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A TWO-PARAMETER CLASS OF COMPLETELY MONOTONIC FUNCTIONS

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ABSTRACT. Let $b \in \mathbb{R}$, let $c > 0$, let $x > 0$, and let

$$G_{b,c}(x) = \frac{e^{-x}}{x^b} P_c(x) \quad \text{with} \quad P_c(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(c+k)}.$$

We prove that $G_{b,c}$ is completely monotonic on $(0, \infty)$ if and only if $b \geq 0$ and $b + c \geq 1$. Moreover, we present various functional inequalities for P_c. Among others, we show that if $c \in (0, 1)$, then, for $x, y > 0$ we have

$$e < \frac{P_c(1/x)^x \, P_c(1/y)^y}{P_c(1/(x+y))^{x+y}}.$$

If $c > 1$, then the reverse inequality holds for $x, y > 0$. In both cases, the constant bound e is best possible.

1. Introduction

Inspired by the work of Gautschi [7] and other mathematicians, in 2000 Elbert and Laforgia [5] published the interesting inequality

$$(1.1) \qquad 1 - \frac{v(x^p)}{p+1} < \frac{1}{x} \int_0^x e^{-t^p} dt, \quad p > 1, \, 0 < x < \Big(\frac{9(3p+1)}{4(2p+1)}\Big)^{1/p},$$

where

$$v(x) = \int_0^x \frac{1 - e^{-t}}{t} dt = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k \cdot k!}.$$

The constant factor $1/(p+1)$ is best possible. The integral in (1.1) has a close connection to Euler's gamma function and the incomplete gamma function

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt.$$

We have

$$\int_0^x e^{-t^p} dt = \frac{1}{p} \big(\Gamma(1/p) - \Gamma(1/p, x^p) \big).$$

In 2006, Laforgia and Natalini [10] extended (1.1) to include all $x > 0$. Since their proof is rather long, it might be of interest to present the following short and simple proof which reveals that (1.1) is valid not only for $p > 1$ and $x > 0$ but even for $p > 0$ and $x > 0$.

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We define for $p > 0$ and $x > 0$,

$$F_p(x) = \int_0^x e^{-t^p}\,dt - x + \frac{xv(x^p)}{p+1} = \int_0^x e^{-t^p}\,dt - x + \frac{px}{p+1}\int_0^x \frac{1-e^{-t^p}}{t}dt.$$

Then,

$$F_p'(x) = \frac{1}{p+1}(e^{-x^p}-1) + \frac{p}{p+1}\int_0^x \frac{1-e^{-t^p}}{t}dt$$

and

$$F_p''(x) = \frac{pe^{-x^p}}{(p+1)x}\bigl(e^{x^p}-1-x^p\bigr). \tag{1.2}$$

Using

$$e^t - 1 - t > 0 \quad \text{for} \quad t > 0$$

yields $F_p''(x) > 0$. Since $F_p(0) = F_p'(0) = 0$, we conclude that (1.1) holds.

From (1.2), we obtain the integral representation

$$\frac{1}{x}\int_0^x e^{-t^p}\,dt - 1 + \frac{v(x^p)}{p+1} = \frac{p}{(p+1)x}\int_0^x\int_0^t \frac{e^{-s^p}}{s}\sum_{k=2}^{\infty}\frac{s^{pk}}{k!}\,ds\,dt.$$

This leads to a one-line proof of (1.1).

The fact that the second derivative of F_p is positive on $(0,\infty)$ plays a central role in our proof. In a private communication, Professor M. E. H. Ismail asked for properties of higher derivatives of F_p. This inspired us to look for all positive parameters p such that the function

$$H_p(x) = \frac{e^{-x^p}}{x}\bigl(e^{x^p}-1-x^p\bigr)$$

is completely monotonic on $(0,\infty)$. We solve this problem in Section 3, Corollary 3.2.

A function f is called completely monotonic on $(0,\infty)$ if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \ge 0, \quad n = 0,1,2,\ldots,\ x > 0. \tag{1.3}$$

Dubourdieu [4] showed that if f is a nonconstant completely monotonic function, then strict inequality holds in (1.3).

Completely monotonic functions play an important role in probability theory, numerical analysis, the theory of special functions, and other branches. The main properties of these functions can be found in Widder [17, chapter IV]. In the recent past, many papers on this subject have been published. The authors proved that certain functions which are defined in terms of the gamma and other classical functions are completely monotonic. We refer to Alzer and Berg [1], [2], Bustoz and Ismail [3], Ismail and Laforgia [8], Ismail et al. [9], Miller and Samko [13], Simon [16], and the references cited therein.

