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DIMENSION DEPENDENCE OF FACTORIZATION PROBLEMS: BIPARAMETER HARDY SPACES

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Abstract. Given $1 \leq p, q < \infty$, and $n \in \mathbb{N}_0$, let $H^p_n(H^q_n)$ denote the finite-dimensional building blocks of the biparameter dyadic Hardy space $H^p(H^q)$. Let $(V_n : n \in \mathbb{N}_0)$ denote either $\big(H^p_n(H^q_n) : n \in \mathbb{N}_0\big)$ or $\big((H^p_n(H^q_n))^* : n \in \mathbb{N}_0\big)$. We show that the identity operator on V_n factors through any operator $T : V_N \to V_N$ which has a large diagonal with respect to the Haar system, where N depends *linearly* on n.

1. Introduction

For each $n \in \mathbb{N}$, suppose that V_n has a normalized 1-unconditional basis e_j, $1 \leq j \leq n$, and let $e^*_j \in V^*_n$, $1 \leq j \leq n$, denote the associated coordinate functionals. This work is concerned with the following question.

Question 1.1. Given $n \in \mathbb{N}$ and $\delta, \Gamma, \eta > 0$, what is the smallest integer $N = N(n, \delta, \Gamma, \eta)$ such that for any operator $T : V_N \to V_N$ satisfying

$$\|T\| \leq \Gamma \qquad \text{and} \qquad |\langle e^*_j, Te_j\rangle| \geq \delta, \quad 1 \leq j \leq N, \tag{1.1}$$

there are operators $E : V_n \to V_N$ and $F : V_N \to V_n$, such that the diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\mathrm{Id}_{V_n}} & V_n \\ {\scriptstyle E}\big\downarrow & & \big\uparrow{\scriptstyle F} \\ V_N & \xrightarrow[T]{} & V_N \end{array} \qquad \|E\|\|F\| \leq \frac{1+\eta}{\delta} \tag{1.2}$$

commutes?

In numerous Banach spaces, there exist quantitative estimates for N (see, e.g., [1–3, 7, 9–11, 13, 15–17]). To illustrate: the estimate for the relationship between N and n is

- ▷ *linear* for $V_n = \ell^p_n$, $1 \leq p \leq \infty$ (see, e.g., [3]);
- ▷ *polynomial* for the one-parameter dyadic Hardy spaces H^p_n, $1 \leq p < \infty$ (see Section 2 for the definition of H^p_n), and their duals (see [9]).

However, in many other Banach spaces the best known estimates for N are often *superexponential*. To illustrate, put $d_n = 2^{n+1} - 1$, $n \in \mathbb{N}_0$, and let $H^p_n(H^q_n)$, $1 \leq$

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$p, q < \infty$, denote the biparameter mixed norm dyadic Hardy space with dimension d_n^2 (see Section 2 for the definition of $H_n^p(H_n^q)$). The best known estimate for $V_{d_n^2} = H_n^p(H_n^q)$ and $V_{d_n^2} = (H_n^p(H_n^q))^*$, $1 \leq p, q < \infty$, is a *nested exponential* (see [10]), e.g., of the form

$$N \leq 2^{8^n 2^{8^{n-1} 2^{8^{n-2} 2^{8^{n-3} 2^{\cdot^{\cdot^{\cdot}}}}}}}. \tag{1.3}$$

In this work, we use the new *probabilistic method* introduced in [9] to improve the *superexponential* estimate (1.3) to the *linear* estimate

$$N \leq cn, \qquad \text{where } c = c(\delta, \Gamma, \eta) > 0. \tag{1.4}$$

2. Notation

Let $\mathcal{D}$ denote the *dyadic intervals* contained in the unit interval $[0,1)$, i.e.,

$$\mathcal{D} = \{[(k-1)2^{-n}, k2^{-n}) : n \in \mathbb{N}_0, 1 \leq k \leq 2^n\}.$$

Let $|\cdot|$ denote the Lebesgue measure. For any $N \in \mathbb{N}_0$, we put

$$\mathcal{D}_N = \{I \in \mathcal{D} : |I| = 2^{-N}\} \qquad \text{and} \qquad \mathcal{D}_{\leq N} = \bigcup_{n=0}^{N} \mathcal{D}_n.$$

Given $n \in \mathbb{N}_0$ and a dyadic interval $I \in \mathcal{D}_n$, we define $I^-, I^+ \in \mathcal{D}_{n+1}$ by

$$I^+ \cup I^- = I \qquad \text{and} \qquad \inf I^+ < \inf I^-.$$

For any two collections $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$, we introduce the following notation:

$$\mathcal{A} \otimes \mathcal{B} = \{I \times J : I \in \mathcal{A},\ J \in \mathcal{B}\}.$$

The L^∞-normalized *Haar system* h_I, $I \in \mathcal{D}$, is given by

$$h_I = \chi_{I^+} - \chi_{I^-}, \qquad I \in \mathcal{D},$$

where χ_A denotes the characteristic function of the set $A \subset [0,1)$. Given $1 \leq p < \infty$, the *one-parameter dyadic Hardy space* H^p is the completion of

$$\operatorname{span}\{h_I : I \in \mathcal{D}\}$$

under the square function norm

$$\|f\|_{H^p} = \Big(\int_0^1 \big(\sum_I |a_I|^2 h_I^2(x) \big)^{p/2} \, \mathrm{d}x \Big)^{1/p}, \tag{2.1}$$

where $f = \sum_I a_I h_I$. For all $n \in \mathbb{N}_0$, we define the finite-dimensional subspaces H_n^p of H^p by

$$H_n^p = \operatorname{span}\{h_I : I \in \mathcal{D}_{\leq n}\}.$$

We write $A \sim_\alpha B$ whenever there exists a constant $C = C(\alpha)$ such that $A/C \leq B \geq CA$. We remark that by Khintchine's inequality we have that

$$\|f\|_{H^p} \sim_p \int_0^1 \Big\| \sum_I r_I(t) a_I h_I \Big\|_{L^p} \, \mathrm{d}t, \tag{2.2}$$

where $f = \sum_I a_I h_I$ and r_I, $I \in \mathcal{D}$, is an independent Rademacher sequence.

