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Strong sequential completeness of the natural domain of a conditional expectation operator in Riesz spaces

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Abstract

Strong convergence and convergence in probability were generalized to the setting of a Riesz space with conditional expectation operator, $T$, in [Y. Azouzi, W.-C. Kuo, K. Ramdane, B. A. Watson, Convergence in Riesz spaces with conditional expectation operators, Positivity, 19 (2015), 647-657] as $T$-strong convergence and convergence in $T$-conditional probability, respectively. Generalized $L^p$ spaces for the cases of $p = 1, 2, \infty$, were discussed in the setting of Riesz spaces as $L^p(T)$ spaces in [C. C. A. Labuschagne, B. A. Watson, Discrete stochastic integration in Riesz spaces, Positivity, 14 (2010), 859-875]. An $R(T)$ valued norm, for the cases of $p = 1, \infty$, was introduced on these spaces in [W. Kuo, M. Rogans, B.A. Watson, Mixing processes in Riesz spaces, Journal of Mathematical Analysis and Application, 456 (2017), 992-1004] where it was also shown that $R(T)$ is a universally complete $f$-algebra and that these spaces are $R(T)$-modules. In [Y. Azouzi, M. Trabelsi, $L^p$-spaces with respect to conditional expectation on Riesz spaces, Journal of Mathematical Analysis and Application, 447 (2017), 798-816] functional calculus was used to consider $L^p(T)$ for $p \in (1, \infty)$. In this paper we prove the strong sequential completeness of the space $L^1(T)$, the natural domain of the conditional expectation operator $T$, and the strong completeness of $L^\infty(T)$.

Keywords: Strong completeness; Riesz spaces; conditional expectation operators. Mathematics subject classification (2010): 46B40; 60F15; 60F25.

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1 Introduction

Strong convergence and convergence in probability were generalized to Dedekind complete Riesz spaces with a conditional expectation operator in [2] as $T$-strong convergence and $T$-conditional convergence in conditional probability, respectively. Generalized $L^p$ spaces for $p = 1, 2, \infty$ were discussed in the setting of Riesz spaces as $L^p(T)$ spaces in [9]. An $R(T)$ valued norm, for the cases of $p = 1, \infty$, was introduced on the $L^p(T)$ spaces in [8] where it was also shown that $R(T)$ is a universally complete $f$-algebra and that these spaces are $R(T)$-modules. More recently, in [3], the $L^p(T)$, for $p \in (1, \infty)$, spaces were considered. We also refer the reader to [13] for an interesting study of sequential order convergence in vector lattices using convergence structures and filters, and to [4] for the well known proof of the strong sequential completeness of $L^1(\Omega, \mathcal{F}, \mu)$.

In this paper we prove the strong sequential completeness of the natural domain, $L^1(T)$, of the Riesz space conditional expectation operator $T$, i.e. that each strong Cauchy sequence in $L^1(T)$ converges strongly in $L^1(T)$. The term strong here means with respect to the vector valued norm induced by the conditional expectation operator $T$ in the given space. These results can be extended to the convergence of strong Cauchy nets which contain a sequence as a subnet. We conclude by showing the strong completeness of $L^\infty(T)$, i.e. that every strong Cauchy net in $L^\infty(T)$ is strongly convergent.

Interest in the completeness studied in this paper came at least in part from [6]. There the Riesz space $L^1(T)$ is endowed with a locally solid, locally convex linear topology where, for every positive order continuous linear functional $\varphi$ on $L^1(T)$, a semi-norm $p_\varphi$, is defined as $p_\varphi(f) := \varphi(T|f|)$. Under the condition that $L^1(T)$ is a perfect Riesz space, in [6] it was proved $L^1(T)$, endowed with the topology generated by these semi-norms, is complete. The results of the current work supply a partial answer to whether one can avoid duality theory and the assumption of the Riesz space being perfect to obtain that $L^1(T)$ is complete with suitable definitions of completeness and convergence.

