

RANDOM DATA FINAL-STATE PROBLEM FOR THE MASS-SUBCRITICAL NLS IN L^2

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ABSTRACT. We study the final-state problem for the mass-subcritical NLS above the Strauss exponent. For $u_+ \in L^2$, we perform a physical-space randomization, yielding random final states $u_+^\omega \in L^2$. We show that for almost every ω , there exists a unique, global solution to NLS that scatters to u_+^ω . This complements the deterministic result of Nakanishi, which proved the existence (but not necessarily uniqueness) of solutions scattering to prescribed L^2 final states.

1. INTRODUCTION

We consider power-type nonlinear Schrödinger equations (NLS) of the form

$$(1.1) \quad (i\partial_t + \Delta)u = \mu|u|^p u,$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^d \rightarrow \mathbb{C}$, $p > 0$, and $\mu \in \{\pm 1\}$. The case $\mu = 1$ is the defocusing case, while $\mu = -1$ gives the focusing case.

The scaling symmetry

$$(1.2) \quad u(t, x) \mapsto \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x)$$

defines a notion of *criticality* for (1.1). In particular, a space of initial data is critical if its norm is invariant under this rescaling. The *mass-critical NLS* refers to the case $p = \frac{4}{d}$. In this case the conserved *mass* of solutions

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t, x)|^2 dx$$

is invariant under rescaling.

We consider the *mass-subcritical* regime, i.e., $0 < p < \frac{4}{d}$. In this case, any initial data $u_0 \in L^2$ leads to a local-in-time solution u . As a consequence of mass-subcriticality, the time of existence depends only on the L^2 -norm of u_0 . In particular, by the conservation of mass, solutions are automatically global-in-time. For further details, we refer the reader to the textbook [8].

The long-time behavior of solutions for the mass-subcritical NLS is a rich and interesting subject. We briefly review some relevant results, noting that our discussion is far from exhaustive.

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The initial-value problem. Many results have been established for the initial-value problem when one selects data from the weighted space Σ , which is defined via the norm

$$\|f\|_{\Sigma}^2 = \|f\|_{H^1}^2 + \|xf\|_{L^2}^2.$$

Tsutsumi and Yajima [29] established that in the defocusing case, for $\frac{2}{d} < p < \frac{4}{d}$ and $u_0 \in \Sigma$, the solution u to (1.1) with data u_0 *scatters* in L^2 ; that is, there exists $u_+ \in L^2$ such that

$$(1.3) \quad \lim_{t \rightarrow \infty} \|u(t) - e^{it\Delta}u_+\|_{L^2} = 0.$$

This result is sharp in the following sense: for $0 < p \leq \frac{2}{d}$, any solution to (1.1) that satisfies (1.3) must be identically zero [1].

Restricting still to the defocusing case, Cazenave and Weissler [9] proved that for a smaller range of p , the solution scatters in Σ , that is,

$$(1.4) \quad \lim_{t \rightarrow \infty} \|e^{-it\Delta}u(t) - u_+\|_{\Sigma} = 0.$$

In particular, (1.4) holds for all $p > \frac{4}{d+2}$ in the small-data regime, while for arbitrary data the result is restricted to $p \in [p_0(d), \frac{4}{d})$, where $p_0(d)$ is the *Strauss exponent* defined by

$$(1.5) \quad p_0(d) = \frac{2-d+\sqrt{d^2+12d+4}}{2d}.$$

The question of scattering in Σ for arbitrary data for $p \in (\frac{4}{d+2}, p_0(d))$ remains open.

For the scattering results of [9, 29], an important role is played by the *pseudoconformal energy estimate*, which implies decay for the potential energy $\|u(t)\|_{L^{p+2}}$ that matches the rate for linear solutions. For $p > p_0(d)$, this estimate implies critical space-time bounds for solutions that can be used to deduce scattering.

Existence of wave operators. A counterpart to the initial-value problem is the *final-state problem*: given u_+ , can one find a (unique) solution u that scatters to u_+ ? If one can do this, one calls the map $u_+ \mapsto u(0)$ the *wave operator*. For $p \in (\frac{4}{d+2}, \frac{4}{d})$, one has the existence of wave operators in the Σ topology [9]. In fact, one can prove results in critical weighted spaces essentially all the way down to $p > \frac{2}{d}$ [26].