The *biparameter* L^∞-normalized *Haar system* $h_{I \times J}$, $I, J \in \mathcal{D}$, is given by

$$h_{I \times J} = h_I \otimes h_J, \qquad I, J \in \mathcal{D},$$

where the tensor product of two functions $f, g : [0,1) \to \mathbb{R}$ is defined by

$$\big(f \otimes g\big)(x,y) = f(x)g(x), \qquad x, y \in [0,1).$$

For $1 \le p, q < \infty$, the *biparameter dyadic Hardy space* $H^p(H^q)$ is the completion of

$$\operatorname{span}\{h_R : R \in \mathcal{D} \otimes \mathcal{D}\}$$

under the square function norm

$$\|f\|_{H^p(H^q)} = \bigg(\int_0^1 \Big(\int_0^1 \big(\sum_{R \in \mathcal{D}\otimes\mathcal{D}} |a_R|^2 h_R^2(x,y)\big)^{q/2} \,\mathrm{d}y\Big)^{p/q} \,\mathrm{d}x\bigg)^{1/p}, \tag{2.3}$$

where $f = \sum_{R \in \mathcal{D}\otimes\mathcal{D}} a_R h_R$. For each $n \in \mathbb{N}_0$, we define the finite-dimensional subspace $H_n^p(H_n^q)$ of $H^p(H^q)$ by

$$H_n^p(H_n^q) = \operatorname{span}\{h_R : R \in \mathcal{D}_{\le n} \otimes \mathcal{D}_{\le n}\}.$$

If X is any Banach space, then a common way of defining $H^p(X)$ is as the completion of

$$\operatorname{span}\Big\{\sum_{I \in \mathcal{D}_{\le n}} a_I h_I : a_I \in X,\ I \in \mathcal{D},\ n \in \mathbb{N}\Big\}$$

under the norm

$$\Big\|\sum_{I \in \mathcal{D}} a_I h_I\Big\|_{*,p} = \int_0^1 \Big\|\sum_{I \in \mathcal{D}} r_I(t) a_I h_I\Big\|_{L^p(X)} \,\mathrm{d}t, \tag{2.4}$$

where r_I, $I \in \mathcal{D}$, are independent Rademacher and $L^p(X)$ is the usual Bochner–Lebesgue space. We will now show that for $X = H^q$, the norms (2.4) and (2.3) are equivalent. Using Kahane's and then Khintchine's inequality yields

$$\begin{aligned}\Big\|\sum_{I \in \mathcal{D}} a_I h_I\Big\|_{*,p} &\sim_p \Big(\int_0^1 \Big\|\sum_{I \in \mathcal{D}} r_I(t) a_I h_I\Big\|_{L^p(H^q)}^p \,\mathrm{d}t\Big)^{1/p} \\ &= \Big(\int_0^1 \int_0^1 \Big\|\sum_{I \in \mathcal{D}} r_I(t) a_I h_I(x)\Big\|_{H^q}^p \,\mathrm{d}t \,\mathrm{d}x\Big)^{1/p} \\ &\sim_q \Big(\int_0^1 \Big(\int_0^1 \Big\|\sum_{I \in \mathcal{D}} r_I(t) a_I h_I(x)\Big\|_{H^q}^q \,\mathrm{d}t\Big)^{p/q} \,\mathrm{d}x\Big)^{1/p}.\end{aligned}$$

By (2.2), we obtain that $\big\|\sum_{I \in \mathcal{D}} a_I h_I\big\|_{*,p}$ is a constant (depending on p and q) multiple of

$$\bigg(\int_0^1 \bigg(\int_0^1 \bigg(\int_0^1 \bigg(\int_0^1 \Big|\sum_{J \in \mathcal{D}} r_I(t) r_J(s) a_{I,J} h_I(x) h_J(y)\Big|^q \,\mathrm{d}y\bigg)^{1/q} \,\mathrm{d}s\bigg)^q \,\mathrm{d}t\bigg)^{p/q} \,\mathrm{d}x\bigg)^{1/p},$$

where $a_I = \sum_{J \in \mathcal{D}} a_{I,J} h_J$. Using Khintchine's inequality again yields the following constant multiple of $\big\|\sum_{I \in \mathcal{D}} a_I h_I\big\|_{*,p}$:

$$\bigg(\int_0^1 \bigg(\int_0^1 \int_0^1 \int_0^1 \Big|\sum_{J \in \mathcal{D}} r_I(t) r_J(s) a_{I,J} h_I(x) h_J(y)\Big|^q \,\mathrm{d}s \,\mathrm{d}t \,\mathrm{d}y\bigg)^{p/q} \,\mathrm{d}x\bigg)^{1/p}. \tag{2.5}$$

Finally, we note that by [12, Proposition 2.d.6]

$$\int_0^1 \int_0^1 \Big| \sum_{J \in \mathcal{D}} r_I(t) r_J(s) a_{I,J} h_I(x) h_J(y) \Big|^q \,\mathrm{d}s\,\mathrm{d}t \ \sim_{p,q} \Big(\sum_{J \in \mathcal{D}} a_{I,J}^2 h_I^2(x) h_J^2(y) \Big)^{q/2}, \tag{2.6}$$

for all $x, y \in [0,1)$. Inserting (2.6) into (2.5) yields the desired result.

3. Main result

Recall that we put $d_n = 2^{n+1} - 1$, $n \in \mathbb{N}_0$, and let $1 \le p, q < \infty$. We give a quantitative estimate for the N appearing in Question 1.1 for the spaces $V_{d_n^2} = H_n^p(H_n^q)$ and $V_{d_n^2} = (H_n^p(H_n^q))^*$. In particular, the relation between N and n is *linear.*

Theorem 3.1. *Let $1 \le p, q < \infty$, and let $(V_k : k \in \mathbb{N}_0)$ denote either*

$$(H_k^p(H_k^q) : k \in \mathbb{N}_0) \qquad \text{or} \qquad \big((H_k^p(H_k^q))^* : k \in \mathbb{N}_0\big). \tag{3.1}$$

Let $n \in \mathbb{N}_0$ and $\delta, \Gamma, \eta > 0$. Define the integer $N = N(n, \delta, \Gamma, \eta)$ by the formula

$$N = 41(n+3) + \big\lfloor 4 \log_2(\Gamma/\delta) + 4 \log_2\big(1 + \eta^{-1}\big)\big\rfloor. \tag{3.2}$$

Then for any operator $T : V_N \to V_N$ satisfying

$$\|T\| \le \Gamma \qquad \text{and} \qquad |\langle T h_Q, h_Q \rangle| \ge \delta |Q|, \quad Q \in \mathcal{D}_{\le N} \otimes \mathcal{D}_{\le N}, \tag{3.3}$$

there exist bounded linear operators $E : V_n \to V_N$ and $F : V_N \to V_n$ such that the diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\ \mathrm{Id}_{V_n}\ } & V_n \\ {\scriptstyle E}\downarrow & & \uparrow{\scriptstyle F} \\ V_N & \xrightarrow[\ T\ ]{} & V_N \end{array} \qquad \|E\|\|F\| \le \frac{1+\eta}{\delta} \tag{3.4}$$

commutes.