The issue of completeness of $L^1(T)$ is important in the theory of stochastic integrals in Riesz spaces, since these integrals are defined to be limits of Cauchy nets in $L^1(T)$. The results also impact on the study of martingales in Riesz spaces, see [11, 12].

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2 Preliminaries

For general background on Riesz spaces and order convergence we refer the reader to [1, 10, 14].
A conditional expectation operator, $T$, on a Dedekind complete Riesz space, $E$, with weak order unit, say $e$, is a positive order continuous projection which maps weak order units to weak order units and has $R(T)$ a Dedekind complete Riesz subspace of $E$, see [7]. In addition we assume in this paper that $T$ is strictly positive, in that if $v \in E_+$ with $v \neq 0$ then $Tv \neq 0$ ($Tv \geq 0$ as $T$ is positive). This last condition is required for both the construction of the $T$-universal completion of $E$, i.e. the natural domain, $L^1(T)$, of $T$ and so that the mapping $f \mapsto T|f|$ defines an $R(T)$ valued norm on $L^1(T)$.

The Riesz space $L^1(T)$ is defined to be the $T$-universal completion of $E$ or natural domain of $T$, see [5] and [7]. We recall that $T$ has a unique extension to $L^1(T)$ as a conditional expectation operator. In particular $L^1(T)$ is characterized by the property that if $(x_\alpha)$ is an upward directed net in $L^1(T)$ with $(Tx_\alpha)$ bounded in $E^\infty$ (the universal completion), then $(Tx_\alpha)$ is order convergent in $L^1(T)$.

We recall from [8] that in $L^1(T)$, $R(T)$ is a universally complete $f$-algebra and that $L^1(T)$ is an $R(T)$-module. It thus makes sense, as was done in [8], to define an $R(T)$-valued norm on $L^1(T)$ by $\|f\|_{T,1} := T|f|$. This norm takes its values in $R(T)^+$, is homogeneous with respect to multiplication by elements of $R(T)^+$, is strictly positive and obeys the triangle inequality. For more details on this norm we refer the reader to [8]. Convergence with respect to this norm was called $T$-strongly convergence in [2] where various of its properties were studied in relation to other modes of convergence.

The other space that will be of interest in this work is

$$L^\infty(T) := \{ f \in L^1(T) : |f| \leq g \text{ for some } g \in R(T) \}$$

with $R(T)$-valued norm

$$\|f\|_{T,\infty} := \inf\{ g \in R(T) : |f| \leq g \}.$$

We refer the reader to [8] for more details and for the readers convenience we give an abbreviated version of the example presented there.

**Example:** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, which to be interesting should have $\mu(\Omega) = \infty$ and suppose that there is $(\Omega_n)_{n \in \mathbb{N}}$ be an $\mathcal{A}$-measurable partition of $\Omega$ into sets of finite positive measure. Let $\Sigma$ be the sub-$\sigma$-algebra of $\mathcal{A}$ generated by $(\Omega_n)_{n \in \mathbb{N}}$. We take as the starting Riesz space $E = L^\infty(\Omega, \mathcal{A}, \mu)$ and the conditional expectation operator $T = \mathbb{E}[\cdot | \Sigma]$.

For $f \in E$ we have

$$Tf(\omega) = \frac{\int_{\Omega_n} f \, d\mu}{\mu(\Omega_n)}, \quad \text{for } \omega \in \Omega_n. \quad (2.1)$$

The universal completion, $E^\infty$, of $E$ is the space of equivalence classes of $\mathcal{A}$-measurable functions. Here the $\mathcal{A}$-measurable functions $f$ and $g$ are equivalent if $f = g$ a.e. with respect to the measure $\mu$. 

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The $T$-universal completion of $E$ is the space

$$\mathcal{L}^1(T) = \left\{ f \in E^u \bigg| \int_{\Omega_n} |f| d\mu < \infty \text{ for all } n \in \mathbb{N} \right\},$$

which is characterized by $f|_{\Omega_n} \in L^1(\Omega, \mathcal{A}, \mu)$, for each $n \in \mathbb{N}$.