In this paper, we study the monotonicity behaviour of the function

$$G_{b,c}(x) = \frac{e^{-x}}{x^b}P_c(x) \quad \text{with} \quad P_c(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(c+k)}, \quad b\in\mathbb{R},\ c>0,$$

which is a solution of the first-order linear differential equation

$$y' + \frac{b+c-1}{x}y = \frac{c-1}{\Gamma(c)}\,\frac{e^{-x}}{x^{b+1}}\,.$$

In Section 3, we provide necessary and sufficient conditions for b and c such that $G_{b,c}$ is completely monotonic on $(0,\infty)$. As an application, we show that the function

$$x \mapsto e^{-x}\sum_{k=0}^{\infty} a_k x^k \quad \text{with} \quad a_k = \int_{\alpha}^{\beta} \frac{\phi(t)}{(t+1)\cdots(t+k+1)}\,dt$$

is completely monotonic on $(0,\infty)$. Moreover, we discuss the complete monotonicity of $c \mapsto \Gamma(c)P_c(x)$. In Section 4, we offer concavity and convexity properties of P_c and closely related functions and use them to prove various functional inequalities. Among others, we show that if $c \in (0,1)$, then, for $x, y > 0$, we have

$$e < \frac{P_c(1/x)^x\, P_c(1/y)^y}{P_c(1/(x+y))^{x+y}}\,.$$

The constant lower bound e is best possible.

Throughout, we maintain the notation introduced in this section.

2. Lemmas

Here, we collect some lemmas which we need to prove the theorems given in Sections 3 and 4. The first two lemmas provide basic properties of completely monotonic functions. Proofs are given in Fink [6], Lorch and Newman [12], Miller and Samko [13], and Widder [17, p. 167].

Lemma 2.1.

(i) *If f and g are completely monotonic, then $f+g$ and $f\cdot g$ are completely monotonic.*

(ii) *Let $f, g : (0,\infty) \to (0,\infty)$. If f and g' are completely monotonic, then $f \circ g$ is completely monotonic.*

Lemma 2.2. *If f is positive and completely monotonic, then f is log-convex.*

The next lemma gives an elegant functional inequality for convex functions which was proved by Petrović [15]; see also Mitrinović [14, section 1.4.7].

Lemma 2.3. *If f is convex on $[0,\infty)$, then, for all $x, y \geq 0$,*

$$f(x) + f(y) \leq f(0) + f(x+y). \tag{2.1}$$

If f is convex on $(-\infty, 0]$, then (2.1) holds for all $x, y \leq 0$. Moreover, if f is strictly convex, then strict inequality is valid for $x, y > 0$ and $x, y < 0$, respectively.

The following lemma is due to Lorch [11].

Lemma 2.4. *If f is log-convex and $\delta > 0$, then $x \mapsto f(x+\delta)/f(x)$ is increasing.*

Finally, we establish a convexity result for the power series which might be of independent interest.

Lemma 2.5. *Let $(b_k)_{k\geq 0}$ be a sequence of nonnegative real numbers such that*

$$b_k\, b_{n+2-k} \leq b_{k+1}\, b_{n+1-k}, \quad 0 \leq k \leq [n/2]. \tag{2.2}$$

If the power series

$$\phi(x) = \sum_{k=0}^{\infty} b_k x^k$$

is convergent and positive on $[0,R)$, then $1/\phi$ is convex on $[0,R)$. If strict inequality holds in (2.2), then $1/\phi$ is strictly convex.

Proof. We have

$$\left(\frac{1}{\phi}\right)'' = \frac{1}{\phi^3}\left(2\phi'^2 - \phi\phi''\right). \tag{2.3}$$

By using termwise differentiation, we obtain for $x \in [0, R)$,
(2.4)

$$2\phi'(x)^2 - \phi(x)\phi''(x) = 2\Big(\sum_{k=1}^{\infty} kb_k x^{k-1}\Big)^2 - \sum_{k=0}^{\infty} b_k x^k \sum_{k=2}^{\infty}(k-1)kb_k x^{k-2} = \sum_{n=0}^{\infty} A_n x^n$$

with

$$A_n = \sum_{k=0}^{[n/2]+1} \lambda_{n,k}\, b_k\, b_{n+2-k}$$

and

$$\lambda_{n,k} = \begin{cases} -6k^2 + 6(n+2)k - (n+1)(n+2), & \text{if} \quad 0 \le k \le (n+1)/2, \\ n^2/4 + 3n/2 + 2, & \text{if} \quad k = n/2 + 1. \end{cases}$$

By direct computation we find

$$\sum_{k=0}^{[n/2]+1} \lambda_{n,k} = 0. \tag{2.5}$$

In order to prove that the coefficients A_n are nonnegative we consider two cases.

