g.f = \sum_{i=1}^{|M_d|} (g.f)_i m_{i,d}.$$

Therefore,

$$\min_{m \in \Xi_{g.x,d}} \left\langle \eta(\gamma) \otimes \frac{1}{\|\gamma\|}, m \right\rangle = \left\langle \eta(\gamma) \otimes \frac{1}{\|\gamma\|}, m_{\alpha,d} \right\rangle, \tag{5}$$

where $\alpha = \max\{1 \leq i \leq |M_d| \mid (g.f)_i \neq 0\}$.

V^P_{d+D} has a basis $\{\bigwedge_{j=1}^{|M_D|} m_{a_j,d+D} | a_j < a_{j+1}, 1 \leq a_j \leq |M_D|\}$. Now the coefficient of $\bigwedge_{j=1}^{|M_D|} m_{a_j,d+D}$ in $g.\phi_{d+D}(x)$ with this basis can be written as follows:

$$\left[g. \bigwedge_{i=1}^{|M_D|} m_{i,D} f \right]_{\bigwedge_{j=1}^{|M_D|} m_{a_j,d+D}} = \left[\bigwedge_{i=1}^{|M_D|} (g.m_{i,D})(g.f) \right]_{\bigwedge_{j=1}^{|M_D|} m_{a_j,d+D}}$$

$$= (\det g)^{\binom{r+D}{r}} \left[\bigwedge_{i=1}^{|M_D|} m_{i,D}(g.f) \right]_{\bigwedge_{j=1}^{|M_D|} m_{a_j,d+D}}$$

$$= (\det g)^{\binom{r+D}{r}} \sum_{\sigma \in P_{r+1}} \mathrm{sgn}(\sigma) \prod_{i=1}^{|M_D|} [m_{i,D}(g.f)]_{m_{a_{\sigma(i)},d+D}}$$

where P_{r+1} is the permutation group on $r+1$. This means that

$$g.\phi_{d+D}(x) = \left[\sum_{1 \le a_1 < a_2 < \cdots < a_{|M_D|} \le |M_{d+D}|} A_{g.f,D}^{a_1,\ldots,a_{|M_D|}} \bigwedge_{j=1}^{|M_D|} m_{a_j,d+D} \right],$$

where $A_{g.f,D}^{a_1,\ldots,a_{|M_D|}}$ is an $|M_D| \times |M_D|$-minor of matrix $A_{g.f,D}$ with the choice of a_ith columns. Here $[A_{g.f,D}]_{ij} = [m_{i,D}(g.f)]_{m_{j,d+D}}$ for all $1 \le i \le |M_D|$ and $1 \le j \le |M_{d+D}|$. If a strictly increasing sequence of integers $\{a_j\}_{j=1}^{|M_D|}$ satisfies $1 \le a_i \le |M_{d+D}|$ and

$$\prod_{i=1}^{|M_D|} m_{a_i,d+D} <_\gamma m_{\alpha,d}^{|M_D|} \prod_{i=1}^{|M_D|} m_{i,D},$$

then $\{i \in \mathbb{Z} | 1 \le i \le |M_D|,\ m_{a_i,d+D} <_\gamma m_{\alpha,d} m_{i,D}\} \neq \emptyset$. Let

$$\beta = \max\{i \in \mathbb{Z} | 1 \le i \le |M_D|,\ m_{a_i,d+D} <_\gamma m_{\alpha,d} m_{i,D}\}.$$

By the definition of α,

$$[A_{g.f,D}]_{ia_j} = 0$$

for all $1 \le i \le \beta$ and $\beta \le j \le |M_D|$. This means that

$$A_{g.f,D}^{a_1,\ldots,a_{|M_D|}} = 0.$$

Therefore,

$$\min_{m \in \Xi_{g.x,d+D}} \left\langle \eta(\gamma) \otimes \frac{1}{\|\gamma\|}, m \right\rangle = \left\langle \eta(\gamma) \otimes \frac{1}{\|\gamma\|}, m_{\alpha,d}^{|M_D|} \prod_{i=1}^{|M_D|} m_{i,D} \right\rangle$$

$$= \tau\left(\left\langle \eta(\gamma) \otimes \frac{1}{\|\gamma\|}, m_{\alpha,d} \right\rangle, D \right) = \tau\left(\min_{m \in \Xi_{g.x,d}} \left\langle \eta(\gamma) \otimes \frac{1}{\|\gamma\|}, m \right\rangle, D \right)$$

by (5). Taking a $\gamma \in \Gamma(T)$ maximizing the above values, we can verify that

$$|\Delta_{g.x,d+D}| = \tau(|\Delta_{g.x,d}|, D)$$

by (3). Taking $g \in G$ maximizing these values, we have $[\lambda'] = [\lambda] = [\lambda_{g.x,d}]$ and $\delta' = \tau(\delta, D)$ by Theorem 2.1. ☐