We note that $E$ has weak order unit $e = 1$, the equivalence class of functions a.e. identically 1 on $\Omega$, which again is a weak order unit for $L^1(T)$. The range of the generalized conditional expectation operator $T$ (as extended to $L^1(T)$) is

$$R(T) = \{ f \in E^u \mid f \text{ a.e. constant on } \Omega_n, n \in \mathbb{N} \},$$

which is an $f$-algebra.

Finally

$$\mathcal{L}^\infty(T) = \{ f \in E^u \mid f \text{ essentially bounded on } \Omega_n \text{ for each } n \in \mathbb{N} \}.$$

The vector norms on $L^1(T)$ and $L^\infty(T)$ are

$$\|f\|_{T,1}(\omega) = T|f|(\omega) = \int_{\Omega_n} |f| d\mu(\Omega_n), \text{ for } \omega \in \Omega_n, f \in L^1(T), \quad (2.2)$$

$$\|f\|_{T,\infty}(\omega) = \text{ ess sup}_{\Omega_n} |f|, \text{ for } \omega \in \Omega_n, f \in L^\infty(T). \quad (2.3)$$

The following lemma will assist in the proof of strong sequential completeness.

**Lemma 2.1** Let $(h_n)$ be a sequence in $L^1(T)$ with $s := \sum_{n=1}^{\infty} T|h_n|$ order convergent in the universal completion of $L^1(T)$, then the summation $\sum_{n=1}^{\infty} h_n$ is order convergent in $L^1(T)$.

**Proof:** Let $s_n^\pm = \sum_{i=1}^{n} h_i^\pm$, then the partial sums $s_n$ of $\sum_{n=1}^{\infty} h_n$ are given by $s_n = s_n^+ - s_n^-$. Here $(s_n^\pm)$ are increasing sequences with

$$Ts_n^\pm = \sum_{i=1}^{n} Th_i^\pm \leq \sum_{i=1}^{n} T|h_i| \leq s.$$

The $T$-universal completeness of $L^1(T)$ now allows us to conclude that $(s_n^\pm)$ are convergent in $L^1(T)$ to limits, say $h^\pm$. Setting $h = h^+ - h^-$ we have that

$$s_n = s_n^+ - s_n^- \to h^+ - h^- = h \in L^1(T)$$

in order as $n \to \infty$. \hfill \blacksquare
Definition 2.2 We say that a net \((f_\alpha)\) in \(L^p(T)\), \(p = 1, \infty\), is a strong Cauchy net if
\[
v_\alpha := \sup_{\beta, \gamma \geq \alpha} \|f_\beta - f_\gamma\|_{T,p}
\]
is eventually defined and has order limit zero.

3 Strong sequential completeness of \(L^1(T)\)

We now show that \(L^1(T)\) is strongly sequentially complete - i.e. that for every sequence \((f_n)\) in \(L^1(T)\) with \(\sup_{i,j \geq n} T|f_i - f_j| \downarrow 0\) there is \(f \in L^1(T)\) so that \(T|f_n - f| \to 0\) in order as \(n \to \infty\).

Theorem 3.1 \(L^1(T)\) is strongly sequentially complete.

Proof: Let \((f_n)\) be a strong \(T\)-Cauchy sequence in \(L^1(T)\). From the definition of a strong Cauchy sequence, we can define
\[
v_n := \sup_{r,s \geq n} T|f_r - f_s|
\]
where the sequence \((v_n) \subset R(T)\) decreases with infimum zero. As \(e, v_n \in R(T)\), it follows that \((\frac{1}{2^j}e - v_n)^+ \in R(T)\) and hence the band projections \(P_{j,n} := P_{\frac{1}{2^j}e - v_n}^+, j, n \in \mathbb{N}\), commute with \(T\), see [7]. For \(n = 0\) define \(P_{j,0} = 0\). We observe that \(P_{j,n}\) is increasing in \(n\) and decreasing in \(j\). In particular, \(\lim_{n \to \infty} P_{j,n} = I\), since \(v_n \downarrow 0\). Hence
\[
\sum_{n=0}^{\infty} (P_{j,n+1} - P_{j,n}) = I \text{ for each } j \in \mathbb{N}.
\]