The result that is most relevant to this note is due to Nakanishi [26]. He showed that in dimensions $d \geq 3$, for any $\frac{2}{d} < p < \frac{4}{d}$ and any $u_+ \in L^2$ there exists a solution u to (1.1) satisfying (1.3). However, his arguments did not yield *uniqueness* of the solution, and thus one cannot conclude existence of wave operators in L^2 .

Nonetheless, Nakanishi’s result is quite remarkable in the sense that working with merely L^2 final states makes the final-state problem essentially a *supercritical* problem. Indeed, the pseudoconformal transformation

$$u(t, x) = (2it)^{-\frac{d}{2}} e^{\frac{ix|^2}{4t}} \bar{v}\left(\frac{1}{2t}, \frac{x}{2t}\right)$$

transforms the final-state problem for (1.1) into the initial-value problem

$$(1.6) \quad \begin{cases} (i\partial_t + \Delta)v = \mu t^{\frac{dp}{2}-2}|v|^p v, \\ v(0) = \widetilde{u_+} \in L^2. \end{cases}$$

The scaling symmetry for (1.6), namely, $v(t, x) \mapsto \lambda^{d-\frac{2}{p}}v(\lambda^2t, \lambda x)$ identifies (1.6) as an L^2 -*supercritical* problem for $p < \frac{4}{d}$.

Main result. Inspired by recent works concerning almost sure well-posedness for supercritical problems (see e.g. [2–7, 10–15, 21, 22, 25, 28, 30] and the references therein), we consider the question of the ‘almost sure existence of wave operators’ for the mass-subcritical NLS in the L^2 -topology. In particular, we will prove existence and uniqueness of solutions scattering to suitably randomized prescribed final states in L^2 . For the probabilistic aspects of this note, we largely follow the presentation of [22].

Definition 1.1 (Randomization). We perform a randomization in physical space. This is similar to the randomization appearing in [22], although there the randomization is in frequency space.

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a smooth bump function such that $\phi = 1$ for $|x| \leq 1$ and $\phi = 0$ for $|x| > 2$. For $k \in \mathbb{Z}^d$, we set $\phi_k(x) = \phi(x - k)$ and define the partition of unity $\{\psi_k\}$ by

$$\psi_k(x) = \frac{\phi_k(x)}{\sum_{\ell \in \mathbb{Z}^d} \phi_\ell(x)}.$$

Next, let $\{g_k\}_{k \in \mathbb{Z}^d}$ be a sequence of independent, mean-zero random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distributions μ_k . We assume that there exists $c > 0$ so that

$$(1.7) \quad \left| \int_{\mathbb{R}} e^{\gamma x} d\mu_k(x) \right| \leq e^{c\gamma^2} \quad \text{for all } \gamma \in \mathbb{R}$$

and all $k \in \mathbb{Z}^d$. For simplicity, one can keep in mind the example of Gaussian random variables.

For $f \in L^2$, we define the *randomization* of f by

$$f^\omega(x) = \sum_{k \in \mathbb{Z}^d} g_k(\omega) \psi_k(x) f(x),$$

which we understand as a limit in $L^2(\Omega; L_x^2(\mathbb{R}^d))$ of sums over $|k| \leq n$ as $n \rightarrow \infty$.

Remark 1.2. This randomization procedure does not imply additional decay for f^ω in the sense of weighted bounds. In particular, if there exists $c > 0$ such that

$$\sup_{k \in \mathbb{Z}^d} \mu_k([-c, c]) < 1,$$

then the following holds for any $\varepsilon > 0$: if $|x|^\varepsilon f \notin L^2$, then $|x|^\varepsilon f^\omega \notin L^2$ almost surely. To prove this, one can mimic the proof of [6, Lemma B.1], relying on the fact that

$$(1.8) \quad \||x|^\varepsilon f\|_{L_x^2}^2 \sim \sum_{k \in \mathbb{Z}^d} \||x|^\varepsilon \psi_k f\|_{L_x^2}^2.$$

On the other hand, one can show that $\widehat{f^\omega}$ almost surely enjoys additional regularity in the sense of higher L^r -norms. This is analogous to the situation of frequency-space randomization, in which case the randomized data does not gain Sobolev regularity but does gain regularity in the sense of higher L^r -norms. See Section 4 for a further discussion.

The result of this note is the following ‘almost sure existence of wave operators’ above the Strauss exponent.

