AN INTEGRAL FORMULA FOR THE $Q'$-PRIME CURVATURE IN 3-DIMENSIONAL CR GEOMETRY

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ABSTRACT. We give an integral formula for the total $Q'$-curvature of a three-dimensional CR manifold with positive CR Yamabe constant and nonnegative Paneitz operator. Our derivation includes a relationship between the Green’s functions of the CR Laplacian and the $P'$-operator.

1. Introduction

The $Q'$-curvature, introduced to three-dimensional CR manifolds by the first- and third-named authors [2] and to higher-dimensional CR manifolds by Hirachi [9], has recently emerged as the natural CR counterpart to Branson’s $Q$-curvature in conformal geometry. The analogies are especially strong in dimension three, where it is known that the total $Q'$-curvature is a biholomorphic invariant — indeed, it is a multiple of the Burns–Epstein invariant [2, 3] — and gives rise to a CR invariant characterization of the standard CR three-sphere.

The above discussion is complicated by the fact that the $Q'$-curvature is most naturally defined only for pseudo-Einstein contact forms. A pseudohermitian manifold $(M^3, J, \theta)$ is pseudo-Einstein if the curvature $R$ and torsion $A_{11}$ of the Tanaka–Webster connection satisfy $R_1 = iA_{11,\bar{1}}$. It is known [8] that if $\theta$ is a pseudo-Einstein contact form, then $\hat{\theta} := e^\Upsilon \theta$ is pseudo-Einstein if and only if $\Upsilon$ is a CR pluriharmonic function. Moreover, if $M^3$ is embedded in $\mathbb{C}^2$, then pseudo-Einstein contact forms arise from solutions of Fefferman’s Monge–Ampère equation [5]. For a pseudo-Einstein manifold $(M^3, J, \theta)$, the $Q'$-curvature is defined by

$$Q' := -2\Delta_b R + R^2 - 4|A_{11}|^2.$$ 

The behavior of $Q'$ under the conformal transformation of $\theta$ to $\hat{\theta}$ is controlled by the $P'$-prime operator $P'$ and the Paneitz operator $P$, which have the local expressions

$$P'(u) := 4\Delta^2_b u - 8 \text{Im}(A_{11}u_{\bar{1}})_{,\bar{1}} - 4 \text{Re}(Ru_{1})_{,\bar{1}},$$

$$P(u) := \Delta^2_b u + T^2 u - 4 \text{Im}(A_{11}u_{1})_{,\bar{1}}.$$ 

More precisely, if $\hat{\theta} = e^\Upsilon \theta$ and $\theta$ are both pseudo-Einstein, then

$$e^{2\Upsilon} Q' = Q' + P'(\Upsilon) + \frac{1}{2} P(\Upsilon^2).$$

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From this formula, it is clear that the total $Q'$-curvature is independent of the choice of pseudo-Einstein contact form. A direct computation also shows that if the holomorphic tangent bundle of $M$ is trivial, then the total $Q'$-curvature is a multiple of the Burns–Epstein invariant [3].

The CR Yamabe constant of a CR manifold $(M^3, J)$ is the infimum of the total (Tanaka–Webster) scalar curvature over all contact forms of volume one. For CR manifolds $(M^3, J)$ with positive CR Yamabe constant and nonnegative Paneitz operator, the first- and third-named authors [2] showed that $\int Q' \leq 16\pi^2$ with equality if and only if $(M^3, J)$ is CR equivalent to the standard CR three-sphere. The main goal of this note is to refine this statement by giving an integral formula for the total $Q'$-curvature in terms of the Green’s function of the CR Laplacian:

**Theorem 1.** Let $(M^3, J, \theta)$ be a pseudo-Einstein manifold with positive CR Yamabe constant and nonnegative Paneitz operator. Assume also that $(M^3, J, \theta)$ is embedded in $\mathbb{C}^2$. Given any $p \in M$, it holds that

$$\int_M Q' = 16\pi^2 - 4 \int_M G^4_L |\hat{A}_{11}|^2 - 12 \int_M \log(G_L) P_4 \log(G_L)$$

where $G_L$ is the Green’s function for the CR Laplacian with pole $p$ and $\hat{\theta} = G^2_L \theta$. In particular,

$$\int_M Q' \leq 16\pi^2$$

with equality if and only if $(M^3, J)$ is CR equivalent to the standard CR three sphere.

