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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF SEMIPOSITONE QUASILINEAR PROBLEMS THROUGH ORLICZ-SOBOLEV SPACE

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ABSTRACT. In this paper we show the existence of weak solutions for a class of semipositone problems of the type

$$\text{(P)} \qquad \begin{cases} -\Delta_\Phi u &= f(u) - a \quad \text{in} \quad \Omega, \\ u(x) &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded domain, $f : [0, +\infty) \to \mathbb{R}$ is a continuous function with subcritical growth, $a > 0$, and $\Delta_\Phi u$ stands for the Φ-Laplacian operator. By using variational methods, we prove the existence of a solution for a small enough.

1. Introduction

In this paper we study the existence of positive weak solutions for the semipositone problem

$$\text{(P)} \qquad \begin{cases} -\Delta_\Phi u &= f(u) - a \quad \text{in} \quad \Omega, \\ u(x) &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded domain with smooth boundary denoted by $\partial\Omega$, $f : [0, +\infty) \to \mathbb{R}$ is a continuous function with subcritical growth, $a > 0$, and $\Delta_\Phi u = div(\phi(|\nabla u|)\nabla u)$ stands for the Φ-Laplacian operator, where $\phi : (0, \infty) \to (0, \infty)$ is an appropriate C^1-function such that

$$\Phi(t) := \int_0^{|t|} \phi(s) s ds, \ t \in \mathbb{R},$$

is an N-function. For more details see Section 2. In what follows, ϕ satisfies the following conditions:

(ϕ_1) $\phi : (0, \infty) \to (0, \infty)$ is a C^1-function;
(ϕ_2) $\phi(t)$, $(\phi(t)t)' > 0$, $t > 0$;
(ϕ_3) there exist $l, m \in (1, N)$ with $m \in [l, l^*)$ and $l^* = \frac{lN}{N-l}$, such that

$$l \leq \frac{\Phi'(t)t}{\Phi(t)} \leq m \quad \forall t > 0;$$

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(ϕ_4) there exist $\bar{l}, \bar{m} > 0$ such that

$$\bar{l} \leq \frac{\Phi''(t)t}{\Phi'(t)} \leq \bar{m} \quad \forall t > 0.$$

Related to the function f, we assume that $f : [0, +\infty) \to \mathbb{R}$ is a continuous function and assume the following conditions:

$$(f_1) \qquad 0 = f(0) = \min_{t \in [0,+\infty)} f(t),$$

$$(f_2) \qquad \lim_{t \to 0^+} \frac{f(t)}{\phi(t)t} = 0.$$

There is $q \in (m, l^*)$ such that

$$(f_3) \qquad \limsup_{|t| \to +\infty} \frac{|f(t)|}{|t|^{q-1}} < +\infty.$$

There are $\theta > m$ and $t_0 > 0$ such that

$$(f_4) \qquad \theta F(t) \leq f(t)t \quad \forall t \geq t_0,$$

where $F(t) = \int_0^t f(\tau)\, d\tau$.

In the sequel, we say that $u \in W_0^{1,\Phi}(\Omega)$ is a *weak solution* for (P) if u is a continuous positive function that verifies

$$\int_\Omega \phi(|\nabla u|) \nabla u \nabla \varphi dx = \int_\Omega (f(u) - a)\varphi dx \quad \forall \varphi \in W_0^{1,\Phi}(\Omega).$$

Hereafter, $W_0^{1,\Phi}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in the norm $\| \ \|_{1,\Phi}$; for more details see Section 2.

The problem (P) for $a = 0$ is well known, and it can be solved by using the mountain pass theorem due to Ambrosetti and Rabinowitz [5]. However, for the case where (P) is semipositone, that is, when $a < 0$, the existence of a positive solution is not so simple, because the standard arguments via the mountain pass theorem combined with the maximum principle do not directly give a positive solution for the problem. Here, the size of a is a key point in our arguments, in the sense that we were able to prove the existence of a positive solution for (P) when a is small enough.

Many authors have studied semipositone problems over the years since the appearance of the paper by Castro and Shivaji [9] who were the first to consider this class of problem. In the literature we find different methods to prove the existence and nonexistence of solutions, such as subsupersolutions, degree theory arguments, fixed point theory, and bifurcation; see for example the [2], [4], [3], [6] and their references. Moreover, of these methods, the variational method was also used in a few papers as can be seen in [15], [7], [8], [10], [12], and [11].

The present work has been mainly motivated by the papers [7] and [8]. In [7], Caldwell, Castro, Shivaji and Unsurangsie have studied the existence of positive solutions for the following class of the semipositone problem:

$$(\mathrm{P1}) \qquad \begin{cases} -\Delta u = \mu h(u) + \lambda f(u) & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded domain, μ, λ are positive parameters, and $h, f : [0, +\infty) \to \mathbb{R}$ are differentiable and nondecreasing functions verifying the following conditions.

Conditions (on h)**.** There exist $A, B > 0$ and $q \in (1, \frac{N+2}{N-2})$ such that

$$At^q \leq h(t) \leq Bt^q \quad \forall t \in [0, +\infty).$$

There exists $\theta > 2$ such that for $t > 0$ large,

$$0 < \theta H(t) \leq h(t)t,$$

where $H(t) = \displaystyle\int_0^t h(s)\, ds$.

Conditions (on f)**.** There is $\alpha \in (0, 1)$ such that

$$\lim_{t \to +\infty} \frac{f(t)}{t^\alpha} = 0$$

and $f(0) < 0$. The existence of the solution has been obtained by applying the mountain pass theorem and subsupersolutions for convenient values of λ and μ.

In [8], Castro, de Figueiredo and Lopera have established the existence of a positive solution for the following class of semipositone involving the p-Laplacian operator:

$$\text{(P2)} \qquad \begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N > p > 2$, is a smooth bounded domain, $\lambda > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(0) < 0$. In that paper, the authors have assumed that there exist $q \in (p-1, \frac{Np}{N-p} - 1), A, B > 0$ such that

$$\left.\begin{array}{l} A(t^q - 1) \leq f(t) \leq B(t^q - 1), \quad \text{for} \quad t > 0 \\ f(t) = 0, \quad \text{for} \quad t \leq -1. \end{array}\right\}$$

The existence of the solution was proved by combining the mountain pass theorem with the regularity theory.