Theorem 1.3. *Let $d \geq 1$, $\mu \in \{\pm 1\}$, and*

$$(1.9) \quad p_0(d) < p < \frac{4}{d},$$

with $p_0(d)$ as in (1.5). Let $u_+ \in L^2$ and define the randomization u_+^ω as in Definition 1.1. For almost every ω , there exists a unique global solution u to (1.1) that scatters to u_+^ω in L^2 , that is,

$$\lim_{t \rightarrow \infty} \|u(t) - e^{it\Delta} u_+^\omega\|_{L^2} = 0.$$

Remark 1.4. Like Nakanishi's result in [26], Theorem 1.3 is valid in both the focusing and defocusing settings. Compared to his result, the novelty here is in the uniqueness of the solutions constructed (while still working at the level of L^2). In addition, our result is valid in dimensions $d \in \{1, 2\}$. Note, however, that we were unable to treat the whole range $\frac{2}{d} < p < \frac{4}{d}$.

Remark 1.5. Uniqueness holds in the class of solutions in $C_t L_x^2$ that belong to $L_t^q L_x^r((T, \infty) \times \mathbb{R}^d)$ for some T , where (q, r) is a certain 'critical' exponent pair (see Section 2.2).

As in previous works treating supercritical problems via probabilistic techniques, an essential ingredient is the fact that solutions to the *linear* Schrödinger equation with randomized initial data obey improved space-time estimates almost surely (see Proposition 3.1). Combining this fact with an 'exotic' inhomogeneous Strichartz estimate, we are able to find a suitable space in which to run a simple contraction mapping argument (see Proposition 3.3). In fact, we are able to work with the space-time norms that one can access with the pseudoconformal energy estimate.¹ We failed to find suitable spaces precisely when p reaches the Strauss exponent; treating the range $\frac{2}{d} < p \leq \alpha_0(d)$ is an interesting open problem.

Whether or not one has deterministic existence of wave operators in L^2 (i.e., uniqueness in the result of Nakanishi [26]) remains an open question. It is also natural to consider ill-posedness results in this setting, given the supercritical nature of the problem.

By now there are many works addressing almost sure well-posedness in supercritical settings via probabilistic techniques. There has also been some recent progress on almost sure *scattering* in supercritical settings (e.g. [5, 12–15, 21, 28]). In particular, the papers [5, 28] prove probabilistic global well-posedness for the NLS with a harmonic potential, using a randomization based on the Hermite functions. By applying the lens transform, they deduce scattering results for the usual NLS. In the mass-subcritical setting, they establish scattering with non-zero probability (for L^2 initial data) in dimensions $d \geq 2$.

The present work considers the final-state problem, which is essentially a local problem with data prescribed at $t = \infty$. By employing a physical-space randomization, we are able to prove a scattering result for L^2 data that holds almost surely. Furthermore, we are able to treat all dimensions $d \geq 1$, while the deterministic result of Nakanishi [26] holds only for $d \geq 3$ (with an extension to H^1 final states for $d = 2$ [18]).

¹This strategy for proving existence of wave operators appears already in the work of Kato [19].

Outline of the paper. In Section 2, we introduce notation, collect estimates related to the linear Schrödinger equation, and introduce the requisite probabilistic results. In Section 3, we prove Theorem 1.3. The main ingredients are improved space-time bounds for the linear Schrödinger equation with randomized data (Proposition 3.1) and a deterministic existence of wave operators result (Proposition 3.3). In Section 4, we conclude with some final remarks and prove almost sure additional ‘Fourier–Lebesgue regularity’ for randomized final states (Proposition 4.2).

2. NOTATION AND USEFUL LEMMAS

We write $A \lesssim B$ to denote $A \leq CB$ for some $C > 0$. We write $A \sim B$ to denote $A \lesssim B \lesssim A$. We write $A \lesssim_\rho B$ to mean $A \leq CB$ for some $C = C(\rho) > 0$. We use the notation $L_t^q L_x^r(I \times \mathbb{R}^d)$ to denote space-time norms

$$\|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)} = \left(\int_I \left(\int_{\mathbb{R}^d} |u(t, x)|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}},$$

with the usual adjustments when q or r is infinite.

We define the *scaling* associated to an exponent pair (q, r) by

$$s(q, r) = \frac{d}{2} - \left(\frac{2}{q} + \frac{d}{r} \right).$$

A space $L_t^q L_x^r$ is *critical* for (1.1) if $s(q, r) = s_c := \frac{d}{2} - \frac{2}{p}$. In this case, the $L_t^q L_x^r$ -norm is invariant under the rescaling (1.2).