Note that the linear relation between N and n amounts to a polynomial relation between the dimensions of the respective spaces, i.e., $\dim V_N$ is a polynomial in $\dim V_n$.

Formula (3.2) is the main focus of this work. Specifically, we improve the previously best known estimate for the relation between N and n in $H_N^p(H_N^q)$ and $(H_N^p(H_N^q))^*$, $1 \le p, q < \infty$ (see [10]), *from superexponential to linear* (which means from (1.3) to (1.4)). The superexponential growth in [10] is caused by the use of *combinatorics.* The same is true even in one-parameter spaces (see, e.g., [7, 11, 13, 15]).

Recently, using a *probabilistic* approach (see [9]), *linear* estimates for N in n were obtained in the context of one-parameter spaces. In this work, we extend this probabilistic method to the biparameter spaces $H_N^p(H_N^q)$ and $(H_N^p(H_N^q))^*$, $1 \le p, q < \infty$, and thereby obtain the formula (3.1).

4. Tensor products, embeddings, and projections in mixed norm spaces

This section consists of two major parts: The first part connects Jones's compatibility condition (J1) to Capon's local product condition (P1)–(P4). In the second part, we show that every operator on a biparameter Hardy space is almost-diagonalized by a properly constructed randomized block basis. Both parts are vital components in the proof of our main result Theorem 3.1.

4.1. **Jones's compatibility condition and Capon's local product condition.**

Given $\mathcal{Z}_I \subset \mathcal{D}$, $I \in \mathcal{D}$, we put $Z_I = \bigcup \mathcal{Z}_I$. We say that the collections $\mathcal{Z}_I$, $I \in \mathcal{D}$, satisfy *Jones's compatibility condition* (J1) (see [5]; see also [14]) with constant $\kappa \geq 1$, if the following four conditions are satisfied:

(J1) For each $I \in \mathcal{D}$, the collection $\mathcal{Z}_I$ consists of finitely many pairwise disjoint dyadic intervals; moreover, $\mathcal{Z}_I \cap \mathcal{Z}_{I'} = \emptyset$ whenever $I, I' \in \mathcal{D}$, $I \neq I'$.

(J2) For every $I \in \mathcal{D}$, we have that $Z_{I^-} \cup Z_{I^+} \subset Z_I$ and $Z_{I^-} \cap Z_{I^+} = \emptyset$.

(J3) $\kappa^{-1}|I| \leq |Z_I| \leq \kappa|I|$, for all $I \in \mathcal{D}$.

(J4) For all $I_0, I \in \mathcal{D}$ with $I_0 \subset I$ and $K \in \mathcal{Z}_I$, we have $\frac{|K \cap Z_{I_0}|}{|K|} \geq \kappa^{-1}\frac{|Z_{I_0}|}{|Z_I|}$.

Jones's compatibility condition (J1) is crucial to construct block bases of the Haar system onto which the natural projection is bounded in H^1; especially (J4). Lemma 4.1 below asserts that the tensor product of collections satisfying Jones' compatibility condition (J1) satisfies Capon's local product condition (P1)–(P4) (see [6]). Capon's local product condition is used to construct block bases of the biparameter Haar system onto which the natural projection onto that block basis is bounded in $H^p(H^q)$, $1 \leq p, q < \infty$; (P4) is crucial for the endpoint spaces $p = 1$ or $q = 1$.

Lemma 4.1. *Let $\mathcal{X}_I \subset \mathcal{D}$, $I \in \mathcal{D}$, and $\mathcal{Y}_J \subset \mathcal{D}$, $J \in \mathcal{D}$, both satisfy condition* (J1) *with constant $\kappa \geq 1$. Define*

$$\mathcal{B}_{I\times J} = \mathcal{X}_I \otimes \mathcal{Y}_J = \{K \times L : K \in \mathcal{X}_I, L \in \mathcal{Y}_J\}, \qquad I, J \in \mathcal{D}, \tag{4.1a}$$

and put

$$X_I = \bigcup \mathcal{X}_I, \quad I \in \mathcal{D}, \qquad \textit{as well as} \qquad Y_J = \bigcup \mathcal{Y}_J, \quad J \in \mathcal{D}. \tag{4.1b}$$

Then $\mathcal{B}_R$, $R \in \mathcal{D} \otimes \mathcal{D}$, satisfies Capon's local product condition (P1)–(P4) *with constants $C_X = C_Y = \kappa$, i.e., the following four properties* (P1), (P2), (P3), *and* (P4) *hold true:*

(P1) *For all $R \in \mathcal{D} \otimes \mathcal{D}$ the collection $\mathcal{B}_R$ consists of pairwise disjoint dyadic rectangles, and for all $R_0, R_1 \in \mathcal{D} \otimes \mathcal{D}$ with $R_0 \neq R_1$, we have $\mathcal{B}_{R_0} \cap \mathcal{B}_{R_1} = \emptyset$.*

(P2) *For all $I, J, I_0, J_0, I_1, J_1 \in \mathcal{D}$ with $I_0 \cap I_1 = \emptyset$, $I_0 \cup I_1 \subset I$ and $J_0 \cap J_1 = \emptyset$, $J_0 \cup J_1 \subset J$, we have*

$$\begin{aligned} X_{I_0} \cap X_{I_1} &= \emptyset, & X_{I_0} \cup X_{I_1} &\subset X_I, \\ Y_{J_0} \cap Y_{J_1} &= \emptyset, & Y_{J_0} \cup Y_{J_1} &\subset Y_J. \end{aligned}$$

(P3) *For every $I, J \in \mathcal{D}$, we have*

$$\kappa^{-1}|I| \leq |X_I| \leq \kappa|I| \qquad \textit{and} \qquad \kappa^{-1}|J| \leq |Y_J| \leq \kappa|J|.$$