Our main result is the following.

Theorem 1.1. *Assume* (ϕ_1)*–*(ϕ_4) *and* (f_1)*–*(f_4)*. Then, there exists* $a^* > 0$ *such that if* $a \in (0, a^*)$*, problem* (P) *has a positive weak solution* $u_a \in C^{1,\gamma}(\Omega)$ *for some* $\gamma \in (0, 1)$*.*

In the proof of Theorem 1.1 we have used variational and regularity results found in Liberman [20, 21]. By using the mountain pass theorem we have found a solution u_a for all $a > 0$. By taking the limit of a goes to 0, we were able to show, via regularity results found in [20] and [21], that u_a is positive for a small enough. We believe that this is the first paper involving the Δ_Φ Laplacian and semipositone problem. Finally, we would like point out that a version of Theorem 1.1 can be done for $N = 1$, by supposing $l, m > 1$ and $q \in (m, +\infty)$ in (f_3), because the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact. For more details about this embedding see [1] and [23].

2. Orlicz–Sobolev spaces and preliminary results

In this section, we present an overview about Orlicz and Orlicz–Sobolev spaces. For more details about Orlicz and Orlicz–Sobolev spaces, see for instance [1] and [23].

A function $\Phi : \mathbb{R} \to [0, +\infty)$ is called an N-function if it is convex and even, $\Phi(t) = 0$ if and only if $t = 0$, $\Phi(t)/t \to 0$ as $t \to 0$, and $\Phi(t)/t \to +\infty$ as $t \to +\infty$. It is well known that an N-function Φ can be rewritten as

$$\Phi(t) = \int_0^{|t|} \varphi(s)ds, \ t \in \mathbb{R}, \tag{2.1}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a right derivative of Φ, nondecreasing, a right continuous function, $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$, and $\lim_{s\to+\infty} \varphi(s) = \infty$. Reciprocally, if φ satisfies the former properties, then Φ defined in (2.1) is an N-function.

For an N-function Φ and an open set $\Omega \subset \mathbb{R}^N$, the Orlicz class is the set of functions defined by

$$K_\Phi(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ / \ \ u \text{ is measurable and } \int_\Omega \Phi(|u(x)|)\, dx < +\infty \right\}$$

and the vector space $L_\Phi(\Omega)$ generated by $K_\Phi(\Omega)$ is called an Orlicz space. When Φ satisfies the Δ_2-condition, namely, there exist constants $k, t_0 > 0$ such that

$$\Phi(2t) \le k\Phi(t), \ \text{for all } t \ge t_0,$$

the Orlicz class $K_\Phi(\Omega)$ is a vector space, and hence equal to $L_\Phi(\Omega)$.

Defining the following norm (Luxemburg norm) on $L_\Phi(\Omega)$ by

$$||u||_\Phi = \inf \left\{ \lambda > 0 \ / \ \int_\Omega \Phi\left(\frac{|u(x)|}{\lambda}\right) dx \le 1 \right\},$$

the space $(L_\Phi(\Omega), ||\cdot||_\Phi)$ is a Banach space. The complement function of Φ is defined by

$$\tilde{\Phi}(t) = \sup_{s>0} \{ts - \Phi(s)\} \ \text{ (note that } \tilde{\tilde{\Phi}} = \Phi)$$

and satisfies $\tilde{\Phi}(\varphi(t)) \le \Phi(2t)$ for all $t \in \mathbb{R}$. Moreover, in the spaces $L_\Phi(\Omega)$ and $L_{\tilde{\Phi}}(\Omega)$ an extension of Hölder's inequality holds:

$$\left| \int_\Omega u(x)v(x)\, dx \right| \le 2||u||_\Phi ||v||_{\tilde{\Phi}}, \ \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\tilde{\Phi}}(\Omega).$$

Another important function related to the function Φ is the Sobolev conjugate function Φ_* of Φ defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} ds, \ \text{for } t > 0.$$

When $\Phi(t) = |t|^p$ for $1 < p < N$, we have $\Phi_*(t) = p^{*p^*}|t|^{p^*}$, where $p^* = \frac{pN}{N-p}$.

Now, the corresponding Orlicz–Sobolev space is defined as

$$W^{1,\Phi}(\Omega) = \Big\{ u \in L^\Phi(\Omega) \ / \ \exists \ v_j \in L_\Phi(\Omega) \ ; \int_\Omega u \frac{\partial \psi}{\partial x_j} dx = -\int_\Omega v_j \psi dx$$
$$\text{for } j = 1, \ldots, N \text{ and } \forall \ \psi \in C_0^\infty(\Omega) \Big\},$$

which endowed by the norm

$$||u||_{1,\Phi} = ||u||_\Phi + ||\nabla u||_\Phi \tag{2.2}$$

is a Banach space.

In what follows, $W_0^{1,\Phi}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm (2.2), and so, it is a Banach space endowed with the norm $\| \ \|_{1,\Phi}$. It is important to recall that $L_\Phi(\Omega)$, $W^{1,\Phi}(\Omega)$, and $W_0^{1,\Phi}(\Omega)$ are separable and reflexive when Φ and $\tilde{\Phi}$ satisfy the Δ_2-condition. Moreover, by the Δ_2-condition, we have that a sequence $\{u_n\}$ in $L_\Phi(\Omega)$ converges to $u \in L_\Phi(\Omega)$ if and only if

$$\int_\Omega \Phi(|u_n - u|)dx \longrightarrow 0 \text{ as } n \to \infty.$$

In addition, there is $\Lambda > 0$ such that

$$\int_\Omega \Phi(|u|)\, dx \leq \Lambda \int_\Omega \Phi(|\nabla u|)\, dx \quad \forall u \in W_0^{1,\Phi}(\Omega).$$

This inequality is well known as *Poincaré's Inequality*, and its proof can be found in [18, Lemma 2]. As a immediate consequence of this inequality, we have that $||u|| := ||\nabla u||_\Phi$ is an equivalent norm to the norm $||u||_{1,\Phi}$ on $W_0^{1,\Phi}(\Omega)$. From now on, we consider $|| \cdot ||$ as the norm on $W_0^{1,\Phi}(\Omega)$.