Case 1. n is odd.
Then,

$$\lambda_{n,0} < \lambda_{n,1} < \cdots < \lambda_{n,[n/2]+1} \quad \text{and} \quad \lambda_{n,0} < 0 < \lambda_{n,[n/2]+1}.$$

Let $k_0 \in \{0, 1, \ldots, [n/2]\}$ be the largest integer such that $\lambda_{n,k_0} \le 0$. Using (2.2) gives

$$b_0\, b_{n+2} \le b_1\, b_{n+1} \le \cdots \le b_{[n/2]+1}\, b_{n+1-[n/2]}. \tag{2.6}$$

If follows that the inequality

$$\lambda_{n,k}\, b_k\, b_{n+2-k} \ge \lambda_{n,k}\, b_{k_0+1}\, b_{n+1-k_0}$$

is valid for $k = 0, \ldots, k_0$ and $k = k_0 + 1, \ldots, [n/2] + 1$. Thus, for $n \ge 0$,

$$A_n \ge \sum_{k=0}^{[n/2]+1} \lambda_{n,k}\, b_{k_0+1}\, b_{n+1-k_0}.$$

Applying (2.4) gives $A_n \ge 0$.

Case 2. n is even.
We have $A_0 = 2(b_1^2 - b_0 b_2) \ge 0$. Let $n \ge 2$. Then,

$$\lambda_{n,0} < \lambda_{n,1} < \cdots < \lambda_{n,[n/2]} \quad \text{and} \quad \lambda_{n,0} < 0 < \lambda_{n,[n/2]}.$$

Let $k_1 \in \{0, 1, \ldots, [n/2] - 1\}$ be the largest integer such that $\lambda_{n,k_1} \le 0$. From (2.2), we obtain

$$\lambda_{n,k}\, b_k\, b_{n+2-k} \ge \lambda_{n,k}\, b_{k_1+1}\, b_{n+1-k_1} \tag{2.7}$$

for $k = 0, \ldots, k_1$ and $k = k_1 + 1, \ldots, [n/2]$. Since $\lambda_{n,[n/2]+1} > 0$ and

$$b_{k_1+1}\, b_{n+1-k_1} \le b_{[n/2]+1}\, b_{n+1-[n/2]},$$

we conclude that (2.7) holds also for $k = [n/2] + 1$. It follows that

$$A_n \geq \sum_{k=0}^{[n/2]+1} \lambda_{n,k}\, b_{k_1+1}\, b_{n+1-k_1} = 0.$$

Using (2.3) and (2.4) reveals that $1/\phi$ is convex on $[0, R)$. If (2.2) holds with $<$ instead of $\leq$, then we obtain $A_n > 0$ for $n \geq 0$. This implies that $1/\phi$ is strictly convex. □

3. Complete monotonicity

The following theorem is the main result of this paper.

Theorem 3.1. *Let b and $c > 0$ be real numbers. The function $G_{b,c}$ is completely monotonic on $(0, \infty)$ if and only if $b \geq 0$ and $b + c \geq 1$.*

Proof. First, we show that if $b \geq 0$ and $b+c \geq 1$, then $G_{b,c}$ is completely monotonic. We consider two cases.

Case 1. $0 < c \leq 1$.

We have

$$G_{b,c}(x) = \frac{1}{x^{b+c-1}} \Phi_c(x) \quad \text{with} \quad \Phi_c(x) = x^{c-1} e^{-x} P_c(x).$$

An application of Lemma 2.1(i) reveals that it suffices to show that Φ_c is completely monotonic.

By differentiation we get

$$x^{2-c} e^x \Phi_c'(x) = (c-1) P_c(x) + x\big(P_c'(x) - P_c(x)\big). \tag{3.1}$$

We have

$$\begin{aligned} P_c'(x) &= \sum_{k=1}^{\infty} \frac{k x^{k-1}}{\Gamma(c+k)} \\ &= \sum_{k=0}^{\infty} (k + c + 1 - c) \frac{x^k}{\Gamma(c+k+1)} \\ &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(c+k)} + (1-c) \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(c+k+1)} \\ &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(c+k)} + \frac{1-c}{x} \Big(\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(c+k)} - \frac{1}{\Gamma(c)} \Big) \\ &= P_c(x) + \frac{1-c}{x} \Big(P_c(x) - \frac{1}{\Gamma(c)} \Big). \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$x^{2-c} e^x \Phi_c'(x) = (c-1) P_c(x) + (1-c) \Big(P_c(x) - \frac{1}{\Gamma(c)} \Big) = \frac{c-1}{\Gamma(c)}.$$

It follows that

$$-\Phi_c'(x) = \frac{1-c}{\Gamma(c)} \cdot \frac{1}{x^{2-c}} \cdot e^{-x}.$$

This means that $-\Phi_c'$ is the product of completely monotonic functions. Thus, $-\Phi_c'$ is completely monotonic, and so is Φ_c.

Case 2. $c > 1$.

We have

$$G_{b,c}(x) = \frac{1}{x^b}\Theta_c(x) \quad \text{with} \quad \Theta_c(x) = e^{-x}P_c(x).$$

From Lemma 2.1(i), we conclude that it is enough to prove that Θ_c is completely monotonic.