(P4) *For all* $I_0, J_0, I, J \in \mathcal{D}$*, with* $I_0 \subset I$*,* $J_0 \subset J$*, and for every* $K \in \mathcal{X}_I$*,* $L \in \mathcal{Y}_J$*, we have*

$$\frac{|K \cap X_{I_0}|}{|K|} \geq \kappa^{-1} \frac{|X_{I_0}|}{|X_I|} \qquad \text{and} \qquad \frac{|L \cap Y_{J_0}|}{|L|} \geq \kappa^{-1} \frac{|Y_{J_0}|}{|Y_J|}.$$

Proof. (P1)–(P4) follow directly from (J1)–(J4). □

Remark 4.2. The conditions (P1)–(P4) were introduced in [6] in a more general form: the collections $\mathcal{B}_{I\times J}$, $I, J \in \mathcal{D}$, in [6] have *local product structure*, i.e., there exist collections $\mathcal{X}_{I\times J}$, $\mathcal{Y}_{I\times J}$, $I, J \in \mathcal{D}$, such that

$$\mathcal{B}_{I\times J} = \{K \times L : K \in \mathcal{X}_{I\times J},\, L \in \mathcal{Y}_{I\times J}\}, \qquad I, J \in \mathcal{D}. \tag{4.2}$$

In Lemma 4.1, we have a special case of (4.2): *true product structure* (see (4.1)). To highlight the distinction explicitly, in Lemma 4.1 we have that $\mathcal{X}_{I\times J}$ does not depend on J and that $\mathcal{Y}_{I\times J}$ does not depend on I.

Lemma 4.3. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *each denote a nonempty, finite collection of pairwise disjoint dyadic intervals, and define* $X = \bigcup \mathcal{X}$ *as well as* $Y = \bigcup \mathcal{Y}$*. Given* $\theta, \varepsilon \in \{\pm 1\}^{\mathcal{D}}$*, put*

$$b^{(\theta,\varepsilon)} = \sum_{\substack{K\in\mathcal{X}\\ L\in\mathcal{Y}}} \theta_K \varepsilon_L h_{K\times L}. \tag{4.3}$$

Then

$$\|b^{(\theta,\varepsilon)}\|_{H^p(H^q)} = |X|^{1/p}|Y|^{1/q} \qquad \text{and} \qquad \|b^{(\theta,\varepsilon)}\|_{(H^p(H^q))^*} = |X|^{1/p'}|Y|^{1/q'}, \tag{4.4}$$

where $1 \leq p, q < \infty$*,* $1 < p', q' \leq \infty$ *with* $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

Lemma 4.3 follows immediately from [6, Lemma 4.1]. Since the proof is short, we include it here for the sake of completeness.

Proof. (4.3) and the disjointness of the collections $\mathcal{X}, \mathcal{Y}$ yields

$$\begin{aligned}
\|b^{(\theta,\varepsilon)}\|_{H^p(H^q)} &= \Big(\int_0^1 \Big(\int_0^1 \sum_{\substack{K\in\mathcal{X}\\ L\in\mathcal{Y}}} h_K^2(x) h_L^2(y)\,\mathrm{d}y\Big)^{p/q} \mathrm{d}x\Big)^{1/p}\\
&= |Y|^{1/q}\Big(\int_0^1 \sum_{K\in\mathcal{X}} h_K^2(x)\,\mathrm{d}x\Big)^{1/p} = |X|^{1/p}|Y|^{1/q}.
\end{aligned}$$

We will now compute $\|b^{(\theta,\varepsilon)}\|_{(H^p(H^q))^*}$. To this end, let $h \in H^p(H^q)$ be given by $h = \sum_{K,L} a_{K\times L} h_{K\times L} \in H^p(H^q)$, and observe that by Hölder's inequality we obtain

$$\begin{aligned}
\langle b^{(\theta,\varepsilon)}, h\rangle &\leq \sum_{K\in\mathcal{X}} |K| \sum_{L\in\mathcal{Y}} |a_{K\times L}||L| \leq |Y|^{1/q'} \sum_{K\in\mathcal{X}} |K| \big(\sum_{L\in\mathcal{Y}} |a_{K\times L}|^q |L|\big)^{1/q}\\
&\leq |X|^{1/p'}|Y|^{1/q'} \Big(\sum_{K\in\mathcal{X}} |K| \big(\sum_{L\in\mathcal{Y}} |a_{K\times L}|^q|L|\big)^{p/q}\Big)^{1/p}\\
&= |X|^{1/p'}|Y|^{1/q'} \|h\|_{H^p(H^q)}.
\end{aligned}$$

Thus, we have $\|b^{(\theta,\varepsilon)}\|_{(H^p(H^q))^*} \le |X|^{1/p'}|Y|^{1/q'}$. Since $\langle b^{(\theta,\varepsilon)}, b^{(\theta,\varepsilon)}\rangle = |X||Y|$ and $\|b^{(\theta,\varepsilon)}\|_{H^p(H^q)} = |X|^{1/p}|Y|^{1/q}$ by the first part of the proof, we obtain

$$\|b^{(\theta,\varepsilon)}\|_{(H^p(H^q))^*} = |X|^{1/p'}|Y|^{1/q'}. \qquad \square$$

The following Theorem 4.4 is one of the two main ingredients in the proof of Theorem 3.1; the other one is the almost-diagonalization of operators using random block bases (see Theorem 4.5).

Theorem 4.4. *Let $\mathcal{X}_I \subset \mathcal{D}$, $I \in \mathcal{D}$, and $\mathcal{Y}_J \subset \mathcal{D}$, $J \in \mathcal{D}$, both satisfy condition* (J1) *with constant $\kappa = 1$, and define the product collections*

$$\mathcal{B}_{I\times J} = \mathcal{X}_I \otimes \mathcal{Y}_J = \{K \times L : K \in \mathcal{X}_I,\, L \in \mathcal{Y}_J\}, \qquad I, J \in \mathcal{D}. \tag{4.5}$$

Given $\theta, \varepsilon \in \{\pm 1\}^{\mathcal{D}}$, we define the tensor product system

$$b_{I\times J}^{(\theta,\varepsilon)} = f_I^{(\theta)} \otimes g_J^{(\varepsilon)}, \qquad I, J \in \mathcal{D}, \tag{4.6}$$

where

$$f_I^{(\theta)} = \sum_{K\in\mathcal{X}_I} \theta_K h_K, \quad I \in \mathcal{D}, \qquad \text{and} \qquad g_J^{(\varepsilon)} = \sum_{L\in\mathcal{Y}_J} \varepsilon_L h_L, \quad J \in \mathcal{D}. \tag{4.7}$$