Another important inequality was proved by Donaldson and Trudinger [14], which establishes that there is a constant $S_N = S(N) > 0$ such that

$$\|u\|_{\Phi_*} \leq S_N \|u\|, \; u \in W_0^{1,\Phi}(\Omega).$$

This inequality shows that the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi_*}(\Omega)$ is continuous. If Ω is a bounded domain and B is an N-function such that

$$\limsup_{|t|\to+\infty} \frac{B(t)}{\Phi_*(t)} = 0, \tag{2.3}$$

then the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L_B(\Omega)$ is compact.

In the sequel, we are assuming that $\varphi(s) = \phi(s)s$ for all $s \in [0, +\infty)$,

$$\Phi(t) = \int_0^{|t|} \phi(s)s\, ds \quad \forall s \in \mathbb{R},$$

where we are extending $\phi(t)t$ at $t = 0$ as zero.

By using some results found in [16], the functions Φ and $\tilde{\Phi}$ are N-functions and satisfy the Δ_2-condition if and only if $(\phi_1) - (\phi_3)$ hold.

The lemma below is due to Fukagai and Narukawa [17].

Lemma 2.1. *Suppose that* (ϕ_1) *and* (ϕ_2) *hold. If*

$$\zeta(t) = \min\{t^l, t^m\} \quad \textit{and} \quad \kappa(t) = \max\{t^l, t^m\}, \; t \geq 0,$$

then

(i) $\zeta(\rho)\Phi(t) \leq \Phi(\rho t) \leq \kappa(\rho)\Phi(t)$, *for* $\rho, t \geq 0$,
(ii) $\zeta(||u||_\Phi) \leq \int_\Omega \Phi(|u|)\, dx \leq \kappa(||u||_\Phi)$, *for all* $u \in L_\Phi(\Omega)$.

The next lemma brings out a general convergence result in the context of Orlicz–Sobolev spaces.

Lemma 2.2. *Assume* (ϕ_1)–(ϕ_3) *hold and that* $\Omega \subset \mathbb{R}^N$ *is a smooth bounded domain. Let* $u \in W_0^{1,\Phi}(\Omega)$ *and* $\{u_n\}$ *be a bounded sequence in* $W_0^{1,\Phi}(\Omega)$ *such that*

$$\int_\Omega \big(\phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u\big)\nabla(u_n - u)dx \longrightarrow 0. \tag{2.4}$$

Then $u_n \to u$ *in* $W_0^{1,\Phi}(\Omega)$.

Proof. Since Φ is strictly convex, we derive

$$\langle \phi(|x|)x - \phi(|y|)y, x - y\rangle > 0 \text{ for all } x, y \in \mathbb{R}^N \text{ with } x \neq y, \tag{2.5}$$

that is, the application $\mathcal{F} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ given by $\mathcal{F}(x) = \phi(|x|)x$ is strictly monotonic. As

$$\langle \phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u, \nabla(u_n - u)\rangle \longrightarrow 0 \text{ in } L^1(\Omega)$$

we have

$$\langle \phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u, \nabla(u_n - u)\rangle \longrightarrow 0 \text{ a.e. in } \Omega, \tag{2.6}$$

for some subsequence of $\{u_n\}$, still denoted by itself. Then, a result in Dalmaso and Murat [13] implies that

$$\nabla u_n(x) \longrightarrow \nabla u(x) \text{ a.e. in } \Omega, \tag{2.7}$$

and so,

$$\Phi(|\nabla u_n - \nabla u|) \longrightarrow 0 \text{ a.e. in } \Omega \tag{2.8}$$

and

$$\phi(|\nabla u_n|)|\nabla u_n|^2 \longrightarrow \phi(|\nabla u|)|\nabla u|^2 \text{ a.e. in } \Omega. \tag{2.9}$$

Using the convexity and monotonicity of Φ together with (ϕ_3), there exists a constant $C > 0$ such that

$$\Phi(|\nabla u_n - \nabla u|) \leq C\big(\phi(|\nabla u_n|)|\nabla u_n|^2 + \phi(|\nabla u|)|\nabla u|^2\big) \in L^1(\Omega). \tag{2.10}$$

Now we will show that

$$\int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 dx \longrightarrow \int_\Omega \phi(|\nabla u|)|\nabla u|^2 dx. \tag{2.11}$$

First, notice that

$$\int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 dx = \int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n - u)dx + \int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla u dx. \tag{2.12}$$

Since the $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ is bounded, for each $i \in \{1, \ldots, N\}$, we have that $\left\{\phi(|\nabla u_n|)\frac{\partial u_n}{\partial x_i}\right\}$ is a bounded sequence in $L_{\tilde{\Phi}}(\Omega)$. Then from (2.7),

$$\phi(|\nabla u_n|)\frac{\partial u_n}{\partial x_i} \longrightarrow \phi(|\nabla u|)\frac{\partial u}{\partial x_i} \text{ a.e in } \Omega.$$

Using Brezis–Lieb for the N-functions (see for instance [18]), we conclude that

$$\phi(|\nabla u_n|)\frac{\partial u_n}{\partial x_i} \rightharpoonup \phi(|\nabla u|)\frac{\partial u}{\partial x_i} \text{ in } L_{\tilde{\Phi}}(\Omega).$$

Hence

$$\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla u dx \longrightarrow \int_\Omega \phi(|\nabla u|)|\nabla u|^2 dx. \tag{2.13}$$