We have

$$\begin{aligned} P_c(x) &= \frac{1}{\Gamma(c)} + \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(c+k)} \\ &= \frac{1}{\Gamma(c)} + \frac{x}{\Gamma(c)} \sum_{k=1}^{\infty} B(c,k) \frac{x^{k-1}}{(k-1)!} \\ &= \frac{1}{\Gamma(c)} + \frac{x}{\Gamma(c)} \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} \int_0^1 t^{c-1}(1-t)^{k-1} dt \\ &= \frac{1}{\Gamma(c)} + \frac{x}{\Gamma(c)} \int_0^1 t^{c-1} \sum_{k=0}^{\infty} \frac{(x(1-t))^k}{k!} dt \\ &= \frac{1}{\Gamma(c)} + \frac{x}{\Gamma(c)} \int_0^1 t^{c-1} e^{x(1-t)} dt. \end{aligned} \tag{3.3}$$

The substitution $t = s/x$ leads to

$$P_c(x) = \frac{1}{\Gamma(c)} + \frac{e^x}{x^{c-1}\Gamma(c)} \int_0^x s^{c-1}e^{-s} ds.$$

Thus,

$$\Theta_c(x)\Gamma(c) = e^{-x} + \frac{1}{x^{c-1}} \int_0^x s^{c-1}e^{-s} ds. \tag{3.4}$$

By induction on N we can easily prove that for $N \geq 0$ we have the representation

$$(-1)^N \Theta_c^{(N)}(x)\Gamma(c) = e^{-x} + \frac{1}{x^{c+N-1}} \int_0^x s^{c+N-1}e^{-s} ds - \frac{N}{x^{c+N-1}} \int_0^x s^{c+N-2}e^{-s} ds.$$

Let

$$Q_{c,N}(x) = x^{c+N-1}(-1)^N \Theta_c^{(N)}(x)\Gamma(c). \tag{3.5}$$

Then,

$$Q_{c,N}(x) = x^{c+N-1}e^{-x} + \int_0^x (s-N)s^{c+N-2}e^{-s} ds$$

and

$$Q'_{c,N}(x) = (c-1)x^{c+N-2}e^{-x} > 0.$$

This yields

$$Q_{c,N}(x) > Q_{c,N}(0) = 0 \quad \text{for} \quad x > 0. \tag{3.6}$$

From (3.5) and (3.6), we conclude that Θ_c is completely monotonic.

Next, we assume that $G_{b,c}$ is completely monotonic. If $b < 0$, then

$$\lim_{x \to 0} G_{b,c}(x) = 0.$$

But, $G_{b,c}$ is positive and decreasing on $(0, \infty)$. Hence, $b \geq 0$.

From (3.4), we obtain the limit relation

$$\lim_{x\to\infty} x^{c-1}\Theta_c(x) = 1. \tag{3.7}$$

Hence, if $b+c<1$, then

$$\lim_{x\to\infty} G_{b,c}(x) = \lim_{x\to\infty} x^{1-b-c}\cdot x^{c-1}\Theta_c(x) = \infty.$$

A contradiction! Thus, $b+c\ge 1$. This completes the proof of Theorem 3.1. □

We are now in a position to solve the monotonicity problem posed in Section 1.

Corollary 3.2. *The function H_p $(p>0)$ is completely monotonic on $(0,\infty)$ if and only if $p\le 1/2$.*

Proof. We first assume that $0<p\le 1/2$. Let $f=G_{1/p-2,3}$ and $g(x)=x^p$. We use Theorem 3.1 and Lemma 2.1(ii). Since

$$H_p(x) = G_{1/p-2,3}(x^p),$$

we conclude that H_p is completely monotonic.

Conversely, if H_p is completely monotonic, then, for $x>0$,

$$x^2e^{x^p}H_p'(x) = 1+x^p+px^{2p}-e^{x^p} = (p-1/2)x^{2p} - \sum_{k=3}^{\infty}\frac{x^{kp}}{k!} \le 0.$$

It follows that $p\le 1/2$. □

Next, we apply Theorem 3.1 to obtain another class of completely monotonic functions. Let $-1<\alpha<\beta$, and let $\phi:(\alpha,\beta)\to(0,\infty)$ be a function such that

$$a_k = \int_\alpha^\beta \frac{\phi(t)}{(t+1)\cdots(t+k+1)}dt \tag{3.8}$$

is defined for all integers $k\ge 0$.

Theorem 3.3. *The function*

$$S_\phi(x) = e^{-x}\sum_{k=0}^{\infty} a_k x^k$$

is completely monotonic on $(0,\infty)$.