Theorem 1 is motivated by similar work in conformal geometry: Gursky [6] used the total $Q$-curvature to characterize the standard four-sphere among all Riemannian manifolds with positive Yamabe constant and Hang–Yang [7] rederived this result by giving an integral formula for the total $Q$-curvature in terms of the Green’s function for the conformal Laplacian.

The key technical difficulty in the proof of Theorem 1 comes from the potential need to consider the $Q'$-curvature of a contact form which is not pseudo-Einstein. On the one hand, $\log G_L$ need not be CR pluri-harmonic, and hence $G^2_L \theta$ need not be pseudo-Einstein; this problem is overcome by adapting ideas from [2]. On the other hand, estimates for $\log G_L$ are usually derived in CR normal coordinates (cf. [10]), but CR normal coordinates need not be specified in terms of a pseudo-Einstein contact form. We overcome the latter issue by using Moser’s contact form, which is necessarily pseudo-Einstein, as a replacement for CR normal coordinates.

Ignoring these technical difficulties, the idea of the proof of Theorem 1 is to observe that $\hat{\theta} := G^2_L \theta$ has vanishing scalar curvature away from the pole, and hence $\hat{Q}'$ has a particularly simple expression. Equation (1) relates $Q'$ and $\hat{Q}'$ in terms of $P'(|\log G_L|)$ and $P'(|(\log G_L)|^2)$. 

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Using normal coordinates, we can compute these latter functions near the pole \( p \), at which point (2) follows from (1) by integration. As an upshot of this approach, we relate \( \log G_L \) and the Green’s function for \( P' \); we expect this relation to be useful for future studies of the \( Q' \)-curvature.

This note is organized as follows: In Section 2, we recall necessary facts about Moser’s contact form and use it to relate \( Q' \) and \( \hat{Q}' \). In Section 3, we integrate this relation to prove Theorem 1.

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2. Moser’s contact form and normal coordinates

Moser’s normal form for a real hypersurface in \( \mathbb{C}^2 \) (see, e.g., [4]) reads

\[
v = |z|^2 - E(u, z, \bar{z})
\]

where \((z, w) \in C^2, w = u + iv, \) and

\[
E(u, z, \bar{z}) = -c_{42}(u)z^4\bar{z}^2 - c_{24}(u)z^2\bar{z}^4 - c_{33}(u)z^3\bar{z}^3 + O(7).
\]

Hereafter we use \( O(k) \) to denote \( O(\rho^k) \) where \( \rho := (|z|^4 + u^2)^{1/4} \).

Associated to the defining function

\[
r = \frac{1}{2i}(w - \bar{w}) - |z|^2 + E(u, z, \bar{z}),
\]

we have Moser’s contact form

\[
\theta = i\partial r = \frac{1}{2} dw - i\bar{z}dz + i(E\bar{z}dz + E_u\frac{1}{2}dw)
\]

in which we have used \( E_w = E_u\frac{1}{2} \) and \( E_z := \partial E/\partial z, E_u := \partial E/\partial u, \) etc..

We call coordinates \((z, u)\) for real hypersurface \( \{r = 0\} \) Moser’s normal coordinates. We are going to compute pseudohermitian quantities with respect to Moser’s contact form in Moser’s normal coordinates. Compute

\[
d\theta = ig_{1\bar{1}}dz \wedge d\bar{z} + \theta \wedge \phi
\]

where

\[
\begin{align*}
g_{1\bar{1}} & = 1 - Ez\bar{z} - \lambda E_u\bar{z} - \bar{\lambda}E_uz - |\lambda|^2E_uu, \\
\phi & = a_1dz + a_{\bar{1}}d\bar{z}
\end{align*}
\]
in which
\begin{align*}
\lambda &= \frac{\bar{z} - E_z}{-i + E_u} = i\bar{z} - iE_z + \bar{z}E_u + O(6) \\
a_1 &= \frac{-E_{uz} - \lambda E_{uu}}{i + E_u}, \ a_{\bar{1}} = (a_1).
\end{align*}