On the other hand, since $\left\{\frac{\partial u_n}{\partial x_i}\right\}$ is bounded sequence in $L_\Phi(\Omega)$ and

$$\frac{\partial u_n}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \text{ a.e in } \Omega,$$

we have that $\dfrac{\partial u_n}{\partial x_i} \rightharpoonup \dfrac{\partial u}{\partial x_i}$ in $L_\Phi(\Omega)$, and so,

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla u_n dx \longrightarrow \int_\Omega \phi(|\nabla u|)|\nabla u|^2 dx. \tag{2.14}$$

By (2.4) and (2.14)

$$\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n - u)dx \longrightarrow 0. \tag{2.15}$$

Now, using (2.8)–(2.11) and the Generalized Lebesgue Theorem (see [22, Exercise 5.4.13]),

$$\int_\Omega \Phi(|\nabla u_n - \nabla u|)dx \longrightarrow 0.$$

Thanks to Δ_2, the last limit leads to $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$. This completes the proof. □

In [20] and [21], Lieberman has considered regularity results of weak solutions for quasilinear problems of the type

$$\begin{cases} \operatorname{div} A(x,u,\nabla u) + B(x,u,\nabla u) = 0, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases}$$

Supposing that (ϕ_1)–(ϕ_4) hold, we can apply the above mentioned result for $A = \Phi$ and $B = 0$ to prove the following result.

Lemma 2.3. *Assume that $(\phi_1)-(\phi_4)$ hold. If $h \in L^\infty(\Omega)$ and $u \in W_0^{1,\Phi}(\Omega)$ is a weak solution of*

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) = h & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \end{cases}$$

then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ that depends on $\bar{l}$ and $\overline{m}$. Moreover, there is a constant $K = K(\| h \|_\infty, \bar{l}, \overline{m}, \Omega) > 0$ such that

$$|u|_{1,\alpha} \le K,$$

and K goes to 0 when $\| h \|_\infty$ goes to 0.

Next, we recall some properties involving maximum principle. Having this in mind, we need the following definition.

Definition 2.1. Let $u, v \in W_0^{1,\Phi}(\Omega)$. We say that

$$-\Delta_\Phi u \le -\Delta_\Phi v \text{ in } \Omega$$

if

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla w dx \le \int_\Omega \phi(|\nabla v|)\nabla v \nabla w dx,$$

for all $w \in W_0^{1,\Phi}(\Omega)$, with $w \ge 0$.

Lemma 2.4 (Comparison Principle)**.** *Let $u, v \in W_0^{1,\Phi}(\Omega)$ with $-\Delta_\Phi u \le -\Delta_\Phi v$. Then, $u \le v$ in Ω.*

Proof. Once that $-\Delta_\Phi u \le -\Delta_\Phi v$ in Ω and $(u-v)^+ \in W_0^{1,\Phi}(\Omega)$, we have

$$\int_\Omega \phi(|\ \nabla u\ |)\nabla u \nabla (u-v)^+ dx \le \int_\Omega \phi(|\ \nabla v\ |)\nabla v \nabla (u-v)^+ dx.$$

In view of this fact, if $\Omega_1 = \{x \in \Omega; u(x) - v(x) \ge 0\}$, then

$$\begin{aligned}\int_{\Omega_1} \langle \phi(|\ \nabla u\ |)\nabla u \quad &- \quad \phi(|\ \nabla v\ |)\nabla v, \nabla(u-v)\rangle dx \\ &= \quad \int_\Omega \langle \phi(|\ \nabla u\ |)\nabla u - \phi(|\ \nabla v\ |)\nabla v, \nabla(u-v)^+\rangle dx \le 0.\end{aligned}$$

By (2.5) $\nabla(u-v) = 0$ a.e in Ω_1, and so $\nabla(u-v)^+ = 0$ a.e in Ω. Now, using the fact that $(u-v)^+ \in W_0^{1,\Phi}(\Omega)$, we get $(u-v)^+ = 0$ in Ω, consequently $u \le v,$ in Ω. □

The next result is a maximum principle whose proof follows as in Guedda and Véron [19], which forms an important role in our paper.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, and let $u \in C^1(\Omega)$ be a nonnegative function verifying*

$$-\Delta_\Phi u \ge 0 \ \textit{in}\ \Omega. \tag{2.16}$$

If $u \ne 0$, then u is positive in Ω. Moreover, if $u \in C^1(\Omega \cup \{x_0\})$, for some $x_0 \in \partial\Omega$, such that $u(x_0) = 0$; then

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν denotes the outer unit normal on $\partial\Omega$ at x_0. In particular, if $u \in C^1(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, there is $c > 0$ such that

$$\frac{\partial u}{\partial \nu}(y) \ge c \quad \forall y \in \partial\Omega.$$

3. Preliminary results

In this section, we denote by $f_a : \mathbb{R} \longrightarrow \mathbb{R}$ the continuous functions given by

$$f_a(t) = \begin{cases} f(t) - a & \text{if} \quad t \ge 0, \\ -a(t+1) & \text{if} \quad t \in [-1, 0], \\ 0 & \text{if} \quad t \le -1, \end{cases}$$

$0 < a < 1$, and $-a = \min_{t\in\mathbb{R}} f_a(t)$. Our intention is to prove the existence of a positive solution for the following auxiliary problem:

$$\text{(AP)} \qquad \begin{cases} -\Delta_\Phi u &= f_a(u) & \text{in} & \Omega, \\ u(x) &> 0 & \text{in} & \Omega, \\ u &= 0 & \text{on} & \partial\Omega, \end{cases}$$

because such a solution is also a solution of (P). Associated with (AP), we have the energy functional $I_a : W_0^{1,\Phi}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$I_a(u) = \int_\Omega \Phi(|\nabla u|)dx - \int_\Omega F_a(u)dx, \ u \in W_0^{1,\Phi}(\Omega),$$

where $F_a(t) = \int_0^t f_a(\tau)d\tau,\ t \in \mathbb{R}$. This functional is Fréchet differentiable (see [16, Lemma A3]) with its derivative I_a' given by

$$\langle I_a'(u), v\rangle = \int_\Omega \phi(|\nabla u|)\nabla u \nabla v dx - \int_\Omega f_a(u) v dx,\ v \in W_0^{1,\Phi}(\Omega).$$

The next two lemmas will be useful to prove that I_a verifies the mountain pass geometry.