Given $1 \le p, q < \infty$, let V denote either $H^p(H^q)$ or $(H^p(H^q))^*$. Then the operators $B^{(\theta,\varepsilon)}, A^{(\theta,\varepsilon)} : V \to V$ given by

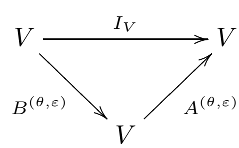
$$B^{(\theta,\varepsilon)} f = \sum_{R\in\mathcal{D}\otimes\mathcal{D}} \frac{\langle f, h_R\rangle}{\|h_R\|_2^2} b_R^{(\theta,\varepsilon)} \qquad \text{and} \qquad A^{(\theta,\varepsilon)} f = \sum_{R\in\mathcal{D}\otimes\mathcal{D}} \frac{\langle f, b_R^{(\theta,\varepsilon)}\rangle}{\|b_R^{(\theta,\varepsilon)}\|_2^2} h_R \tag{4.8}$$

satisfy the estimates

$$\begin{aligned} \|B^{(\theta,\varepsilon)} f\|_V &\le \|f\|_V, \quad f \in V,\\ \|A^{(\theta,\varepsilon)} f\|_V &\le \|f\|_V, \quad f \in V. \end{aligned} \tag{4.9}$$

Moreover, the diagram

(4.10)



commutes and the composition $P^{(\theta,\varepsilon)} = B^{(\theta,\varepsilon)}A^{(\theta,\varepsilon)}$ is the norm 1 projection $P^{(\theta,\varepsilon)} : V \to V$ given by

$$P^{(\theta,\varepsilon)}(f) = \sum_{R\in\mathcal{D}\otimes\mathcal{D}} \frac{\langle f, b_R^{(\theta,\varepsilon)}\rangle}{\|b_R\|_2^2} b_R^{(\theta,\varepsilon)}. \tag{4.11}$$

Consequently, the range of $B^{(\theta,\varepsilon)}$ is complemented (by $P^{(\theta,\varepsilon)}$), and $B^{(\theta,\varepsilon)}$ is an isometric isomorphism onto its range.

Proof. By Lemma 4.1, we know that the collections $\mathcal{B}_{I\times J}$, $I, J \in \mathcal{D}$, have the properties (P1), (P2), (P3), and (P4). In case $V = H^p(H^q)$, we apply [6, Theorem 4.3] to obtain Theorem 4.4. If $V = (H^p(H^q))^*$, the theorem follows from the previous case and the observation that $(B^{(\theta,\varepsilon)})^* = A^{(\theta,\varepsilon)}$ and $(A^{(\theta,\varepsilon)})^* = B^{(\theta,\varepsilon)}$. $\square$

4.2. **Random block bases with tensor product structure.** Let μ denote the uniform probability measure on $\{\pm 1\}$, i.e., $\mu(\{-1\}) = \mu(\{+1\}) = 1/2$. Take the infinite product of μ over the set $\mathcal{D}$, and denote that product $\mathbb{P}_\theta$. We note that $\mathbb{P}_\theta$ is a probability measure on $\Omega_\theta = \{\pm 1\}^{\mathcal{D}}$. Now, take an independent copy $(\Omega_\varepsilon, \mathbb{P}_\varepsilon)$ of $(\Omega_\theta, \mathbb{P}_\theta)$, define $\mathbb{P}_{\theta,\varepsilon}$ as the product measure of $\mathbb{P}_\delta$ and $\mathbb{P}_\varepsilon$, and note that $\mathbb{P}_{\theta,\varepsilon}$ is a probability measure on $\Omega_\theta \times \Omega_\varepsilon$. Moreover, $\mathbb{E}_\theta$, $\mathbb{E}_\varepsilon$, and $\mathbb{E}_{\theta,\varepsilon}$ are the expectations with respect to the probability measures $\mathbb{P}_\theta$, $\mathbb{P}_\varepsilon$, and $\mathbb{P}_{\theta,\varepsilon}$.

Given $n, N \in \mathbb{N}$, $I, J \in \mathcal{D}_{\leq n}$, and $\mathcal{X}_I, \mathcal{Y}_J \subset \mathcal{D}_{\leq N}$, define the functions

$$f_I^{(\theta)} = \sum_{K \in \mathcal{X}_I} \theta_K h_K, \quad \theta \in \Omega_\theta, \qquad \text{and} \qquad g_J^{(\varepsilon)} = \sum_{L \in \mathcal{Y}_J} \varepsilon_L h_L, \quad \varepsilon \in \Omega_\varepsilon. \tag{4.12}$$

Hence, their tensor product $b_{I\times J}^{(\theta,\varepsilon)}$ is given by

$$b_{I\times J}^{(\theta,\varepsilon)} = f_I^{(\theta)} \otimes g_J^{(\varepsilon)} = \sum_{\substack{K \in \mathcal{X}_I \\ L \in \mathcal{Y}_J}} \theta_K \varepsilon_L h_{K\times L}, \qquad (\theta, \varepsilon) \in \Omega_\theta \times \Omega_\varepsilon. \tag{4.13}$$

Let $1 \leq p, q < \infty$, and let V_N denote either $H_N^p(H_N^q)$ or $(H_N^p(H_N^q))^*$. Given a bounded linear operator $T : V_N \to V_N$, we put

$$W_{I,I',J,J'}(\theta,\varepsilon) = \langle T b_{I\times J}^{(\theta,\varepsilon)}, b_{I'\times J'}^{(\theta,\varepsilon)} \rangle, \qquad I, J, I', J' \in \mathcal{D}_{\leq n},\ I \neq I',\ J \neq J', \tag{4.14a}$$

$$X_{I,I',J}(\theta,\varepsilon) = \langle T b_{I\times J}^{(\theta,\varepsilon)}, b_{I'\times J'}^{(\theta,\varepsilon)} \rangle, \qquad I, J, I' \in \mathcal{D}_{\leq n},\ I \neq I', \tag{4.14b}$$

$$Y_{I,J,J'}(\theta,\varepsilon) = \langle T b_{I\times J}^{(\theta,\varepsilon)}, b_{I'\times J'}^{(\theta,\varepsilon)} \rangle, \qquad I, J, J' \in \mathcal{D}_{\leq n},\ J \neq J', \tag{4.14c}$$

$$Z_{I,J}(\theta,\varepsilon) = \langle T b_{I\times J}^{(\theta,\varepsilon)}, b_{I\times J}^{(\theta,\varepsilon)} \rangle - \sum_{\substack{K \in \mathcal{X}_I \\ L \in \mathcal{Y}_J}} \langle T h_{K\times L}, h_{K\times L} \rangle, \quad I, J \in \mathcal{D}_{\leq n}, \tag{4.14d}$$

for all $(\theta, \varepsilon) \in \Omega_\theta \times \Omega_\varepsilon$. From here on, we will regularly omit the subindices of the above random variables, i.e., $W = W_{I,I',J,J'}$, $X = X_{I,I',J}$, $Y = Y_{I,J,J'}$, and $Z = Z_{I,J}$.