The order counting follows the rule that $z$, $\bar{z}$ are of order 1 and $u$ is of order 2. Here we have also used the relation between $dw$ and $\theta$:

$$dw = \frac{2}{1 + iE_u}(\theta + i(\bar{z} - E_z)dz).$$

Take a pseudohermitian coframe
\begin{align*}
\theta^1 &:= dz - i\bar{a}^1\theta, \\
a^1 &:= g^{1\bar{1}}a_{\bar{1}},
\end{align*}

where $g^{1\bar{1}} := (g_{1\bar{1}})^{-1}$, such that

$$d\theta = ig_{1\bar{1}}\theta^1 \land \theta^{\bar{1}}.$$  

The dual frame $Z_1$ (such that $\theta(Z_1) = 0$, $\theta^1(Z_1) = 1$, and $\theta^{\bar{1}}(Z_1) = 0$) reads

$$Z_1 = \frac{\partial}{\partial z} + \lambda \frac{\partial}{\partial u} = \dot{Z}_1 + O(5) \frac{\partial}{\partial u}$$

where $\dot{Z}_1 := \partial_z + i\bar{z}\partial_u$.

Differentiating $\theta^1$ from (6) gives

$$d\theta^1 = \theta^1 \land \dot{\omega}_1^1 + iZ_1(a^1)\theta \land \theta^{\bar{1}}$$

by (7), where

$$\dot{\omega}_1^1 = a_1\theta^{\bar{1}} - iZ_1(a^1)\theta.$$

Differentiating (7) yields

$$\left(dg_{1\bar{1}} - g_{1\bar{1}}\dot{\omega}_1^1 - g_{\bar{1}\bar{1}}\dot{\omega}_1^{\bar{1}}\right) \land \theta^1 \land \theta^1 = 0,$$

where $\dot{\omega}_1^1$ is the complex conjugate of $\dot{\omega}_1^{\bar{1}}$. Therefore

$$dg_{1\bar{1}} - g_{1\bar{1}}\dot{\omega}_1^1 - g_{\bar{1}\bar{1}}\dot{\omega}_1^{\bar{1}} = [Z_1g_{1\bar{1}} - g_{1\bar{1}}a_1]\theta^1 + \text{conjugate}.$$ 

It follows that the pseudohermitian connection form $\omega_1^1$ reads

$$\omega_1^1 = \dot{\omega}_1^1 + (g^{1\bar{1}}Z_1g_{1\bar{1}} - a_1)\theta^1.$$ 

We also conclude from (9) that

$$A_1^1 = iZ_1(a^1).$$

Substituting (11) into the structure equation $d\omega_1^1 = Rg_{1\bar{1}}\theta^1 \land \theta^{\bar{1}}$ mod $\theta$, we obtain the Tanaka-Webster (scalar) curvature

$$R = \frac{1}{4}(Z_1a_1 + Z_1a^1 + Z^1a_1 - Z^1Z_1g_{1\bar{1}} + a^1(Z_1g_{1\bar{1}}) - a^1a_1).$$

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where we have used $g^{11}$ to raise the indices, e.g., $Z^1 := g^{11} Z_1 = (g^{11} Z_1)$, $a^1 := g^{11} a_1$. We then compute the lowest order terms of $Z^1 a_1$, $Z^1 g_{11}$ as follows:

$$
Z^1 a_1 = - E_{uu} - i E_{u\bar{z}} - z E_{uz} + \bar{z} E_{u\bar{z}} - i |z|^2 E_{uuu} + O(3),
$$

$$
Z^1 g_{11} = - E_{z\bar{z}} - 2i z E_{z\bar{z}u} + iz E_{z\bar{z}z} + z^2 E_{uuu} + i E_{uz}
- 2 |z|^2 E_{u\bar{z}u} - \bar{z} E_{u\bar{z}} - i \bar{z} |z|^2 E_{uuu} + O(4).
$$