Lemma 3.1. *There exists $r > 0$, such that if $\rho \in (0, r)$ and $||u|| = \rho$, then there exist $\alpha = \alpha(\rho) > 0$ and $a_1 = a_1(\rho)$ such that $I_a(u) \geq \alpha$ for all $a \in (0, a_1)$. Moreover, the constants r, ρ are independent of $a \in (0, a_1)$.*

Proof. Notice that $F_a(t) \leq \frac{1}{2\Lambda}\Phi(t) + C\frac{|t|^q}{q} + a/2$ for all $t \in \mathbb{R}$ and some $C > 0$. Therefore,

$$\begin{aligned} I_a(u) &= \int_\Omega \Phi(|\nabla u|)dx - \int_\Omega F_a(u)dx \\ &\geq \frac{1}{2}\int_\Omega \Phi(|\nabla u|)dx - \frac{C}{q}\int_\Omega |u|^q\, dx - \frac{a}{2}|\Omega| \\ &\geq \frac{1}{2}\zeta(||u||) - \frac{C}{q}||u||_q^q - \frac{a}{2}|\Omega|, \end{aligned}$$

where ζ was given in Lemma 2.1. Thus, there is $C_1 > 0$ verifying

$$I_a(u) \geq \frac{1}{2}\zeta(||u||) - C_1||u||^q - \frac{a}{2}|\Omega|.$$

Taking $||u|| = \rho$ with ρ small enough and using Lemma 2.1, we get

$$I_a(u) \geq \rho^m\big(1/2 - C_1\rho^{q-m}\big) - \frac{a}{2}|\Omega|.$$

Now, we fix $a_1 = a_1(\rho, q, m) > 0$ and $r > 0$ such that

$$\rho^m\big(1/2 - C_1\rho^{q-m}\big) - \frac{a}{2}|\Omega| \geq \frac{\rho^m\big(1/2 - C_1\rho^{q-m}\big)}{2} > 0 \quad \forall a \in (0, a_1) \quad \text{and} \quad \rho \in (0, r).$$

From this, $I_a(u) \geq \alpha > 0$ if $||u|| = \rho$, where $\alpha := \frac{\rho^m\left(1/2 - C_1\rho^{q-m}\right)}{2}$. □

Lemma 3.2. *There exists $v \in W_0^{1,\Phi}(\Omega)$ such that $||v|| > \rho$ and $I_a(v) < 0$, for all $a \in (0, a_1)$.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ be a function verifying

$$\varphi > 0 \text{ in } \quad \Omega \quad \text{and } ||\varphi|| = 1.$$

Note that for all $t > 1$,

$$\begin{aligned} I_a(t\varphi) &= \int_\Omega \Phi(|\nabla t\varphi|)dx - \int_\Omega F_a(t\varphi)dx \\ &= \int_\Omega \Phi(|\nabla t\varphi|)dx - \int_\Omega F(t\varphi)\, dx + a\int_\Omega t\varphi dx. \end{aligned}$$

By (f_4), there are $A_1, B_1 > 0$ verifying

$$F(t) \geq A_1|t|^\theta - B_1 \quad \forall t \in \mathbb{R}. \tag{3.1}$$

The last inequality together with Lemma 2.1 leads to

$$\begin{aligned} I_a(t\varphi) &\leq \kappa(t)\kappa\big(||\varphi||\big) - A_1 t^\theta \int_\Omega \varphi^\theta dx + ta\|\varphi\|_1 + B_1|\Omega| \\ &\leq t^m - A_1 t^\theta ||\varphi||_\theta^\theta + ta\|\varphi\|_1 + B_1|\Omega|, \end{aligned}$$

where κ was given in Lemma 2.1. Since $\theta > m$ and $a \in (0, a_1)$, we can fix $t_0 > 1$ large enough so that $I_a(v) < 0$, where $v = t_0\varphi \in W_0^{1,\Phi}(\Omega)$. □

In the sequel, we are going to study the boundedness of a (PS) sequence of I_a. To do this, we recall that (f_4) yields f_a satisfies the famous condition due to Ambrosetti–Rabinowitz, that is, there exist $\theta > m$ and $M \in \mathbb{R}$ such that

$$\theta F_a(t) \leq t f_a(t) + M \quad \forall t \in \mathbb{R}. \tag{3.2}$$

It is very important to point out that θ and M are also independent of $a \in (0, a_1)$.

Lemma 3.3. *The functional I_a satisfies the Palais-Smale condition for all $a > 0$.*

Proof. Let $\{u_n\}$ be a sequence in $W_0^{1,\Phi}(\Omega)$ such that $\{I_a(u_n)\}$ is bounded and $I_a'(u_n) \to 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that $|\langle I_a'(u_n), u_n\rangle| \leq ||u_n||$ for $n > n_0$. Thus,

$$\begin{aligned} -||u_n|| &\leq \langle I_a'(u_n), u_n\rangle \\ &= \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 dx - \int_\Omega f_a(u_n)u_n dx \\ &\leq m\int_\Omega \Phi(|\nabla u_n|)dx - \int_\Omega f_a(u_n)u_n dx, \end{aligned}$$

so

$$-||u_n|| - m\int_\Omega \Phi(|\nabla u_n|)dx \leq -\int_\Omega f_a(u_n)u_n dx. \tag{3.3}$$

On the other hand, as there exists $K > 0$ such that $|I_a(u_n)| \leq K$ for all $n = 1, 2, \ldots$, it follows that

$$\int_\Omega \Phi(|\nabla u_n|)dx - \int_\Omega F_a(u_n)dx \leq K \quad \forall n \in \mathbb{N}. \tag{3.4}$$