Theorem 4.5. *Given $n, N \in \mathbb{N}$, let $\mathcal{X}_I \subset \mathcal{D}_{\leq N}$, $I \in \mathcal{D}_{\leq n}$, and $\mathcal{Y}_J \subset \mathcal{D}_{\leq N}$, $J \in \mathcal{D}_{\leq n}$, both denote nonempty collections which satisfy* (J1)*. Define $\alpha > 0$ by putting*

$$\alpha = \max\{|K|, |L| : K \in \mathcal{X}_I,\ L \in \mathcal{Y}_J,\ I, J \in \mathcal{D}_{\leq n}\}. \tag{4.15}$$

Given $1 \leq p, q < \infty$, let V_N denote either $H_N^p(H_N^q)$ or $(H_N^p(H_N^q))^*$. Then for any bounded operator $T : V_N \to V_N$ we have

$$\mathbb{E}_{\theta,\varepsilon} W = \mathbb{E}_{\theta,\varepsilon} X = \mathbb{E}_{\theta,\varepsilon} Y = \mathbb{E}_{\theta,\varepsilon} Z = 0, \tag{4.16}$$

as well as the estimates

$$\mathbb{E}_{\theta,\varepsilon} W^2 \leq \|T\|^2 \alpha^{1/2}, \qquad \mathbb{E}_{\theta,\varepsilon} X^2 \leq 4\|T\|^2 \alpha^{1/2}, \tag{4.17a}$$

$$\mathbb{E}_{\theta,\varepsilon} Y^2 \leq 4\|T\|^2 \alpha^{1/2}, \qquad \mathbb{E}_{\theta,\varepsilon} Z^2 \leq 12\|T\|^2 \alpha^{1/2}, \tag{4.17b}$$

where the random variables W, X, Y, Z are defined in (4.14)*.*

The proof is given in [8, Section 6]. For the one-parameter version of Theorem 4.5 we refer to [9].

5. Proof of the main result Theorem 3.1

Here we prove our main result Theorem 3.1 by extending the probabilistic method introduced in [9, Theorem 3.1] for one-parameter Hardy spaces H^p, to the biparameter Hardy spaces $H^p_N(H^q_N)$. The proof heavily relies on the results of Section 4.

For convenience of the reader we repeat Theorem 3.1 here.

Theorem (Main result of Theorem 3.1)**.** *Let $1 \leq p, q < \infty$, and let $(V_k : k \in \mathbb{N}_0)$ denote either*

$$(H^p_k(H^q_k) : k \in \mathbb{N}_0) \qquad \text{or} \qquad \big((H^p_k(H^q_k))^* : k \in \mathbb{N}_0\big). \tag{5.1}$$

Let $n \in \mathbb{N}_0$ and $\delta, \Gamma, \eta > 0$. Define the integer $N = N(n, \delta, \Gamma, \eta)$ by the formula

$$N = 41(n + 3) + \big\lfloor 4 \log_2(\Gamma/\delta) + 4 \log_2\big(1 + \eta^{-1}\big)\big\rfloor. \tag{5.2}$$

Then for any operator $T : V_N \to V_N$ satisfying

$$\|T\| \leq \Gamma \qquad \text{and} \qquad |\langle Th_Q, h_Q\rangle| \geq \delta|Q|, \quad Q \in \mathcal{D}_{\leq N} \otimes \mathcal{D}_{\leq N}, \tag{5.3}$$

there exist bounded linear operators $E : V_n \to V_N$ and $F : V_N \to V_n$ such that the diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\mathrm{Id}_{V_n}} & V_n \\ {\scriptstyle E}\big\downarrow & & \big\uparrow{\scriptstyle F} \\ V_N & \xrightarrow[T]{} & V_N \end{array} \qquad \|E\|\|F\| \leq \frac{1 + \eta}{\delta} \tag{5.4}$$

commutes.

Proof. Let $M : V_N \to V_N$ denote the norm 1 multiplication operator given by the linear extension of

$$h_Q \mapsto \operatorname{sign}(\langle Th_Q, h_Q\rangle)h_Q, \qquad Q \in \mathcal{D}_{\leq N} \otimes \mathcal{D}_{\leq N}.$$

By (5.3), we obtain

$$\langle TMh_Q, h_Q\rangle = |\langle Th_Q, h_Q\rangle| \geq \delta|Q|, \qquad Q \in \mathcal{D}_{\leq N} \otimes \mathcal{D}_{\leq N},$$

and therefore we can assume

$$\langle Th_Q, h_Q\rangle \geq \delta|Q|, \qquad Q \in \mathcal{D}_{\leq N} \otimes \mathcal{D}_{\leq N}. \tag{5.5}$$

Before we proceed to Step 5 of the proof, we define the constants m_0 and η_0: Let $m_0 \in \mathbb{N}_0$ denote the smallest integer such that

$$2^{m_0} > \frac{2^{8(n+3)}\Gamma^4}{\eta_0^4}, \qquad \text{where} \quad \eta_0 = \frac{\eta\delta}{(1 + \eta)2^{8(n+2)}}. \tag{5.6}$$