Here we have counted $z, \bar{z}$ of order 1, $u$ of order 2, and used $g^{11} = 1 + O(4)$, $\lambda = i \bar{z} - i E_{z} + \bar{z} E_{u} + h.o.t., a_1 = i E_{uz} - \bar{z} E_{u\bar{z}} + h.o.t.$, $Z_1 = \partial_z + i \bar{z} \partial_u + h.o.t.$ From (12) we compute

$$
A_1^1 = E_{u\bar{z}z} - 2iz E_{u\bar{z}u} + z^2 E_{uuu} + O(3).
$$

By (14) and alike formulas, we can compute $R$ through (13):

$$
R = -2 E_{uu} + E_{z\bar{z}z} - 2i z E_{z\bar{z}u} + 2i \bar{z} E_{z\bar{z}u}
+ 4 |z|^2 E_{u\bar{z}u} - z^2 E_{u\bar{z}z} - \bar{z}^2 E_{uuu} + 2iz |z|^2 E_{uuu}
- 2iz |z|^2 E_{uuu} + |z|^4 E_{uuu} + O(3).
$$

We can then compute $R_{1} = Z_1 R$, $A_{1,1}^1$, and obtain the pseudo-Einstein tensor as follows:

$$
R_{1} - i A_{1,1}^1 = E_{z\bar{z}z} - 4i z E_{ uu} + 3iz E_{z\bar{z}u}
-3iz E_{z\bar{z}u} - 2iz E_{z\bar{z}z} + 6 |z|^2 E_{z\bar{z}uu}
-3z^2 E_{u\bar{z}z} + 6iz E_{u\bar{z}u} + 6i z |z|^2 E_{z\bar{z}uu}
-3z E_{u\bar{z}z} - z^2 E_{u\bar{z}u} - 3iz |z|^2 E_{uuu}
-iz^3 E_{uuu} + iz^2 E_{uuu} - 2iz |z|^2 E_{uuu}
-6iz z^2 E_{ uuu} + 3 |z|^4 E_{uuu} + iz |z|^2 E_{uuu}
+iz |z|^4 E_{uuu} + O(2).
$$

From (17) along the $u$-curve (a chain) where $z = 0$, we conclude that $R_{1} - i A_{1,1}^1 = 0$ (terms in $O(2)$ all vanish because of the special structure of Moser’s normal form) and does not vanish identically in general. The reason is that the coefficient of $z$ in $E_{z\bar{z}z} z$ is $c_{42}(u)$ which is proportional to the Cartan tensor.

In general a pseudo-Einstein contact form may not be a “normalized” contact form that gives CR normal coordinates. So we take the contact form associated to the solution $\psi$ to the complex Monge–Ampère equation:

$$
J[\psi] := \det \begin{pmatrix}
\psi & \psi_{\bar{z}} & \psi_{\bar{w}} \\
\psi_{z} & \psi_{zz} & \psi_{z\bar{w}} \\
\psi_{w} & \psi_{w\bar{z}} & \psi_{ww}
\end{pmatrix} = 1
$$

in $\Omega$ and $\psi = 0$ on $\partial \Omega$. The contact form $\theta := i \partial \bar{\psi}$ is pseudo-Einstein. We want to compute $\Delta_b$, $P$, $P$ w.r.t. this $\theta$, but in Moser’s normal
coordinates \((z, u)\). For \(r\) having a form of (3) multiplied by \(4^{1/3}\), we have

\[
J[r] = 1 + O(\rho^4).
\]

Lee-Melrose’s asymptotic expansion [12] reads

\[
\psi \sim r \sum_{k \geq 0} \eta_k (r^3 \log r)^k \text{ near } \partial \Omega = \{r = 0\} \subset C^2
\]

with \(\eta_k \in C^\infty(\bar{\Omega})\). This means that for large \(N\), \(\psi - r \sum_{k=0}^N \eta_k (r^3 \log r)^k\) has many continuous derivatives on \(\bar{\Omega}\) and vanishes to high order at \(\partial \Omega\). It follows from (18), (19), and (20) that

\[
J[r \eta_0] = 1 + O(\rho^4) \quad \text{and} \quad \eta_0 = 1 + O(\rho^4).
\]

So we have

\[
\psi \sim r \eta_0 + \eta_1 r^4 \log r + h.o.t.
\]

\[
\sim r + O(\rho^6).
\]