From (3.2) and (3.4),

$$\int_\Omega \Phi(|\nabla u_n|)dx - \frac{1}{\theta}\int_\Omega f_a(u_n)u_n + \frac{1}{\theta}M|\Omega| \leq K \quad \forall n \in \mathbb{N}. \tag{3.5}$$

Thereby, by (3.3) and (3.5),

$$\left(1 - \frac{m}{\theta}\right)\int_\Omega \Phi(|\nabla u_n|)dx - \frac{1}{\theta}||u_n|| \leq K - \frac{1}{\theta}M|\Omega|.$$

Therefore, by Lemma 2.1,

$$\left(1 - \frac{m}{\theta}\right)\zeta(||u_n||) - \frac{1}{\theta}||u_n|| \leq K - \frac{1}{\theta}M|\Omega|.$$

If $||u_n|| \to +\infty$, we must have

$$\left(1 - \frac{m}{\theta}\right)||u_n||^l\left(1 - \frac{1}{\theta - m}||u_n||^{1-l}\right) \leq K - \frac{1}{\theta}M|\Omega|,$$

for n large enough, which is a contradiction. This shows that $\{u_n\}$ is bounded in $W_0^{1,\Phi}(\Omega)$. Thus, without loss of generality, we may assume that $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$, and since that $B(t) = |t|^q$ verifies (2.3), we get $u_n \to u$ in $L^q(\Omega)$, $m < q < l^*$. By conditions on f,

$$\int_\Omega f_a(u_n)(u_n - u)dx \to 0.$$

Since

$$o_n(1) = I_a'(u_n)(u_n - u) = \int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n - u)dx + \int_\Omega f_a(u_n)(u_n - u)dx,$$

we have

$$\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n - u)dx \to 0. \tag{3.6}$$

The weak convergence $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$ yields

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla(u_n - u)dx \to 0. \tag{3.7}$$

From (3.6) and (3.7),

$$\int_\Omega \big(\phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u\big)\nabla(u_n - u)dx \to 0.$$

Therefore, by Lemma 2.2, $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$, which completes the proof. □

Lemma 3.4. *If $a \in (0, a_1)$, then* (P) *has a solution $u_a \in W_0^{1,\Phi}(\Omega)$ satisfying $I_a(u_a) \le C$ where $C = C(a_1, \theta, m, |\Omega|) > 0$.*

Proof. Lemmas 3.1, 3.2, and 3.3 guarantee that we can apply the Mountain Pass Theorem due to Ambrosetti–Rabinowitz [5] to show the existence of a solution $u_a \in W_0^{1,\Phi}(\Omega)$ for all $a \in (0, a_1)$ with $I_a(u_a) = d_a > 0$, where d_a is the mountain pass level associated with I_a.

Now, letting $\varphi \in C_0^\infty(\Omega)$, $\varphi \ge 0$, $t > 0$, and using (3.1), we obtain

$$\begin{aligned} I_a(t\varphi) &= \int_\Omega \Phi(|\nabla t\varphi|)dx - \int_\Omega F_a(t\varphi)dx \\ &\le \kappa(t)\kappa\big(||\varphi||\big) - A_1 \int_\Omega (t\varphi)^\theta dx + B_1|\Omega| + \int_\Omega at\varphi dx \ \forall\ a \in (0, a_1). \end{aligned}$$

Then,

$$I_a(t\varphi) \le C_1\kappa(t) - C_2 t^\theta + C_3 t + C_4,$$

where $C_1 = \kappa(||\varphi||)$, $C_2 = A_1||\varphi||_\theta^\theta$, $C_3 = a_1\|\varphi\|_1$, and $C_4 = B_1|\Omega|$. Setting $h(t) = C_1\kappa(t) - C_2 t^\theta + C_3 t + C_4$, we find

$$d_a \le \max\{I_a(t\varphi); t \ge 0\} \le \max_{t\ge 0} h(t) = C\big(a_1, m, \theta, |\Omega|\big) > 0.$$

Thus, $I_a(u_a) \le C\big(a_1, m, \theta, |\Omega|\big)$. □

The next lemma establishes a very important estimate involving the solution u_a for $a \in (0, a_1)$.

Lemma 3.5. *There exists $K = K(a_1, q, m, |\Omega|) > 0$ such that $||u_a|| \le K$ for all $a \in (0, a_1)$.*

Proof. To begin with, recall that

$$\begin{aligned} C\big(a_1, m, q, |\Omega|\big) &\ge I_a(u_a) - \frac{1}{\theta} I_a'(u_a)u_a \\ &= \left(\int_\Omega \Phi(|\nabla u_a|)dx - \frac{1}{\theta}\int_\Omega \phi(|\nabla u_a|)|\nabla u_a|^2 dx\right) \\ &\quad + \int_\Omega \left(\frac{1}{\theta} f_a(u_a)u_a - F_a(u_a)\right) dx. \end{aligned}$$

From (3.2),

$$C\big(a_1, m, q, |\Omega|\big) \ge \left(1 - \frac{m}{\theta}\right)\int_\Omega \Phi(|\nabla u_a|)dx - \frac{M}{\theta}|\Omega|.$$

Now the Δ_2-condition combines with the last inequality to gives

$$||u_a|| \leq K \quad \forall a \in (0, a_1),$$

for some $K = K(a_1, q, m, |\Omega|)$. □

Our next result ensures that u_a belongs to $L^\infty(\Omega)$ and that the family $\{u_a\}$ is a bounded set in $L^\infty(\Omega)$ for a small enough. This fact is crucial in our approach.