We now construct a sequence \((g_j) \in L^1(T)\) that is both asymptotically close to \((f_n)\) and is strongly convergent in \(L^1(T)\). As band projections commute with Riesz space absolute value, we have
\[
T|(P_j,n - P_{j,n-1})f_{\max\{j,n\}}| = (P_j,n - P_{j,n-1})T|f_{\max\{j,n\}}|, \quad n, j \in \mathbb{N}.
\]
Here, for \(m \neq n\), \((P_j,n - P_{j,n-1}) \land (P_{j,m} - P_{j,m-1}) = 0\) so
\[
\sum_{n=1}^{\infty} T|(P_j,n - P_{j,n-1})f_{\max\{j,n\}}| = \sum_{n=1}^{\infty} (P_j,n - P_{j,n-1})T|f_{\max\{j,n\}}| = \sup_{n \in \mathbb{N}} (P_j,n - P_{j,n-1})T|f_{\max\{j,n\}}| =: K \in E^u
\]

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exists in the universal completion $E^u$. Lemma 2.1 can now be applied to give that the summation
\[ g_j = \sum_{n=1}^{\infty} (P_{j,n} - P_{j,n-1})f_{\max(j,n)}, \quad j \in \mathbb{N}, \]
converges in order in $L^1(T)$.

We now show that the sequence $(g_j)$ converges in $L^1(T)$. Consider $T|g_j - g_{j+1}|$. Because $\sum_{n=0}^{\infty} (P_{j,n+1} - P_{j,n}) = I$ for each $j \in \mathbb{N}$, we have that
\[
T|g_j - g_{j+1}| = T \left| \sum_{n=1}^{\infty} (P_{j,n} - P_{j,n-1})f_{\max(j,n)} - \sum_{m=1}^{\infty} (P_{j+1,m} - P_{j+1,m-1})f_{\max(j+1,m)} \right| \\
= T \left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (P_{j+1,m} - P_{j+1,m-1})(P_{j,n} - P_{j,n-1})(f_{\max(j,n)} - f_{\max(j+1,m)}) \right| \\
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (P_{j+1,m} - P_{j+1,m-1})(P_{j,n} - P_{j,n-1})T|f_{\max(j,n)} - f_{\max(j+1,m)}|.
\]
Here we have used that
\[
(P_{j+1,m} - P_{j+1,m-1})(P_{j,n} - P_{j,n-1}) \wedge (P_{j+1,x} - P_{j+1,x-1})(P_{j,y} - P_{j,y-1}) = 0
\]
for $(m, n) \neq (x, y)$.

For $m \geq n$ we have
\[
(P_{j,n} - P_{j,n-1})T|f_{\max(j,n)} - f_{\max(j+1,m)}| \leq (P_{j,n} - P_{j,n-1})v_n \\
\leq \frac{1}{2^j}(P_{j,n} - P_{j,n-1})e
\]
while for $m < n$ we have
\[
(P_{j+1,m} - P_{j+1,m-1})T|f_{\max(j,n)} - f_{\max(j+1,m)}| \leq (P_{j+1,m} - P_{j+1,m-1})v_m \\
\leq \frac{1}{2^{j+1}}(P_{j+1,m} - P_{j+1,m-1})e.
\]
Thus
\[
T|g_j - g_{j+1}| \leq \frac{1}{2^j} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (P_{j+1,m} - P_{j+1,m-1})(P_{j,n} - P_{j,n-1})e = \frac{1}{2^j} e
\]
and the summation $\sum_{j=1}^{\infty} T|g_j - g_{j+1}|$ is $e$-uniformly (and hence order) convergent. Application of Lemma 2.1 gives that the summation $\sum_{j=1}^{\infty} (g_j - g_{j+1})$ is order convergent.
which is equivalent to the order limit \( \lim_{j \to \infty} (g_1 - g_{j+1}) \) existing. We can thus define \( g \) to be the order limit of the sequence \((g_j)\) in \( L^1(T) \). Order continuity of \( T \) now gives that \( \lim_{n \to \infty} T|g_n - g| = 0 \) and that \((g_n)\) converges strongly to \( g \) in \( L^1(T) \).