Proof. Let $-1<\alpha<t<\beta$, and let $x>0$. Then,

$$\Gamma(t+1)\phi(t)G_{0,t+2}(x) = e^{-x}\sum_{k=0}^{\infty}\frac{\phi(t)x^k}{(t+1)\cdots(t+k+1)}.$$

Integration gives

$$\int_\alpha^\beta \Gamma(t+1)\phi(t)dt\, G_{0,t+2}(x) = e^{-x}\sum_{k=0}^{\infty} a_k x^k.$$

Since $t>-1$, we conclude from Theorem 3.1 that S_ϕ is completely monotonic on $(0,\infty)$. □

As a specific example of Theorem 3.3 we offer the following result.

Theorem 3.4. *Let*

$$a_k^* = \sum_{j=1}^{k+1} (-1)^{j+1} \frac{\log(j+1)}{j!(k+1-j)!}, \quad k = 0, 1, 2, \ldots .$$

The function

$$S^*(x) = e^{-x} \sum_{k=0}^{\infty} a_k^* x^k$$

is completely monotonic on $(0, \infty)$.

Proof. Since

$$\frac{1}{t(t+1)\cdots(t+k+1)} = \sum_{j=0}^{k+1} \frac{u_j(k)}{t+j} \quad \text{with} \quad u_j(k) = \frac{(-1)^j}{j!(k+1-j)!},$$

we obtain for $x > 1$,

$$\begin{aligned} \int_1^x \frac{1}{t(t+1)\cdots(t+k+1)} dt &= \int_1^x \sum_{j=0}^{k+1} \frac{u_j(k)}{t+j} dt \\ &= \sum_{j=0}^{k} U_j(k) \log\Big(\frac{x+j}{x+j+1}\Big) - \sum_{j=1}^{k+1} u_j(k) \log(j+1) \end{aligned}$$

with

$$U_j(k) = \sum_{\nu=0}^{j} u_\nu(k).$$

Thus, for $k = 0, 1, 2, \ldots$,

$$\int_1^\infty \frac{1}{t(t+1)\cdots(t+k+1)} dt = -\sum_{j=1}^{k+1} u_j(k) \log(j+1) = a_k^*. \tag{3.9}$$

We apply (3.8) with $\phi(t) = 1/t$ and $\alpha = 1$, $\beta = \infty$ and conclude from Theorem 3.3 that S^* is completely monotonic on $(0, \infty)$. □

Remark 3.5. Using (3.9) gives that $(a_k^*)_{k \geq 0}$ is strictly decreasing and strictly convex.

We ask: does

$$\tilde{a}_k = (k+1)! a_k^* = \sum_{j=1}^{k+1} (-1)^{k+1} \binom{k+1}{j} \log(j+1)$$

have a combinatorial interpretation?

Applying Theorem 3.4 and Lemmas 2.2 and 2.3 with $f = \log S^*$ we obtain the following result.

Theorem 3.6. *For all* $x, y \geq 0$, *we have*

$$\sum_{k=0}^{\infty} a_k^* x^k \sum_{k=0}^{\infty} a_k^* y^k \leq \log(2) \sum_{k=0}^{\infty} a_k^* (x+y)^k.$$

The constant factor $\log(2)$ *is best possible.*

We conclude this section with a monotonicity property of $\Gamma(c) P_c(x)$.

Theorem 3.7. *Let $x > 0$. The function*

$$c \mapsto \Gamma(c)P_c(x) = \sum_{k=1}^{\infty} \frac{x^k}{c(c+1)\cdots(c+k-1)}$$

is completely monotonic on $(0, \infty)$.

Proof. Let $x > 0$, and let $c > 0$. From (3.3), we obtain

$$\Gamma(c)P_c(x) = 1 + x\int_0^1 t^{c-1}e^{x(1-t)}dt > 0$$

and

$$(-1)^n \frac{\partial^n}{\partial c^n}\Gamma(c)P_c(x) = x\int_0^1 t^{c-1}e^{x(1-t)}(-\log(t))^n dt > 0, \quad n = 1, 2, \ldots.$$

This implies that $c \mapsto \Gamma(c)P_c(x)$ is completely monotonic on $(0, \infty)$. □

4. Convexity, concavity, and inequalities

In this section, we focus on the function P_c which can be expressed in terms of the incomplete gamma function

$$\gamma(a, x) = \int_0^x t^{a-1}e^{-t}dt.$$

Indeed, from (3.3) we obtain the representation

$$P_c(x) = \frac{1}{\Gamma(c)}\big(1 + e^x x^{1-c}\gamma(c, x)\big).$$

Moreover, (3.2) reveals that P_c solves the differential equation

$$xy' - (x + 1 - c)y = \frac{c-1}{\Gamma(c)}.$$

In what follows, we present convexity and concavity properties of P_c, $1/P_c$, $x \mapsto P_c(e^x)$ and we apply our results to obtain various functional inequalities.