Lemma 3.6. *There is $a_2 \in (0, a_1)$ such that $u_a \in L^\infty(\Omega)$ for all $a \in (0, a_2)$. Moreover, there is $C > 0$ such that*

$$\|u_a\|_\infty \leq C \quad \forall a \in (0, a_2).$$

Proof. In order to prove the lemma, it is enough to show that for any sequence $a_j \to 0$, the sequence of solution $u_j = u_{a_j}$ possesses a subsequence, still denoted by itself, which is bounded in $L^\infty(\Omega)$. As f has a subcritical growth, a straightforward computation implies that there is a subsequence of $\{u_j\}$, still denoted by itself, and $u \in W_0^{1,\Phi}(\Omega)$ such that $u_j \to u$ in $W_0^{1,\Phi}(\Omega)$. Now, the same type of arguments explored in [24, Theorem 3.1] yield $u_j \in L^\infty(\Omega)$ for all $j \in \mathbb{N}$ and $\|u_j\|_\infty \leq C \quad \forall j \in \mathbb{N}$, for some $C > 0$. □

In what follows, we show an estimate from below of the norm $L^\infty(\Omega)$ of u_a for a small enough.

Lemma 3.7. *There exists $a_3 \in (0, a_2)$ and $\delta > 0$ that does dependent on $a \in (0, a_3)$ such that $||u_a||_\infty \geq \delta$ for all $a \in (0, a_2)$.*

Proof. If u_a is a solution of (P), then

$$\int_\Omega \phi(|\nabla u_a|)\nabla u_a \nabla \varphi dx = \int_\Omega f_a(u_a)\varphi dx \quad \forall \varphi \in W_0^{1,\Phi}(\Omega).$$

For $\varphi = u_a$,

$$\int_\Omega f_a(u_a)u_a dx = \int_\Omega \phi(|\nabla u_a|)|\nabla u_a|^2 dx \geq l \int_\Omega \Phi(|\nabla u_a|)dx.$$

By the Δ_2-condition,

$$\int_\Omega \Phi(|\nabla u_a|)\, dx \to 0 \quad \text{as} \quad a \to 0 \Longleftrightarrow \|u_a\| \to 0 \quad \text{as} \quad a \to 0.$$

Since $I_a(u_a) \geq \alpha$ for all $a \in (0, a_2)$, there exists $a_3 \in (0, a_2)$ and $\alpha_0 > 0$ such that

$$\int_\Omega \Phi(|\nabla u_a|)dx \geq \alpha_0 \quad \forall a \in (0, a_3). \tag{3.8}$$

Thus,

$$\int_\Omega f_a(u_a)u_a dx \geq l\alpha := A > 0 \quad \forall a \in (0, a_3).$$

From conditions on f, there is $C > 0$ such that

$$|f_a(t)| \leq C(|t|^q + \phi(|t|)|t|) + a \quad \forall t \in \mathbb{R} \quad \text{and} \quad \forall a \in (0, a_3).$$

Therefore,

$$A \leq \int_\Omega \big(C(|u_a|^q + \phi(|u_a|)|u_a|) + a|u_a|\big)dx \leq \big(C(||u_a||_\infty^{q-1} + \phi(\|u_a\|_\infty)) + a\big)\, \|u_a\|_\infty |\Omega|.$$

This implies that $||u_a||_\infty \geq \delta$ for some $\delta > 0$ for all $a \in (0, a_3)$, decreasing a_3 if necessary. □

4. Proof of Theorem 1.1

In order to conclude the proof of Theorem 1.1, we need to show that the solution u_a is positive for $a \in (0, a_3)$, decreasing a_3 if necessary. Indeed, let $\{a_j\} \subset (0, a_3)$ be a sequence with $a_j \to 0$ as $j \to \infty$, and let u_j be a solution of (P) with $a = a_j$. Setting $f_j(u_j) = f_{a_j}(u_j)$, we have

$$\begin{cases} -\Delta_\Phi u_j &= f_j(u_j) \quad \text{in} \quad \Omega, \\ u_j &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

By Lemma 3.6, there is $C > 0$ such that $||u_j||_\infty \leq C$ for all $j \in \mathbb{N}$; then $||f_j(u_j)||_\infty \leq C_1$ for all $j \in \mathbb{N}$ and some $C_1 > 0$. Thereby, by Lemma 2.3, $u_j \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$, and

$$|u_j|_{1,\beta} \leq M(||f_j(u_j)||_\infty) \leq M_2, \quad \forall j \in \mathbb{N},$$

for some $M_2 > 0$. From compactness embedding $C^{1,\beta}(\overline{\Omega}) \hookrightarrow C^{1,\tau}(\overline{\Omega})$ for $\tau \in (0, \beta)$, there exists a subsequence of $\{u_j\}$, still denoted by $\{u_j\}$, and $u \in C^{1,\tau}(\overline{\Omega})$ such that $u_j \to u$ in $C^{1,\tau}(\overline{\Omega})$. Let v_j be the solution of problem

$$\begin{cases} -\Delta_\Phi v_j &= k_j \quad \text{in} \quad \Omega, \\ v_j &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $k_j = \min\{f_j(t); t \in \mathbb{R}\} = -a_j \to 0^-$ as $j \to \infty$. Then, $u_j, v_j \in W_0^{1,\Phi}(\Omega)$ and

$$-\Delta_\Phi v_j \leq -\Delta_\Phi u_j \quad \text{in} \quad \Omega,$$

and by the comparison principle

$$v_j \leq u_j \text{ in } \Omega \quad \forall j \in \mathbb{N}. \tag{4.1}$$

On the other hand, as $k_j \to 0$, Lemma 2.3 gives

$$||v_j||_\infty \to 0. \tag{4.2}$$

Then, (4.1) combines with (4.2) to give $u \geq 0$ in Ω. Notice that

- $\nabla u_j(x) \to \nabla u(x)$ uniformly in $\overline{\Omega}$,
- $\{f_j(u_j)\}$ is bounded in $L^s(\Omega), s > 1$,
- $f_j(u_j) \rightharpoonup z$ in $L^s(\Omega)$,
- $f_j(u_j(x)) \to f_0(u(x))$ a.e. $x \in \Omega$,

where $f_0(t) = f(t)$ if $t \geq 0$, and $f_0(t) = 0$ if $t < 0$. Having this in mind, we deduce that $z = f_0(u)$, and for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \int_\Omega \phi(|\nabla u|)\nabla u \nabla\varphi \, dx &= \lim_{j\to+\infty} \int_\Omega \phi(|\nabla u_j|)\nabla u_j \nabla\varphi \, dx \\ &= \lim_{j\to+\infty} \int_\Omega f_j(u_j)\varphi dx = \int_\Omega z\varphi dx. \end{aligned}$$