For an exponent $r \in [1, \infty]$, we write $r' \in [1, \infty]$ to denote the dual exponent, i.e., the solution to $\frac{1}{r} + \frac{1}{r'} = 1$.

We write \widehat{f} for the Fourier transform of a function f .

2.1. The linear Schrödinger equation. Solutions to the linear Schrödinger equation are generated by $e^{it\Delta}$, where

$$(2.1) \quad [e^{it\Delta} f](x) = (4\pi it)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) dy.$$

Using this identity together with unitarity on L^2 , one immediately deduces the following dispersive estimate by interpolation:

$$(2.2) \quad \|e^{it\Delta}\|_{L_x^{r'} \rightarrow L_x^r} \lesssim |t|^{-\left(\frac{d}{2} - \frac{d}{r}\right)} \quad \text{for } 2 \leq r \leq \infty.$$

Solutions to the nonlinear equation (1.1) satisfy the *Duhamel formula*

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) - i\mu \int_{t_0}^t e^{i(t-s)\Delta} (|u|^p u)(s) ds$$

for any $t_0, t \in I$.

We recall the standard Strichartz estimates. We call a pair (a, b) *admissible* if $s(a, b) = 0$ and $2 \leq a \leq \infty$. One also excludes the triple $(d, a, b) = (2, 2, \infty)$ and requires $4 \leq a \leq \infty$ in dimension $d = 1$. We call (α, β) *dual admissible* if (α', β') is admissible.

Proposition 2.1 (Strichartz estimates [17, 20, 31]). *Let I be a time interval and let $t_0 \in \bar{I}$. Then for any admissible (a, b) and any dual admissible (α, β) , we have*

$$(2.3) \quad \begin{aligned} & \|e^{it\Delta}\varphi\|_{C_t L_x^2 \cap L_t^\alpha L_x^b(I \times \mathbb{R}^d)} \lesssim \|\varphi\|_{L_x^\beta}, \\ & \left\| \int_{t_0}^t e^{i(t-s)\Delta} F(s) ds \right\|_{C_t L_x^2 \cap L_t^\alpha L_x^b(I \times \mathbb{R}^d)} \lesssim \|F\|_{L_t^\alpha L_x^\beta(I \times \mathbb{R}^d)}. \end{aligned}$$

The estimate (2.3) holds for more exponents than the ones given in Proposition 2.1. The sharp range of ‘exotic’ inhomogeneous estimates was studied by Foschi [16]. We record here one particular estimate that can be proven simply (see e.g. [8, Chapter 2.4]), which will be essential to our arguments below.

Proposition 2.2 (Inhomogeneous estimate). *Let $0 < p < \frac{4}{d-2}$ if $d \geq 3$ and $0 < p < \infty$ if $d \in \{1, 2\}$. Let I be a time interval and let $t_0 \in \bar{I}$. Let $r = p + 2$ and suppose $1 < q, \bar{q} < \infty$ satisfy*

$$(2.4) \quad \frac{1}{q} + \frac{1}{\bar{q}} = \frac{d}{2} - \frac{d}{r}.$$

Then

$$\left\| \int_{t_0}^t e^{i(t-s)\Delta} F(s) ds \right\|_{L_t^q L_x^r(I \times \mathbb{R}^d)} \lesssim \|F\|_{L_t^{\bar{q}'} L_x^{r'}(I \times \mathbb{R}^d)}.$$

To prove this result, one uses (2.2) and the Hardy–Littlewood–Sobolev inequality.

2.2. Function spaces. We now introduce the specific exponents that we will use in the estimates below. We first define

$$r = p + 2, \quad q = \frac{2p(p+2)}{4-p(d-2)}, \quad \bar{q} = \frac{2p(p+2)}{dp^2 - (4-p(d-2))}.$$

Then one can check that $s(q, r) = s_c$ and that (r, q, \bar{q}) satisfies the scaling relation (2.4).

It is easy to verify that

$$\max\{1, p\} < q < \infty$$

for p satisfying (1.9); in fact this holds for a wider range of p than the one appearing in (1.9), for example $\frac{4}{d+2} < p < \frac{4}{d-2}$.