Theorem 4.1. *If $0 < c < 1$, then P_c is strictly log-concave on $[0, \infty)$, whereas if $c > 1$, then P_c is strictly log-convex on $[0, \infty)$.*

Proof. Let $x > 0$. Using

$$xP_c'(x) = xP_c(x) + (1-c)\big(P_c(x) - P_c(0)\big)$$

and

$$xP_c''(x) = P_c(x) + (x - c)P_c'(x)$$

yields

(4.1)

$$P_c^2(x)\frac{d^2}{dx^2}\log P_c(x) = P_c(x)P_c''(x) - P_c'(x)^2 = \frac{c-1}{x^2}\Big[P_c^2(x) - P_c(0)\big(P_c(x) + xP_c'(x)\big)\Big].$$

We obtain the series representation

$$P_c^2(x) - P_c(0)\big(P_c(x) + xP_c'(x)\big) = \sum_{k=0}^{\infty} I_k(c)x^k$$

with

$$I_k(c) = \sum_{\nu=0}^{k} \frac{1}{\Gamma(c+\nu)\Gamma(c+k-\nu)} - \frac{k+1}{\Gamma(c)\Gamma(c+k)}.$$

We have $I_0(c) = I_1(c) = 0$. Let $k \geq 2$. Since

$$\Gamma(c)\Gamma(c+k)I_k(c) = \sum_{\nu=0}^{k} \prod_{j=0}^{\nu-1} \Big(1 + \frac{k-\nu}{c+j}\Big) - (k+1) > 0,$$

we find $I_k(c) > 0$. This yields

$$P_c^2(x) > P_c(0)\big(P_c(x) + xP_c'(x)\big). \tag{4.2}$$

From (4.1) and (4.2), we conclude that if $0 < c \neq 1$, then

$$(c-1)\frac{d^2}{dx^2} \log P_c(x) > 0. \tag{4.3}$$

Thus, if $0 < c < 1$, then $\log P_c$ is strictly concave on $[0, \infty)$, and if $c > 1$, then $\log P_c$ is strictly convex on $[0, \infty)$. □

An application of Theorem 4.1 and Lemma 2.3 gives the following Cauchy-type functional inequality.

Theorem 4.2. *If $0 < c < 1$, then, for all $x, y > 0$ we have*

$$\frac{1}{\Gamma(c)} < \frac{P_c(x)\,P_c(y)}{P_c(x+y)}. \tag{4.4}$$

If $c > 1$, then the reverse inequality holds for all $x, y > 0$. In both cases, the constant bound $1/\Gamma(c)$ is best possible.

Next, we present a relative of (4.4).

Theorem 4.3. *Let $0 < c < 1$. For all $x, y > 0$, we have*

$$e < \frac{P_c(1/x)^x\, P_c(1/y)^y}{P_c(1/(x+y))^{x+y}}. \tag{4.5}$$

If $c > 1$, then the reverse inequality holds for all $x, y > 0$. In both cases, the constant bound e is best possible.

Proof. Let $x > 0$ and

$$J_c(x) = x \log P_c(1/x).$$

We obtain

$$J_c''(x) = \frac{1}{x^3} \frac{d^2}{dt^2} \log P_c(t)\Big|_{t=1/x}.$$

From (4.3), we conclude that if $0 < c < 1$, then $-J_c$ is strictly convex on $(0, \infty)$, whereas if $c > 1$, then J_c is strictly convex on $(0, \infty)$.

Using the asymptotic formula (3.7) gives

$$J_c(0) = \lim_{x\to\infty} J_c(1/x) = \lim_{x\to\infty} \Big(1 - (c-1)\frac{\log(x)}{x} + \frac{\log(x^{c-1}\Theta_c(x))}{x}\Big) = 1. \tag{4.6}$$

Next, we apply Lemma 2.3 and (4.6). Then we find for $x, y > 0$,

$$-J_c(x) - J_c(y) < -1 - J_c(x+y), \quad \text{if} \quad 0 < c < 1$$

and

$$J_c(x) + J_c(y) < 1 + J_c(x+y), \quad \text{if} \quad c > 1.$$

This leads to (4.5). Moreover, since

$$\lim_{x\to 0} P_c(1/x)^x = e,$$

we obtain that e is the best possible constant bound in (4.5) and its reverse. □

Theorem 4.4. *Let $c>0$. The function $1/P_c$ is strictly convex on $[0,\infty)$.*

Proof. Let $b_k = 1/\Gamma(c+k)$ $(k \geq 0)$. Then, for $0 \leq k \leq [n/2]$,

$$\frac{b_{k+1}\, b_{n+1-k}}{b_k\, b_{n+2-k}} = \frac{c+n+1-k}{c+k} > 1.$$

From Lemma 2.5, we conclude that $1/P_c$ is strictly convex on $[0,\infty)$. □

An application of Theorem 4.4 and Lemma 2.3 leads to an additive counterpart of (4.4).