3.2. **Optimization preserving property of lower-triangular matrix.** Once we choose a solution $g \in G$ to the optimization problem arising from Theorem 2.1, we can check that lg is also a solution for an arbitrary lower-triangular matrix l if we assume that

$$\eta(\lambda) = (a_0, a_1, \ldots, a_r) \in X(T_0) \otimes \mathbb{R} \cong \mathbb{R}^{r+1}$$

and $a_i \le a_{i+1}$ for all $0 \le i < r$. Furthermore, $h_{g.x,d} = h_{lg.x,d}$. Actually, this is a consequence of [5, Theorem 4.2].

Lemma 3.2. *Suppose g satisfies*

$$|\Delta_{g.x,d}| = \max_{h\in G} |\Delta_{h.x,d}|,$$

$$\eta(\lambda_{g.x,d}) = (a_0, \dots, a_r) \in X(T_0) \otimes \mathbb{R} \cong \mathbb{R}^{r+1},$$

and $a_i \leq a_{i+1}$ for all $0 \leq i < r$. Then, $\lambda_{lg.x,d} = \lambda_{g.x,d} \in \Lambda_{g.x,d}$ for every lower-triangular matrix $l \in H$.

Proof. Let $\lambda = \lambda_{g.x,d}$. For every lower triangular matrix $l \in H$,

$$l^{-1} \in P(\lambda) := \{q \in H | \exists \lim_{t\to 0} \lambda(t) q \lambda(t)^{-1} \in H\}.$$

Thus,

$$\mu(\phi_d(lg.x), \lambda) = \mu(\phi_d(g.x), l^{-1} \star \lambda) = \mu(\phi_d(g.x), \lambda)$$

by [6, Lemma 4.2]. Since the norm $\|\cdot\|$ on $\Gamma(G)$ is invariant under the conjugate action, $\lambda \in \Lambda_{lg.x,d} \cap \Gamma(T)$ so that $\lambda_{lg.x,d} = \lambda_{g.x,d}$ by [5, Theorem 4.2.b)(4)]. □

4. A relation between instability and singularity

From now on, let's assume (4) for Hilbert polynomial P. In previous sections, we proved that Hesselink stratifications of a Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^r_k)$ of hypersurfaces is unique in some sense. In this section, we will show that for $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)^{\mathrm{us}}_d$, there is a positive relation between the multiplicity n_{H_x} of the hypersurface H_x represented by x and the Hesselink strata $E_{d,[\lambda],\delta}$ which contains x. We see that the multiplicity of the hypersurface H_x at $e = [1 : 0 : \dots : 0] \in \mathbb{P}^r_k$ determines a supporting hyperplane of $\Delta_{x,d}$. Here we choose the homogeneous coordinate $[x_0 : x_1 : \dots : x_r]$ of $\mathbb{P}^r_k$.

Lemma 4.1. *In the above situation,*

$$\Xi_{x,d} \cap x_0^{d-n_{e,X}+1} M_{n_{e,X}-1} = \emptyset$$

and

$$\Xi_{x,d} \cap x_0^{d-n_{e,X}} M_{n_{e,X}} \neq \emptyset.$$

Proof. It is trivial by the definition of $\Xi_{x,d}$ and (2). □

Lemma 4.1 means that there is a relation between a state and the multiplicity of a fixed point. We know that the set $\{\Xi_{g.x} | g \in G\}$ determines the Hesselink strata $E_{d,[\lambda],\delta}$ which contains x for any unstable Hilbert point x and that the subgroup of G generated by upper triangular matrices and permutation matrices acts on $\mathbb{P}^r_k$ transitively, whose dual action is induced by the canonical action on V^P_d. n_{H_x} is a geometric invariant, so it is independent under the choice of coordinate. These facts lead us to

Lemma 4.2. *For any choice of unstable $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)$, there is $g_x \in G$ satisfying*

$$|\Delta_{g_x.x,d}| = \max_{h\in G} |\Delta_{h.x,d}|$$

and

$$n_{H_x} = n_{H_{g_x.x}} = n_{e,H_{g_x.x}}.$$

Proof. By Theorem 2.1, there is a g' satisfying

$$|\Delta_{g'.x,d}| = \max_{h\in G} |\Delta_{h.x,d}|.$$

Without loss of generality, we can guess that

$$\lambda_{g'.x,d} = (a_0, a_1, \dots, a_r)$$

satisfies $a_i \leq a_{i+1}$ for all $0 \leq i < r$. If it doesn't, we may take wg' instead of g' for some permutation matrix w. There is $y = [b_0 : b_1 : \dots : b_r] \in H_{g'.x}$ satisfying $n_{H_{g'.x}} = n_{y,H_{g'.x}}$. We can choose a lower-triangular matrix l such that some row of l is equal to a representative vector $(b_0, b_1, \dots, b_r)$ of y. By Lemma 3.2,

$$|\Delta_{lg'.x,d}| = \max_{h\in G} |\Delta_{h.x,d}|.$$

There is a permutation matrix q satisfying $e.(ql) = (eq).l = y$ by the construction. Now $g_x = qlg'$ has every desired property. □