Similar argument as for \(r\) before works for \(\psi\). Therefore, with respect to the pseudo-Einstein contact form defined by \(\psi\), we still have

\[
\theta = (1 + O(\rho^4)) \hat{\theta} + O(\rho^5) dz + O(\rho^5) d\bar{z},
\]

\[
\theta^1 = O(\rho^3) \hat{\theta} + (1 + O(\rho^8)) dz + O(\rho^8) d\bar{z},
\]

\[
Z_1 = \hat{Z}_1 + O(\rho^5) \frac{\partial}{\partial u},
\]

\[
\omega_1^1 = O(\rho^2) \hat{\omega} + O(\rho^3) dz + O(\rho^7) d\bar{z},
\]

\[
A_1^1 = O(\rho^2), \quad R = O(\rho^2),
\]

\[
g_{11} = 1 + O(\rho^4), \quad g^{11} = 1 + O(\rho^4)
\]

in view of (5), (8), (10), (11), (15), (16), and (4). Now let \(L\) denote the CR Laplacian:

\[
L := -4 \triangle_b + R
\]

where \(\triangle_b\) is the (positive) sublaplacian given by

\[
\triangle_b = Z^1 Z_1 - \omega_1^1 (Z^1)Z_1 + \text{conjugate}.
\]

Let \(G_L\) denote the Green’s function of \(L\) with pole at \(p\); i.e.,

\[
L G_L = -4 \triangle_b G_L + R G_L = 16 \delta_p.
\]

Let \(\hat{P}' := 4 \hat{\triangle}_b^2\), \(\hat{L} := -4 \hat{\triangle}_b\) denote the \(P'\) operator, the CR Laplacian for the Heisenberg group \(\mathbb{H}^1\), respectively. Observe that (cf. [1])

\[
\hat{P}'(\log G_L) = \hat{P}'(\log \frac{1}{2\pi \rho^2}) = 8\pi^2 S_p
\]

with \(S_p := S(p, \cdot)\), where \(S(p, \cdot)\) is the kernel of the orthogonal projection \(\pi: L^2(\mathbb{H}^1) \to \mathcal{P}(\mathbb{H}^1)\) onto the space of CR pluriharmonic functions.
where we have used \( G_L = \frac{1}{2\pi \rho^2} \). From (21) and (22) we obtain

\[
\Delta_b = (1 + O(\rho^4)) \Delta_b + O(\rho^{10}) \frac{\partial^2}{\partial u^2} + O(\rho^4) \frac{\partial}{\partial u} \\
+ O(\rho^5) \frac{\partial}{\partial u} \circ \hat{Z}_1 + O(\rho^7) \hat{Z}_1 \\
+ O(\rho^5) \frac{\partial}{\partial u} \circ \hat{Z}_1 + O(\rho^7) \hat{Z}_1.
\]

From (21) and (22) we obtain

\[
\Delta_b = \frac{1}{2\pi \rho^2} \pi^2 + \omega.
\]

Write \( L \omega = a \) bounded function near \( p \).

Therefore from subelliptic regularity theory of \( L \), we see that \( \omega \) is in the Folland–Stein space \( S^{2,q} \) for any \( q > 1 \), and hence \( w \in C^{1,\gamma} \). In fact, \( \omega \) is \( C^\infty \) smooth \([10]\). Recall that

\[
P' = 4 \Delta_b - 8 \Im \nabla^1(A_1 \nabla_1) - 4 \Re \nabla^1(R \nabla_1)
\]

\[
= \hat{P}' + 4(\Delta_b^2 - \hat{\Delta}_b^2) \\
- 8 \Im \nabla^1(A_1 \nabla_1) - 4 \Re \nabla^1(R \nabla_1).
\]

Write

\[
\log G_L = \log \left( \frac{1}{2\pi \rho^2} + \omega \right)
\]

\[
= \log \left( \frac{1}{2\pi \rho^2} \right) + \log(1 + 2\pi \rho^2 \omega).
\]

We can now compute

\[
P'(\log G_L) = \hat{P}'(\log \left( \frac{1}{2\pi \rho^2} \right)) + (P' - \hat{P}') \log(\frac{1}{2\pi \rho^2})
\]

\[
+ P'(1 + 2\pi \rho^2 \omega)
\]

\[
= 8\pi^2 S_p + \{4(\Delta_b^2 - \hat{\Delta}_b^2) - 8 \Im \nabla^1(A_1 \nabla_1)
\]

\[
- 4 \Re \nabla^1(R \nabla_1)\} \log(\frac{1}{2\pi \rho^2})
\]

\[
+ P'(1 + 2\pi \rho^2 \omega).
\]

Since \( \omega \) is \( C^\infty \) smooth, the third term is a bounded function near \( p \).