Theorem 4.5. *Let $c>0$. For all $x,y>0$, we have*

$$\frac{1}{P_c(x)} + \frac{1}{P_c(y)} < \Gamma(c) + \frac{1}{P_c(x+y)}.$$

The constant $\Gamma(c)$ is best possible.

Theorem 4.6. *Let $c>0$. The function $x \mapsto P_c(e^x)$ is strictly log-convex on $\mathbb{R}$.*

Proof. Let $t = e^x$. We consider two cases.

Case 1. $0 < c \leq 1$.

We obtain

$$\frac{P_c(t)^2}{t}\frac{d^2}{dx^2}\log P_c(e^x) = P_c(t)\big(P_c'(t) + tP_c''(t)\big) - tP_c'(t)^2 = P_c(t)^2 + \frac{1-c}{\Gamma(c)}P_c'(t) > 0.$$

Case 2. $c > 1$.

We apply (4.3) and get

$$\frac{d^2}{dx^2}\log P_c(e^x) = t^2\frac{d^2}{dt^2}\log P_c(t) + t\frac{P_c'(t)}{P_c(t)} > 0. \qquad \square$$

Using Theorem 4.6 and Lemma 2.3, we obtain the following companion of (4.4).

Theorem 4.7. *Let $c>0$. For all $x,y\in(0,1)$ and for all $x,y>1$, we have*

$$\frac{P_c(x)\, P_c(y)}{P_c(xy)} < P_c(1) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(c+k)}. \tag{4.7}$$

The constant $P_c(1)$ is best possible.

Remark 4.8. Inequality (4.7) is not true for all $x,y>0$. Indeed, if we set $y=1/x$, then

$$\lim_{x\to\infty}\frac{P_c(x)P_c(1/x)}{P_c(1)} = \infty.$$

From Theorems 4.1, 4.4, and 4.6 and Jensen's inequality for convex functions, we obtain inequalities involving the classical arithmetic, geometric, and harmonic means of two numbers.

Theorem 4.9. *Let $0 < c < 1$. For all $x, y \geq 0$, we have*

$$P_c(\sqrt{xy}) \leq \sqrt{P_c(x)P_c(y)} \leq P_c\Big(\frac{x+y}{2}\Big) \leq \frac{P_c(x)+P_c(y)}{2}. \tag{4.8}$$

Let $c > 1$. For all $x, y \geq 0$, we have

$$\frac{2}{1/P_c(x)+1/P_c(y)} \leq P_c\Big(\frac{x+y}{2}\Big) \leq \sqrt{P_c(x)P_c(y)}. \tag{4.9}$$

The sign of equality holds in each inequality if and only if $x = y$.

Remark 4.10. The first and the third inequality in (4.8) and the first inequality in (4.9) hold for all $c > 0$.

The next functional inequality is related to (4.4).

Theorem 4.11. *Let $0 < c \leq 1$. For all $x, y > 0$, we have*

$$\frac{P_c(x+y)^2}{P_c(2x)\,P_c(2y)} \leq 4^{c-1}\Big(\frac{(x+y)^2}{xy}\Big)^{1-c}. \tag{4.10}$$

The constant factor 4^{c-1} is best possible.

Proof. From Theorem 3.1, we conclude that if $c \in (0, 1]$, then

$$G_{1-c,c}(x) = e^{-x}x^{c-1}P_c(x)$$

is completely monotonic on $(0,\infty)$. We apply Lemma 2.2 and Jensen's inequality. This gives for $s, t > 0$,

$$\log G_{1-c,c}\Big(\frac{s+t}{2}\Big) \leq \frac{\log G_{1-c,c}(s) + \log G_{1-c,c}(t)}{2}. \tag{4.11}$$

We set $s = 2x$ and $t = 2y$. Then, (4.11) leads to (4.10). If we set $x = y$ in (4.10), then the sign of equality holds. This implies that the constant factor 4^{c-1} is sharp. □

We conclude the paper with an application of Theorem 3.7.

Theorem 4.12. *Let $0 < a < b$. For all $x \geq 0$, we have*

$$\frac{a}{b} \leq \frac{P_a(x)\,P_{b+1}(x)}{P_{a+1}(x)\,P_b(x)} < 1. \tag{4.12}$$

Both bounds are best possible.

Proof. Let $0 < a < b$. From Theorem 3.7 and Lemmas 2.2 and 2.4, we conclude that the function

$$c \mapsto \frac{\Gamma(c+1)P_{c+1}(x)}{\Gamma(c)P_c(x)} = c\frac{P_{c+1}(x)}{P_c(x)}, \quad x > 0,$$

is increasing on $(0,\infty)$. It follows that the first inequality in (4.12) holds for $x \geq 0$.