Choosing $g_x \in G$ as in Lemma 4.2 for a fixed Hilbert point $x \in E_{d,[\lambda],\delta}$, we can compare n_{H_x} and the pair $([\lambda], \delta)$.

Theorem 4.3. *Suppose $x \in E_{d,[\lambda],\delta}$ for an unstable Hilbert point $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)$ where*

$$\lambda = (a_0, a_1, \dots, a_r) \in \Gamma(T)$$

satisfies $\sum_{i=0}^r a_i = 0$. If $b = \max_{0\leq i\leq r} a_i$ and $a = \min_{0\leq i \leq r} a_i$, then

$$\frac{\|\lambda\|\delta - ad}{b-a} \leq n_{H_x} \leq \frac{rd}{r+1} - \delta\frac{a}{\|\lambda\|}. \tag{6}$$

Proof. By Lemma 4.2, we can assume that

$$|\Delta_{x,d}| = \max_{h\in G} |\Delta_{h.x,d}|$$

and

$$n_{H_x} = n_{e,H_x}.$$

Without loss of generality, $\lambda \in \Lambda_{x,d}$. It it doesn't, we may choose a permutation matrix q satisfying $q \star \lambda \in \Lambda_{x,d}$ and let $q\star\lambda$ be another representative of $[\lambda]$. It is possible because $x \in E_{d,[\lambda],\delta}$, by Theorem 2.1. By definition, $\Delta_{x,d}$ contains

$$h_{x,d} = \frac{d}{r+1}\mathbb{1} + \frac{\delta}{\|\lambda\|}\eta(\lambda).$$

Since $\Delta_{x,d}$ is the convex hull of $\Xi_{x,d}$, Lemma 4.1 and our assumption on x imply that

$$n_{H_x} = n_{e,H_x} \leq d - \frac{d}{r+1} - \frac{\delta a}{\|\lambda\|} = \frac{rd}{r+1} - \delta\frac{a}{\|\lambda\|}$$

if we consider the maximum value of the degree of x_0 in each monomial $m \in \Xi_{x,d}$. Now suppose that

$$n_{H_x} < \frac{\|\lambda\|\delta - ad}{b-a}.$$

Then for all $0 \leq j \leq r$ satisfying $a = a_j$ and for all $(b_0, \dots, b_r) \in \Delta_{x,d}$,

$$b(d-b_j) + a(b_j - d) + ad \geq \sum_{i=0}^r a_i b_i \geq \delta\|\lambda\|$$

by (3) so that

$$d - b_j \geq \frac{\|\lambda\|\delta - ad}{b-a} > n_{H_x}.$$

This means that $n_{e,H_{p.x}} > n_{H_x} = n_{H_{p.x}}$, for the transposition matrix p which permutes 0 and j. This is a contradiction. □

If $\lambda = p \star (-r, 1, \ldots, 1) \in \Gamma(T)$ for some permutation matrix $p \in G$, then we see that

$$\frac{\|\lambda\|\delta - ad}{b-a} = \frac{rd}{r+1} - \delta \frac{a}{\|\lambda\|}$$

so that n_{H_x} must be a fixed value by (6). We also see that

$$\frac{\|\lambda\|\delta - ad}{b-a} > \frac{d}{r+1}.$$

This implies that every unstable hypersurface of degree $d \geq r+1$ is singular. This is also a weaker version of [8, Chapter 3, Proposition 4.2]. Equation (6) has been derived by the existence of certain coordinates, but smoothness of $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)$ requires a general property of each state polytope in $\{\Delta_{g.x,d} | g \in G, |\Delta_{g.x,d}| = \max_{h \in G} |\Delta_{h.x,d}|\}$. However, we can check that (6) is sharp. If $r = 3$, $d = 4$, and $\phi_d(x) = [x_0^4] \in \mathbb{P}(k[x_0, x_1, x_2, x_3]_4)$, then $\lambda = (3, -1, -1, -1) \in \Lambda_{x,d}$ so that

$$\frac{\|\lambda\|\delta - ad}{b-a} = \frac{rd}{r+1} - \delta \frac{a}{\|\lambda\|} = 4 = n_{H_x}.$$

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