One can also check that $\bar{q} > 1$ for $p < \frac{4}{d-2}$, while $\bar{q} < \infty$ precisely when $p > p_0(d)$ (cf. (1.5)). This is one reason why we work above the Strauss exponent (see also Proposition 3.1).

In particular, we can apply Proposition 2.2 with (r, q, \bar{q}) . As one can check that

$$r' = \frac{r}{p+1} \quad \text{and} \quad \bar{q}' = \frac{q}{p+1},$$

we get the nonlinear estimate

$$(2.5) \quad \left\| \int_{t_0}^t e^{i(t-s)\Delta} (|u|^p u)(s) ds \right\|_{L_t^q L_x^r} \lesssim \| |u|^p u \|_{L_t^{\bar{q}'} L_x^{r'}} \lesssim \|u\|_{L_t^q L_x^r}^{p+1}.$$

We next define an admissible and dual admissible pair. For $d \geq 2$, we first take a satisfying

$$(2.6) \quad \max\{\frac{1}{2} - \frac{p}{q}, 0\} < \frac{1}{a} < \min\{1 - \frac{p}{q}, \frac{1}{2}\}$$

and choose b so that $s(a, b) = 0$; in particular, (a, b) is an admissible pair. For $d = 1$, we instead impose

$$(2.6) \quad \max\{\frac{3}{4} - \frac{p}{q}, 0\} < \frac{1}{a} < \min\{1 - \frac{p}{q}, \frac{1}{4}\}.$$

We now define (α, β) via

$$\frac{1}{\alpha} = \frac{p}{q} + \frac{1}{a} \quad \text{and} \quad \frac{1}{\beta} = \frac{p}{r} + \frac{1}{b}.$$

Then (2.6) implies $1 < \alpha < 2$ in dimensions $d \geq 2$ and $1 < \alpha < \frac{4}{3}$ in $d = 1$. The scaling relations $s(q, r) = s_c$ and $s(a, b) = 0$ then guarantee that (α, β) is a dual admissible pair.

We have the following nonlinear estimate via Proposition 2.1:

$$(2.7) \quad \left\| \int_{t_0}^t e^{i(t-s)\Delta} (|u|^p u)(s) ds \right\|_{C_t L_x^2 \cap L_t^q L_x^b} \lesssim \| |u|^p u \|_{L_t^q L_x^\beta} \lesssim \| u \|_{L_t^q L_x^r}^p \| u \|_{L_t^2 L_x^b}.$$

2.3. Probabilistic results. We next import a few probabilistic results that will play a role in establishing improved space-time estimates for linear solutions with randomized data.

The first result appears in [6, Lemma 3.1].

Proposition 2.3 (Large deviation estimate [6]). *Let $\{\ell_k\}$ be a sequence of independent random variables with distributions $\{\mu_k\}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying (1.7). Then for $\alpha \geq 2$ and $\{c_k\} \in \ell^2$,*

$$\left\| \sum_k c_k \ell_k(\omega) \right\|_{L_\omega^\alpha(\Omega)} \lesssim \sqrt{\alpha} \left(\sum_k |c_k|^2 \right)^{\frac{1}{2}}.$$

The next lemma appears in [22, Lemma 2.5], which is in turn an adaptation of [30, Lemma 4.5].

Lemma 2.4. *Suppose F is a measurable function on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that there exist $C > 0$, $A > 0$, and $\alpha_0 \geq 1$ such that for $\alpha \geq \alpha_0$, we have*

$$\|F\|_{L_\omega^\alpha(\Omega)} \leq C\sqrt{\alpha}A.$$

Then there exists $C' = C'(C, \alpha_0) > 0$ and $c = c(C, \alpha_0) > 0$ such that

$$\mathbb{P}(\omega \in \Omega : |F(\omega)| > \eta) \leq C' \exp\{-c\eta^2 A^{-2}\}.$$

3. PROOF OF THEOREM 1.3

This section contains the proof of Theorem 1.3. We first prove some improved space-time estimates for solutions to the linear Schrödinger equation with randomized data (Proposition 3.1). We then prove a deterministic existence of wave operators result (Proposition 3.3). These two results together will quickly imply Theorem 1.3.