We have

$$P_{a+1}(x)\,P_b(x) = \sum_{k=0}^{\infty} R_k(a,b)x^k$$

with

$$R_k(a,b) = \sum_{\nu=0}^{k} \frac{1}{\Gamma(a+1+\nu)\Gamma(b+k-\nu)}.$$

Since

$$R_k(a,b) - R_k(b,a) = \frac{1}{\Gamma(a)\Gamma(b)}\Big(\prod_{j=0}^{k}\frac{1}{a+j} - \prod_{j=0}^{k}\frac{1}{b+j}\Big) > 0, \quad k = 0,1,2,\ldots,$$

we obtain

$$P_{a+1}(x)\,P_b(x) - P_a(x)\,P_{b+1}(x) = \sum_{k=0}^{\infty}\big(R_k(a,b) - R_k(b,a)\big)x^k > 0, \quad x \geq 0.$$

This proves the right-hand side of (4.12).

We denote the middle term in (4.12) by $Z_{a,b}(x)$. Then, $Z_{a,b}(0) = a/b$, and from (3.7) we conclude that

$$\lim_{x\to\infty} Z_{a,b}(x) = 1.$$

This implies that both bounds given in (4.12) are sharp. □

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References

[1] Horst Alzer and Christian Berg, *Some classes of completely monotonic functions*, Ann. Acad. Sci. Fenn. Math. **27** (2002), no. 2, 445–460. MR1922200

[2] Horst Alzer and Christian Berg, *Some classes of completely monotonic functions. II*, Ramanujan J. **11** (2006), no. 2, 225–248, DOI 10.1007/s11139-006-6510-5. MR2267677

[3] Joaquin Bustoz and Mourad E. H. Ismail, *On gamma function inequalities*, Math. Comp. **47** (1986), no. 176, 659–667, DOI 10.2307/2008180. MR856710

[4] M. J. Dubourdieu, *Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace-Stieltjes* (French), Compositio Math. **7** (1939), 96–111. MR0000436

[5] Árpád Elbert and Andrea Laforgia, *An inequality for the product of two integrals relating to the incomplete gamma function*, J. Inequal. Appl. **5** (2000), no. 1, 39–51, DOI 10.1155/S1025583400000035. MR1740211

[6] A. M. Fink, *Kolmogorov-Landau inequalities for monotone functions*, J. Math. Anal. Appl. **90** (1982), no. 1, 251–258, DOI 10.1016/0022-247X(82)90057-9. MR680877

[7] Walter Gautschi, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. and Phys. **38** (1959/60), 77–81, DOI 10.1002/sapm195938177. MR0103289

[8] Mourad E. H. Ismail and Andrea Laforgia, *Monotonicity properties of determinants of special functions*, Constr. Approx. **26** (2007), no. 1, 1–9, DOI 10.1007/s00365-005-0627-4. MR2310684

[9] Mourad E. H. Ismail, Lee Lorch, and Martin E. Muldoon, *Completely monotonic functions associated with the gamma function and its q-analogues*, J. Math. Anal. Appl. **116** (1986), no. 1, 1–9, DOI 10.1016/0022-247X(86)90042-9. MR837337

[10] A. Laforgia and P. Natalini, *Supplements to known monotonicity results and inequalities for the gamma and incomplete gamma functions*, J. Inequal. Appl., posted on 2006, Art. ID 48727, 8, DOI 10.1155/JIA/2006/48727. MR2215490

[11] Lee Lorch, *A property of completely [absolutely] monotonic and other logarithmically convex [concave] functions*, Indian J. Math. **45** (2003), no. 2, 207–210. MR2035907

[12] Lee Lorch and Donald J. Newman, *On the composition of completely monotonic functions and completely monotonic sequences and related questions*, J. London Math. Soc. (2) **28** (1983), no. 1, 31–45, DOI 10.1112/jlms/s2-28.1.31. MR703462

[13] K. S. Miller and S. G. Samko, *Completely monotonic functions*, Integral Transforms Spec. Funct. **12** (2001), no. 4, 389–402, DOI 10.1080/10652460108819360. MR1872377

[14] D. S. Mitrinović, *Analytic Inequalities*, In cooperation with P. M. Vasić. Die Grundlehren der mathematischen Wissenschaften, Band 165, Springer-Verlag, New York-Berlin, 1970. MR0274686

[15] M. Petrović, *Sur une équation fonctionnelle*, Publ. Math. Univ. Belgrade **1** (1932), 149-156.

[16] Thomas Simon, *Mittag-Leffler functions and complete monotonicity*, Integral Transforms Spec. Funct. **26** (2015), no. 1, 36–50, DOI 10.1080/10652469.2014.965704. MR3275447
[17] David Vernon Widder, *The Laplace Transform*, Princeton Mathematical Series, v. 6, Princeton University Press, Princeton, N. J., 1941. MR0005923

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