Consequently,

$$\begin{cases} -\Delta_\Phi u &= z \quad \text{in} \quad \Omega, \\ u &\geq 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

As $||u_j||_\infty \geq C_0$ for all $j \in \mathbb{N}$, we derive that $||u||_\infty \geq C_0$, and so $u \neq 0$. As $z \geq 0$, by Theorem 2.1,

$$u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \eta} < 0 \text{ on } \partial\Omega,$$

where $\partial/\partial\eta$ denotes the exterior normal derivative. This information together with the limit

$$u_j \to u \text{ in } C^{1,\tau}(\overline{\Omega})$$

leads to $u_j(x) > 0$, $x \in \Omega$, for j large enough. Decreasing a_3 if necessary, the above analysis guarantees that u_a is positive for $a \in (0, a_3)$. This completes the proof of Theorem 1.1.

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References

[1] Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, 2nd ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003. MR2424078

[2] Ismael Ali, Alfonso Castro, and R. Shivaji, *Uniqueness and stability of nonnegative solutions for semipositone problems in a ball*, Proc. Amer. Math. Soc. **117** (1993), no. 3, 775–782, DOI 10.2307/2159143. MR1116249

[3] A. Ambrosetti, D. Arcoya, and B. Buffoni, *Positive solutions for some semi-positone problems via bifurcation theory*, Differential Integral Equations **7** (1994), no. 3-4, 655–663. MR1270096

[4] W. Allegretto, P. Nistri, and P. Zecca, *Positive solutions of elliptic nonpositone problems*, Differential Integral Equations **5** (1992), no. 1, 95–101. MR1141729

[5] Antonio Ambrosetti and Paul H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381. MR0370183

[6] V. Anuradha, D. D. Hai, and R. Shivaji, *Existence results for superlinear semipositone BVP's*, Proc. Amer. Math. Soc. **124** (1996), no. 3, 757–763, DOI 10.1090/S0002-9939-96-03256-X. MR1317029

[7] Scott Caldwell, Alfonso Castro, Ratnasingham Shivaji, and Sumalee Unsurangsie, *Positive solutions for classes of multiparameter elliptic semipositone problems*, Electron. J. Differential Equations (2007), No. 96, 10. MR2328697

[8] Alfonso Castro, Djairo G. de Figueredo, and Emer Lopera, *Existence of positive solutions for a semipositone p-Laplacian problem*, Proc. Roy. Soc. Edinburgh Sect. A **146** (2016), no. 3, 475–482, DOI 10.1017/S0308210515000657. MR3507282

[9] Alfonso Castro and R. Shivaji, *Nonnegative solutions for a class of nonpositone problems*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), no. 3-4, 291–302, DOI 10.1017/S0308210500014670. MR943804

[10] M. Chhetri, P. Drábek, and R. Shivaji, *Existence of positive solutions for a class of p-Laplacian superlinear semipositone problems*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 5, 925–936, DOI 10.1017/S0308210515000220. MR3406455

[11] David G. Costa, Humberto Ramos Quoirin, and Hossein Tehrani, *A variational approach to superlinear semipositone elliptic problems*, Proc. Amer. Math. Soc. **145** (2017), no. 6, 2661–2675, DOI 10.1090/proc/13426. MR3626519

[12] David G. Costa, Hossein Tehrani, and Jianfu Yang, *On a variational approach to existence and multiplicity results for semipositone problems*, Electron. J. Differential Equations (2006), No. 11, 10. MR2198924

[13] Gianni Dal Maso and François Murat, *Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems*, Nonlinear Anal. **31** (1998), no. 3-4, 405–412, DOI 10.1016/S0362-546X(96)00317-3. MR1487552

[14] Thomas K. Donaldson and Neil S. Trudinger, *Orlicz-Sobolev spaces and imbedding theorems*, J. Functional Analysis **8** (1971), 52–75. MR0301500

[15] Abdou K. Drame and David G. Costa, *On positive solutions of one-dimensional semipositone equations with nonlinear boundary conditions*, Appl. Math. Lett. **25** (2012), no. 12, 2411–2416, DOI 10.1016/j.aml.2012.07.015. MR2967854

[16] Nobuyoshi Fukagai, Masayuki Ito, and Kimiaki Narukawa, *Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on* $\mathbf{R}^N$, Funkcial. Ekvac. **49** (2006), no. 2, 235–267, DOI 10.1619/fesi.49.235. MR2271234

[17] Nobuyoshi Fukagai and Kimiaki Narukawa, *On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems*, Ann. Mat. Pura Appl. (4) **186** (2007), no. 3, 539–564, DOI 10.1007/s10231-006-0018-x. MR2317653

[18] J.P. Gossez, *Orlicz-Sobolev spaces and nonlinear elliptic boundary value problems.* Nonlinear Analysis, Function Spaces and Applications. Leipzig: BSB B. G. Teubner Verlagsgesellschaft, 1979. 59-94. <http://eudml.org/doc/220389>

[19] Mohammed Guedda and Laurent Véron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13** (1989), no. 8, 879–902, DOI 10.1016/0362-546X(89)90020-5. MR1009077

[20] Gary M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219, DOI 10.1016/0362-546X(88)90053-3. MR969499

[21] Gary M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural′tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), no. 2-3, 311–361, DOI 10.1080/03605309108820761. MR1104103

[22] Inder K. Rana, *An introduction to measure and integration*, 2nd ed., Graduate Studies in Mathematics, vol. 45, American Mathematical Society, Providence, RI, 2002. MR1934675

[23] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146, Marcel Dekker, Inc., New York, 1991. MR1113700

[24] Zhong Tan and Fei Fang, *Orlicz-Sobolev versus Hölder local minimizer and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **402** (2013), no. 1, 348–370, DOI 10.1016/j.jmaa.2013.01.029. MR3023263

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