Space-time estimates with randomized data. Recall the exponents (q, r) defined in Section 2.2. We define

$$\varepsilon_0 := \left(\frac{d}{2} - \frac{d}{r}\right) - \frac{1}{q} = \frac{dp^2 + p(d-2) - 4}{2p(p+2)}$$

and note that $\varepsilon_0 > 0$ precisely when $p > p_0(d)$ (cf. (1.5)). This is another reason that we work above the Strauss exponent.

Proposition 3.1. *Let $T \geq 1$ and $u_+ \in L^2$. Define u_+^ω as in Definition 1.1 and consider the set*

$$\Omega_{\eta, T} := \{\omega \in \Omega : \|e^{it\Delta} u_+^\omega\|_{L_t^q L_x^r((T, \infty) \times \mathbb{R}^d)} < \eta\}$$

for some $\eta > 0$. There exists $C > 0$ such that

$$(3.1) \quad \mathbb{P}(\Omega_{\eta,T}^c) \lesssim \exp\{-C\eta^2 T^{2\varepsilon_0} \|u_+\|_{L^2}^{-2}\}.$$

Proof. Fix $\alpha > \max\{q, r\}$. Changing the order of integration and using the large deviation estimate (Proposition 2.3), the dispersive estimate (2.2), and Hölder’s inequality, we estimate

$$\begin{aligned} \|e^{it\Delta} u_+^\omega\|_{L_\omega^\alpha(\Omega; L_t^q L_x^r((T,\infty)\times\mathbb{R}^d))} &\lesssim \left\| \left\| \sum_{k\in\mathbb{Z}^d} g_k(\omega) [e^{it\Delta} \psi_k u_+] \right\|_{L_\omega^\alpha} \right\|_{L_t^q L_x^r((T,\infty)\times\mathbb{R}^d)} \\ &\lesssim \sqrt{\alpha} \left\| \left(\sum_{k\in\mathbb{Z}^d} |e^{it\Delta} \psi_k u_+|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q L_x^r((T,\infty)\times\mathbb{R}^d)} \\ &\lesssim \sqrt{\alpha} \left\| \left(\sum_{k\in\mathbb{Z}^d} \|e^{it\Delta} \psi_k u_+\|_{L_x^r}^2 \right)^{\frac{1}{2}} \right\|_{L_t^q((T,\infty))} \\ &\lesssim \sqrt{\alpha} \| |t|^{-(\frac{d}{2}-\frac{d}{r})} \|_{L_t^q((T,\infty))} \left(\sum_{k\in\mathbb{Z}^d} \|\psi_k u_+\|_{L_x^{r'}}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sqrt{\alpha} T^{-\varepsilon_0} \left(\sum_{k\in\mathbb{Z}^d} \|\psi_k u_+\|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \sqrt{\alpha} T^{-\varepsilon_0} \|u_+\|_{L_x^2}. \end{aligned}$$

To pass from this estimate to (3.1) we now use Lemma 2.4. □

Remark 3.2. The use of Hölder’s inequality in the argument above is akin to the ‘unit scale Bernstein estimate’ in [22].

In Section 4, we discuss some conditions on u_+ that imply $L_t^q L_x^r$ bounds for $e^{it\Delta} u_+$ in the deterministic setting.

Final-state problem. We next prove a deterministic result concerning existence of wave operators. We recall the exponents defined in Section 2.2.

Proposition 3.3. *Let $d \geq 1$ and $p_0(d) < p < \frac{4}{d}$, with $p_0(d)$ as in (1.5). There exists $\eta_0 > 0$ such that the following holds: If $\varphi \in L^2$ and T is such that*

$$(3.2) \quad \|e^{it\Delta} \varphi\|_{L_t^q L_x^r((T,\infty)\times\mathbb{R}^d)} < \eta \quad \text{for some } 0 < \eta < \eta_0,$$

then there exists a unique global solution u to (1.1) satisfying

$$(3.3) \quad \lim_{t\rightarrow\infty} \|u(t) - e^{it\Delta} \varphi\|_{L^2} = 0.$$

Proof. We define

$$\begin{aligned} X &= \{u \in C_t L_x^2 \cap L_t^a L_x^b \cap L_t^q L_x^r((T,\infty)\times\mathbb{R}^d) : \\ &\quad \|u\|_{C_t L_x^2 \cap L_t^a L_x^b((T,\infty)\times\mathbb{R}^d)} \leq 2C \|\varphi\|_{L^2}, \\ &\quad \|u\|_{L_t^q L_x^r((T,\infty)\times\mathbb{R}^d)} \leq C\eta\}, \end{aligned}$$

which is complete with respect to

$$d(u, v) = \|u - v\|_{L_t^a L_x^b((T,\infty)\times\mathbb{R}^d)}.$$

Here C encodes various constants appearing in the estimates below. We will show that for η sufficiently small,

$$[\Phi u](t) := e^{it\Delta} \varphi + i\mu \int_t^\infty e^{i(t-s)\Delta} (|u|^p u)(s) ds$$

is a contraction from X to X with respect to $d(\cdot, \cdot)$. In the following, we take all space-time norms over $(T, \infty) \times \mathbb{R}^d$.