The second term is also bounded near \( p \) in view of (21) and (24). So we conclude that

\[
P'(\log G_L) = 8\pi^2 S_p + a \text{ bounded function}.
\]

Similarly we can show

\[
P'((\log G_L)^2) = 8\pi^2 (\delta_p - S_p) + a \text{ bounded function}.
\]
On the other hand, we reduce computing the most singular term in $P_3(\log G_L)$ to computing $P_3(\log(\frac{1}{2\pi\rho^2}))$ by (26). In view of (21) we find that the most singular term in $P_3(\log(\frac{1}{2\pi\rho^2}))$ is a constant multiple of $\hat{P}_3(\log \rho)$ where $\hat{P}_3 = \hat{Z}_1\hat{Z}_1\hat{Z}_1$ is the $P_3$-operator w.r.t. the Heisenberg group $H_1$. Observe that $|z|^2 - iu$ is a CR function on $H_1$, i.e.,

$$\hat{Z}_1(|z|^2 - iu) = (\partial_z - iz\partial_u)(|z|^2 - iu) = 0.$$ 

It follows that the real part of $\log(|z|^2 - iu)$ is CR pluriharmonic. By [11] we have

$$\hat{P}_3 ((\log ||z|^2 - iu|) = \hat{Z}_1\hat{Z}_1\hat{Z}_1 (\log ||z|^2 - iu|) = 0.$$ 

Since $\log ||z|^2 - iu| = 2\log \rho$, we conclude that

$$\hat{P}_3(\log \rho) = 0.$$ 

It follows that

(29) $P_3(\log G_L) = \hat{P}_3(\log(\frac{1}{2\pi\rho^2}))$

$$+ (P_3 - \hat{P}_3)(\log(\frac{1}{2\pi\rho^2})) + P_3(\log(1 + 2\pi\rho^2\omega))$$

$$= (P_3 - \hat{P}_3)(\log(\frac{1}{2\pi\rho^2})) + P_3(\log(1 + 2\pi\rho^2\omega))$$

$$= O(\rho).$$

by (21). It follows that $(\log G_L)P(\log G_L) = O(\log \rho)$ near the pole $p$. Hence it is integrable with respect to the volume $\theta \wedge d\theta$ which has vanishing order $\rho^3$ near $p$.

3. A FORMULA FOR THE INTEGRAL OF $Q'$ CURVATURE

Let $\theta$ be a pseudo-Einstein contact form on $(M^3, J)$. By [2, Proposition 6.1], for any $\Upsilon \in C^\infty(M)$, it holds that $\hat{\Upsilon} := e^\Upsilon \theta$ satisfies

(30) $e^{2\Upsilon} \hat{Q}' = Q' + P'(\Upsilon) + \frac{1}{2}P(\Upsilon^2)$

$$- \Upsilon P(\Upsilon) - 16 \text{Re}(\nabla^1 \Upsilon)(P_3 \Upsilon)_1$$

where $P_3$ is the operator characterizing CR pluriharmonics. Recall that $P(\Upsilon) = 4\nabla^1(P_3 \Upsilon)_1$.

Let $G_L$ be the Green’s function of the CR Laplacian (we assume $Y(J) > 0$). Set $\hat{\theta} = G_L^2 \theta$. Then $\hat{\theta}$ has vanishing scalar curvature away from the pole $p$. In particular, we have

$$\hat{Q}' = -4|\hat{A}_{11}|_p^2$$

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away from the pole \( p \). Plugging this into (30), we see that away from \( p \),

\[
-4G^4_L|\hat{A}_{11}|^2_\bar{\theta} = Q' + 2P'(\log G_L) \\
+ 2P((\log G_L)^2) - 4(\log G_L)P(\log G_L) \\
- 64 \text{Re}(\nabla^1 \log G_L)(P_3(\log G_L))_1.
\]