Step 1: Constructing the block basis $b^{(\theta,\varepsilon)}_R$, $R \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n}$. In this step, we will define a random block basis $(\theta, \varepsilon) \mapsto b^{(\theta,\varepsilon)}_R$, $R \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n}$, of the Haar system h_Q, $Q \in \mathcal{D}_{\leq N} \otimes \mathcal{D}_{\leq N}$, given by

$$b^{(\theta,\varepsilon)}_{I\times J} = f^{(\theta)}_I \otimes g^{(\varepsilon)}_J = \sum_{K\in\mathcal{X}_I} \theta_K h_K \otimes \sum_{L\in\mathcal{Y}_J} \varepsilon_L h_L, \qquad \theta \in \Omega_\theta,\ \varepsilon \in \Omega_\varepsilon, \tag{5.7}$$

where $\mathcal{X}_I \subset \mathcal{D}_{\leq N}$, $I \in \mathcal{D}_{\leq n}$, and $\mathcal{Y}_J \subset \mathcal{D}_{\leq N}$, $J \in \mathcal{D}_{\leq n}$, both satisfy condition (J1) with constant $\kappa = 1$. The collections will be selected by a minimalist Gamlen–Gaudet construction. Then, using Theorem 4.5, we will find signs $(\theta, \varepsilon) \in \Omega_\theta \times \Omega_\varepsilon$ such that

$$\text{(5.8a)} \qquad |\langle Tb_R^{(\theta,\varepsilon)}, b_{R'}^{(\theta,\varepsilon)}\rangle| \leq \eta_0, \qquad R, R' \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n},\ R \neq R',$$

$$\text{(5.8b)} \qquad \langle Tb_R^{(\theta,\varepsilon)}, b_R^{(\theta,\varepsilon)}\rangle \geq (\delta - \eta_0 2^{2n}) \|b_R^{(\theta,\varepsilon)}\|_2^2, \qquad R \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n}.$$

We will now inductively define the collections $\mathcal{X}_I$, $I \in \mathcal{D}_{\leq n}$, and $\mathcal{Y}_J$, $J \in \mathcal{D}_{\leq n}$, using the Gamlen–Gaudet construction. We begin by putting

$$\text{(5.9)} \qquad \mathcal{X}_{[0,1)} = \mathcal{Y}_{[0,1)} = \mathcal{D}_{m_0}.$$

Now, let $0 \leq k \leq n-1$, assuming that we have already constructed the collections $\mathcal{X}_I$, $I \in \mathcal{D}_{\leq k}$, and $\mathcal{Y}_J$, $J \in \mathcal{D}_{\leq k}$. Then we define

$$\text{(5.10a)} \qquad \mathcal{X}_{I^+} = \{K^+ : K \in \mathcal{X}_I\}, \qquad \mathcal{X}_{I^-} = \{K^- : K \in \mathcal{X}_I\}, \qquad I \in \mathcal{D}_k,$$

$$\text{(5.10b)} \qquad \mathcal{Y}_{J^+} = \{K^+ : K \in \mathcal{Y}_J\}, \qquad \mathcal{Y}_{J^-} = \{K^- : K \in \mathcal{Y}_J\}, \qquad J \in \mathcal{D}_k.$$

Clearly, $\mathcal{X}_I$, $I \in \mathcal{D}_{\leq n}$, and $\mathcal{Y}_J$, $J \in \mathcal{D}_{\leq n}$, both satisfy condition (J1) with constant $\kappa = 1$. (We remark that the minimalist part in the above Gamlen–Gaudet construction refers to the fact that the collections $\mathcal{X}_{I^\pm}$, $\mathcal{Y}_{J^\pm}$ consist of intervals of constant length that are exactly half the length of the intervals that form the collections $\mathcal{X}_I$, $\mathcal{Y}_J$. This is in contrast to the general Gamlen–Gaudet construction: For example, the length of the intervals in $\mathcal{X}_{I^\pm}$, $\mathcal{Y}_{J^\pm}$ can vary within each collection and they are much smaller than the length of the intervals in $\mathcal{X}_I$, $\mathcal{Y}_J$. Especially in the finite-dimensional setting, these collections are usually missing pieces, i.e., $\bigcup \mathcal{X}_{I^+} \cup \bigcup \mathcal{X}_{I^-} \subsetneq \bigcup \mathcal{X}_I$; see the proof of [14, Theorem 5.2.2] and [14, Lemma 5.2.4]. For more details on the Gamlen–Gaudet construction we refer to [4]; see also [14]. For a biparameter version we refer to [10, 11].)

Next, we will use the probabilistic Theorem 4.5 to find signs $(\theta, \varepsilon) \in \Omega_\theta \times \Omega_\varepsilon$ such that (5.8) is satisfied. To this end, we define the off-diagonal events

$$O_{R,R'} = \big\{(\theta,\varepsilon) : |\langle Tb_R^{(\theta,\varepsilon)}, b_{R'}^{(\theta,\varepsilon)}\rangle| > \eta_0\big\}, \qquad R, R' \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n},\ R \neq R',$$

and the diagonal events

$$D_{I,J} = \Bigg\{(\theta,\varepsilon) : \Big|\langle Tb_{I\times J}^{(\theta,\varepsilon)}, b_{I\times J}^{(\theta,\varepsilon)}\rangle - \sum_{\substack{K\in\mathcal{X}_I\\ L\in\mathcal{Y}_J}} \langle Th_{K\times L}, h_{K\times L}\rangle\Big| > \eta_0\Bigg\}, \qquad I, J \in \mathcal{D}_{\leq n}.$$

By Theorem 4.5 and the definition of the random variables W, X, Y, Z (see (4.14)), we obtain

$$\text{(5.11a)} \qquad \mathbb{P}_{\theta,\varepsilon}(O_{R,R'}) \leq \frac{4\Gamma^2}{2^{m_0/2}\eta_0^2}, \qquad R, R' \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n},\ R \neq R',$$

$$\text{(5.11b)} \qquad \mathbb{P}_{\theta,\varepsilon}(D_{I,J}) \leq \frac{12\Gamma^2}{2^{m_0/2}\eta_0^2}, \qquad I, J \in \mathcal{D}_{\leq n}.$$

Combining (5.11) with (5.6) yields

$$\text{(5.12)} \qquad \mathbb{P}_{\theta,\varepsilon}\Bigg(\bigcup_{\substack{R,R'\in\mathcal{D}_{\leq n}\otimes\mathcal{D}_{\leq n}\\ R\neq R'}} O_{R,R'} \cup \bigcup_{I,J\in\mathcal{D}_{\leq n}} D_{I,J}\Bigg) \leq 2^{4(n+3)} \frac{\Gamma^2}{2^{m_0/2}\eta_0^2} < 1.$$

Hence, we can find at least one $(\theta,\varepsilon)\in\Omega_\theta\times\Omega_\varepsilon$ such that

$$|\langle Tb_R^{(\theta,\varepsilon)}, b_{R'}^{(\theta,\varepsilon)}\rangle| \le \eta_0, \qquad R,R'\in\mathcal{D}_{\le n}\otimes\mathcal{D}_{\le n},\ R\neq R', \tag{5.13a}$$

$$\Big|\langle Tb_{I\times J}^{(\theta,\varepsilon)}, b_{I\times J}^{(\theta,\varepsilon)}\rangle - \sum_{\substack{K\in\mathcal{X}_I\\ L\in\mathcal{Y}_J}}\langle Th_{K\times L}, h_{K\times L}\rangle\Big| \le \eta_0, \qquad I,J\in\mathcal{D}_{\le n}. \tag{5.13b}$$