First note that by Strichartz (Proposition 2.1) and (3.2),

$$\|e^{it\Delta}\varphi\|_{L_t^q L_x^b} \lesssim \|\varphi\|_{L^2}, \quad \|e^{it\Delta}\varphi\|_{L_t^q L_x^r} < \eta.$$

Now let $u \in X$. Estimating as in (2.7) via Proposition 2.1, we have

$$(3.4) \quad \left\| \int_t^\infty e^{i(t-s)\Delta} (|u|^p u)(s) ds \right\|_{C_t L_x^2 \cap L_t^q L_x^b} \lesssim \|u\|_{L_t^q L_x^r}^p \|u\|_{L_t^q L_x^b} \lesssim \eta^p \|\varphi\|_{L^2}.$$

Similarly, estimating as in (2.5) via Proposition 2.2,

$$\left\| \int_t^\infty e^{i(t-s)\Delta} (|u|^p u)(s) ds \right\|_{L_t^q L_x^r} \lesssim \|u\|_{L_t^q L_x^r}^{p+1} \lesssim \eta^{p+1}.$$

Thus, for η sufficiently small, $\Phi : X \rightarrow X$. Furthermore, estimating similarly to (3.4) yields

$$d(\Phi u, \Phi v) \lesssim \eta^p d(u, v),$$

so that Φ is a contraction for η small enough.

Consequently, Φ has a unique fixed point $u \in X$. It is now standard to verify that u is a solution to (1.1) on (T, ∞) that satisfies (3.3). Furthermore, as described in the introduction, u is automatically global-in-time. \square

We are now ready to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Fix $u_+ \in L^2$ and define the randomization u_+^ω as in Definition 1.1. Now choose η_0 as in Proposition 3.3 and fix $0 < \eta < \eta_0$. Define the sets $\Omega_{\eta, T}$ as in Proposition 3.1. Then, using Proposition 3.1, we find that the set $\Omega_\eta := \bigcup_{T=1}^\infty \Omega_{\eta, T}$ has $\mathbb{P}(\Omega_\eta) = 1$.

Furthermore, for each $\omega \in \Omega_\eta$, we may apply Proposition 3.3 with $\varphi = u_+^\omega$ and T chosen so that $\omega \in \Omega_{\eta, T}$. In particular, we find that there exists a unique global solution u that scatters to u_+^ω . This completes the proof of Theorem 1.3. \square

4. REMARKS

4.1. Deterministic existence of wave operators. For a restricted range of mass-subcritical powers in dimensions $d \geq 3$, Proposition 3.3 actually implies a new (deterministic) existence of wave operators result. Specifically, we have the existence of wave operators for final states $u_+ \in L^2$ satisfying the additional condition

$$(4.1) \quad \widehat{u}_+ \in L_\xi^{\rho_0}(\mathbb{R}^d), \quad \text{where } \rho_0 := \frac{dp}{d-2}.$$

Indeed, this is a consequence of the following Strichartz estimate (see [24, Proposition 2.4]) and the monotone convergence theorem:

Proposition 4.1 (Fourier–Lebesgue–Strichartz estimate [24]). *Let $d \geq 3$ and take (q, r) as in Section 2.2. Then*

$$\|e^{it\Delta}\varphi\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\widehat{\varphi}\|_{L_\xi^{\rho_0}(\mathbb{R}^d)},$$

provided p is chosen so that the following constraints are satisfied:

$$(4.2) \quad \frac{1}{q} \leq \frac{d}{d-2} \frac{1}{r}, \quad \frac{1}{q} < \frac{d+1}{2(d+3)} - \frac{d+1}{2} \left(\frac{1}{r} - \frac{d+1}{2(d+3)} \right).$$

This estimate is proven using Fourier restriction estimates, which are ultimately the source of the constraints (4.2). Note that (4.2) is always satisfied in a neighborhood of $p = \frac{4}{d}$, although not typically in our full range of interest $p_0(d) < p < \frac{4}{d}$.