Now assume \((M^3, J)\) is embedded in \( \mathbb{C}^2 \). Take \( \theta \) to be the pseudo-Einstein contact form associated to the solution to complex Monge–Ampère equation (18). We look at the order of 

\[
-4G^4_L|\hat{A}_{11}|^2_\bar{\theta}
\]

near \( p \). The transformation law of torsion reads

\[
\hat{A}_{11} = G^{-2}_L(A_{11} + 2i(\log G_L),_{11} - 4i(\log G_L),_1(\log G_L),_1
\]

(see [11, p. 421]). Recall \( \check{Z}_1 := \partial_z + i\bar{z}\partial_u \). Observe that

\[
\hat{Z}_1 \log \rho^4 = \frac{2\bar{z}}{|z|^2 - iu}, \\
\check{Z}_1 \hat{Z}_1 \log \rho^4 = \frac{-4\bar{z}^2}{(|z|^2 - iu)^2} = -(\check{Z}_1 \log \rho^4)^2.
\]

Therefore we have

\[
\hat{Z}_1 \hat{Z}_1 \log \frac{1}{2\pi \rho^2} - 2(\hat{Z}_1 \log \frac{1}{2\pi \rho^2})^2 = 0
\]

It follows from (21) and (33) that

\[
A_{11} = O(\rho^2) \\
2i(\log G_L),_{11} - 4i(\log G_L),_1(\log G_L),_1 = O(\rho^2)
\]

near \( p \). So from (32) and (34), we learn that

\[
G^4_L|\hat{A}_{11}|^2_\bar{\theta} = O(\rho^4)
\]

near \( p \). By (29), we obtain that the last two terms in (31) are \( L^1 \) and bounded near \( p \), respectively. In view of (27), (28), (35), and (31), we then have

\[
2P'(\log G_L) + 2P((\log G_L)^2) \\
= 16\pi^2 \delta_p - Q' - 4G^4_L|\hat{A}_{11}|^2_\bar{\theta} \\
+ 4(\log G_L)P(\log G_L) + 64 \text{Re}(\nabla^1 \log G_L)(P_3(\log G_L))_1.
\]

in the distribution sense. Integrating the last term in (36) gives

\[
-16 \int (\log G_L)P(\log G_L) + 64 \text{Re} \int_{\text{around } p} (\log G_L)P_3(\log G_L)i\theta \wedge \theta^1.
\]

Here we have omitted the lower index “1” for the \( P_3 \) term. The boundary term in (37) vanishes by (29) and that \( \theta \wedge \theta^1 \) has vanishing order
of $\rho^3$ near $p$. Applying (36) to the constant function 1 yields

$$0 = 16\pi^2 - \int Q' - 4\int G^4_L |\hat{A}_{11}|^2_{\hat{\theta}} - 12\int (\log G_L)P(\log G_L)$$

by (37). Here notice that, since $P'(1) = 0$, it holds that

$$P'(\log G_L)(1) := \int (\log G_L)P'(1) = 0;$$

i.e. $P'(\log G_L)$ annihilates constants as a distribution. Similarly, since $P(1) = 0$, we obtain that $2P((\log G_L)^2)(1) = 0$. Therefore

$$\int Q' = 16\pi^2 - 4\int G^4_L |\hat{A}_{11}|^2_{\hat{\theta}} - 12\int (\log G_L)P(\log G_L).$$

(38) Since $P \geq 0$ and $(\log G_L)P(\log G_L)$ is integrable (cf. (29)), we conclude that

$$\int Q' \leq 16\pi^2.$$ (39)

Moreover, since the total $Q'$-curvature is independent of the choice of pseudo-Einstein contact form [2], both (38) and (39) inequality hold for any pseudo-Einstein contact form on $M$.

Finally, equality holds in (39) if and only if $\hat{A}_{11} \equiv 0$ and $\log G_L$ is pluriharmonic. Since also $\hat{R} \equiv 0$, we conclude that $(M \setminus \{p\}, \hat{\theta})$ is isometric to the Heisenberg group $\mathbb{H}^3$. Indeed, the developing map identifies the universal cover of $M \setminus \{p\}$ with $\mathbb{H}^3$, while the fact that a neighborhood of $p$ (equivalently, a neighborhood of infinity in $(M \setminus \{p\}, \hat{\theta})$) is simply connected implies that the covering map is trivial. By adding back the point $p$, we conclude that $(M, J)$ is CR equivalent to the standard CR three-sphere.

**REFERENCES**


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