From the order continuity of \( T \) and the order convergence of \((g_n)\) to \( g \) we have that \( T|g_n - g| \to 0 \) in order. Hence to show that \( g \) is the strong limit of the \((f_n)\) it suffices to prove that \( T|g_n - f_n| \to 0 \) in order. As \( \sum_{n=0}^{\infty} (P_{j,n+1} - P_{j,n}) = I \) we have

\[
g_j - f_j = \sum_{n=1}^{\infty} (P_{j,n} - P_{j,n-1}) (f_{\max_{j,n}} - f_j) = \sum_{n=1}^{j} (P_{j,n} - P_{j,n-1}) (f_j - f_j) + \sum_{n=j+1}^{\infty} (P_{j,n} - P_{j,n-1}) (f_n - f_j)
\]

The order continuity of \( T \) gives

\[
T|g_j - f_j| \leq \sum_{n=j+1}^{\infty} (P_{j,n} - P_{j,n-1}) T|f_n - f_j| \leq \sum_{n=j+1}^{\infty} (P_{j,n} - P_{j,n-1}) v_j \leq v_j
\]

and \( v_j \downarrow 0 \) as \( j \to \infty \). Thus \( T|f_j - g| \to 0 \) in order as \( n \to \infty \).

These results extended to the convergence of strong Cauchy nets which contain a sequence as a subnet. More can be said in the case of \( p = \infty \), as see in the following section.

### 4 Strong completeness of \( \mathcal{L}^\infty(T) \)

For the case of \( \mathcal{L}^\infty(T) \) we can prove a stronger result, i.e. that \( \mathcal{L}^\infty(T) \) is strongly complete. The proof, unfortunately, cannot be extended to \( \mathcal{L}^p(T) \) for \( p \in [1, \infty) \).

**Theorem 4.1** Each strong Cauchy net in \( \mathcal{L}^\infty(T) \) is strongly convergent in \( \mathcal{L}^\infty(T) \).
Proof: Let $(f_\alpha)$ be a strong Cauchy net in $L^\infty(T)$, then eventually

$$v_\alpha := \sup_{\beta, \gamma \geq \alpha} \|f_\beta - f_\gamma\|_{T, \infty} = \inf\{g \in R(T) : |f_\beta - f_\gamma| \leq g \text{ for all } \beta, \gamma \geq \alpha\}$$

exists as an element of $R(T)$ and $v_\alpha \downarrow 0$. Hence eventually $|f_\beta - f_\gamma| \leq v_\alpha$, for $\beta, \gamma \geq \alpha$, and the Cauchy net $(f_\alpha)$ is eventually bounded. We can thus define $\underline{f} := \liminf_\alpha f_\alpha$, $\overline{f} := \limsup_\alpha f_\alpha$ in $L^\infty(T)$. Now

$$0 \leq \overline{f} - \underline{f} = \lim_\alpha (\sup_{\beta \geq \alpha} f_\beta - \inf_{\gamma \geq \alpha} f_\gamma) = \lim_\alpha \sup_{\beta, \gamma \geq \alpha} (f_\beta - f_\gamma) \leq \lim_\alpha v_\alpha = 0.$$ 

So $\overline{f} = f$ and we can set $f := \overline{f} = f$ with $(f_\alpha)$ converging in order $f$, see [2]. However $|f_\beta - f_\gamma| \leq v_\alpha$ for all $\beta, \gamma \geq \alpha$, so taking the order limit in the index $\gamma$ we have $|f_\alpha - f| \leq v_\alpha$ and hence $\|f_\alpha - f\|_{T, \infty} \leq v_\alpha \downarrow 0.$ \(\blacksquare\)

References