One also has existence of wave operators if $u_+ \in L^2$ satisfies the weighted assumption

$$(4.3) \quad |x|^{|s_c|} u_+ \in L^2(\mathbb{R}^d), \quad s_c = \frac{d}{2} - \frac{2}{p},$$

as is already well-known in the literature (see e.g. [23, 26, 27]). In this case, one does not need to use exotic inhomogeneous estimates; one can instead work with the quantity $e^{it\Delta} |x|^{|s_c|} e^{-it\Delta} u$ in admissible (Lorentz-modified) Strichartz spaces.

The weighted assumption in (4.3) is a strictly stronger assumption than the ‘Fourier–Lebesgue regularity’ in (4.1). Indeed, by Sobolev embedding and Plancherel, one has

$$(4.4) \quad \|\widehat{u_+}\|_{L_\xi^{\rho_0}} \lesssim \|\widehat{|\nabla|^{|s_c|} u_+}\|_{L_\xi^2} \sim \| |x|^{|s_c|} u_+ \|_{L_x^2}.$$

We failed to find a simple proof of boundedness of $e^{it\Delta} u_+$ in the particular space $L_t^q L_x^r$ under the assumption (4.3) in the range of interest $p_0(d) < p < \frac{4}{d}$. Of course, whenever (4.2) is satisfied, boundedness follows from Proposition 4.1 and (4.4).

4.2. Fourier–Lebesgue regularity. Finally, returning to the discussion in Remark 1.2, we note that while the randomization in Definition 1.1 does not yield weighted bounds, it does imply additional Fourier–Lebesgue regularity almost surely.

Proposition 4.2 (Fourier–Lebesgue regularity). *Let $u_+ \in L^2$ and $\rho \in (2, \infty)$. Define u_+^ω as in Definition 1.1 and consider the set*

$$\Omega_{M,\rho} := \{\omega \in \Omega : \|\widehat{u_+^\omega}\|_{L_\xi^\rho} \leq M\}$$

for $M > 0$. There exists $c = c(\rho) > 0$ such that

$$\mathbb{P}(\Omega_{M,\rho}^c) \lesssim_\rho \exp\{-cM^2 \|u_+\|_{L^2}^{-2}\}.$$

Consequently, for any $\frac{2}{d} < p < \frac{4}{d}$, we have $\widehat{u_+^\omega} \in L_\xi^{\rho_0}$ almost surely.

Proof. The argument is similar to the proof of Proposition 3.1. We fix $\alpha > \rho$. We change the order of integration and use the large deviation estimate (Proposition 2.3), the Hausdorff–Young inequality, and Hölder’s inequality to estimate

$$\begin{aligned} \|\widehat{u_+^\omega}\|_{L_\omega^\alpha L_\xi^\rho} &\lesssim \left\| \left\| \sum_{k \in \mathbb{Z}^d} g_k(\omega) \widehat{\psi_k u_+} \right\|_{L_\omega^\alpha} \right\|_{L_\xi^\rho} \lesssim \sqrt{\alpha} \left\| \left(\sum_{k \in \mathbb{Z}^d} |\widehat{\psi_k u_+}|^2 \right)^{\frac{1}{2}} \right\|_{L_\xi^\rho} \\ &\lesssim \sqrt{\alpha} \left(\sum_{k \in \mathbb{Z}^d} \|\widehat{\psi_k u_+}\|_{L_\xi^\rho}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sqrt{\alpha} \left(\sum_{k \in \mathbb{Z}^d} \|\psi_k u_+\|_{L_x^{\rho'}}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sqrt{\alpha} \left(\sum_{k \in \mathbb{Z}^d} \|\psi_k u_+\|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \sqrt{\alpha} \|u_+\|_{L_x^2}. \end{aligned}$$

Thus the desired estimate follows from Lemma 2.4.

To get the final conclusion we note that $\Omega_0 := \bigcup_{M=1}^{\infty} \Omega_{M,\rho_0}$ satisfies $\mathbb{P}(\Omega_0) = 1$ and $\widehat{u}_+^{\omega} \in L_{\xi}^{\rho_0}$ for all $\omega \in \Omega_0$. \square

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