Recall that $\kappa = 1$ by construction of $\mathcal{X}_I$ and $\mathcal{Y}_J$, $I,J\in\mathcal{D}_{\le n}$ (see (5.10)). Hence, by (5.3), (J1), and (J3) we obtain

$$\sum_{\substack{K\in\mathcal{X}_I\\ L\in\mathcal{Y}_J}}\langle Th_{K\times L}, h_{K\times L}\rangle \ge \sum_{\substack{K\in\mathcal{X}_I\\ L\in\mathcal{Y}_J}}\delta|K\times L| = \delta|X_I\times Y_J| = \delta|I\times J|, \qquad I,J\in\mathcal{D}_{\le n}.$$

The latter estimate and (5.13b) give us

$$\langle Tb_R^{(\theta,\varepsilon)}, b_R^{(\theta,\varepsilon)}\rangle \ge \delta|R| - \eta_0, \qquad R\in\mathcal{D}_{\le n}\otimes\mathcal{D}_{\le n}. \tag{5.14}$$

Note that by Lemma 4.3 we have $\|b_R^{(\theta,\varepsilon)}\|_2^2 = R$, thus we obtain from (5.14)

$$\langle Tb_R^{(\theta,\varepsilon)}, b_R^{(\theta,\varepsilon)}\rangle \ge (\delta - \eta_0 2^{2n})\|b_R^{(\theta,\varepsilon)}\|_2^2, \qquad R\in\mathcal{D}_{\le n}\otimes\mathcal{D}_{\le n}. \tag{5.15}$$

Combining (5.13a) with (5.15) yields

$$|\langle Tb_R^{(\theta,\varepsilon)}, b_{R'}^{(\theta,\varepsilon)}\rangle| \le \eta_0, \qquad R,R'\in\mathcal{D}_{\le n}\otimes\mathcal{D}_{\le n},\ R\neq R', \tag{5.16a}$$

$$\langle Tb_R^{(\theta,\varepsilon)}, b_R^{(\theta,\varepsilon)}\rangle \ge (\delta - \eta_0 2^{2n})\|b_R^{(\theta,\varepsilon)}\|_2^2, \qquad R\in\mathcal{D}_{\le n}\otimes\mathcal{D}_{\le n}. \tag{5.16b}$$

Step 2: Constructing the operators. Here, we will use the basic operators $B^{(\theta,\varepsilon)} : V_n\to V_N$ and $A^{(\theta,\varepsilon)} : V_N\to V_n$ given by

$$B^{(\theta,\varepsilon)}f = \sum_{R\in\mathcal{D}_{\le n}\otimes\mathcal{D}_{\le n}} \frac{\langle f, h_R\rangle}{\|h_R\|_2^2} b_R^{(\theta,\varepsilon)}, \qquad f\in V_n, \tag{5.17a}$$

$$A^{(\theta,\varepsilon)}f = \sum_{R\in\mathcal{D}_{\le n}\otimes\mathcal{D}_{\le n}} \frac{\langle f, b_R^{(\theta,\varepsilon)}\rangle}{\|b_R^{(\theta,\varepsilon)}\|_2^2} h_R, \qquad f\in V_N, \tag{5.17b}$$

as building blocks for the operators E and F in diagram (5.4). Let us recall that by Theorem 4.4, the operators $B^{(\theta,\varepsilon)}$ and $A^{(\theta,\varepsilon)}$ satisfy the estimates

$$\|B^{(\theta,\varepsilon)}\| \le 1 \qquad\text{and}\qquad \|A^{(\theta,\varepsilon)}\| \le 1, \tag{5.18}$$

and $P^{(\theta,\varepsilon)} : V_N\to V_N$ defined as $P^{(\theta,\varepsilon)} = B^{(\theta,\varepsilon)}A^{(\theta,\varepsilon)}$ is a norm 1 projection given by

$$P^{(\theta,\varepsilon)}f = \sum_{R\in\mathcal{D}_{\le n}\otimes\mathcal{D}_{\le n}} \frac{\langle f, b_R^{(\theta,\varepsilon)}\rangle}{\|b_R^{(\theta,\varepsilon)}\|_2^2} b_R^{(\theta,\varepsilon)}, \qquad f\in V_N. \tag{5.19}$$

Now put $Y = P^{(\theta,\varepsilon)}(V_N)$ and note that the following diagram commutes:

$$\begin{array}{ccc} V_n & \xrightarrow{\ \mathrm{Id}_{V_n}\ } & V_n \\ {\scriptstyle B^{(\theta,\varepsilon)}}\big\downarrow & & \big\uparrow{\scriptstyle A^{(\theta,\varepsilon)}_{|Y}} \\ Y & \xrightarrow[\ \mathrm{Id}_Y\ ]{} & Y \end{array} \qquad \|B^{(\theta,\varepsilon)}\|, \|A^{(\theta,\varepsilon)}_{|Y}\| \le 1. \tag{5.20}$$

Observe that T almost acts as a multiplication operator on Y (see (5.16)). Next, we define $U^{(\theta,\varepsilon)} : V_N \to Y$ by putting

$$U^{(\theta,\varepsilon)} f = \sum_{R \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n}} \frac{\langle f, b_R^{(\theta,\varepsilon)} \rangle}{\langle T b_R^{(\theta,\varepsilon)}, b_R^{(\theta,\varepsilon)} \rangle} b_R^{(\theta,\varepsilon)}, \qquad f \in V_N. \tag{5.21}$$

By the 1-unconditionality of the biparameter Haar system in V_N and the definition of the norm 1 projection $P^{(\theta,\varepsilon)}$ (see (5.19) and (5.18)), we obtain

$$\|U^{(\theta,\varepsilon)}\| \leq \frac{\|P^{(\theta,\varepsilon)}\|}{\delta - \eta_0 2^{2n}} \leq \frac{1}{\delta - \eta_0 2^{2n}}. \tag{5.22}$$

We will now show that $U^{(\theta,\varepsilon)} : V_N \to Y$ almost acts as an inverse of T restricted to Y. Firstly, for all $g = \sum_{R \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n}} a_R b_R^{(\theta,\varepsilon)} \in Y$, we have the following identity:

$$U^{(\theta,\varepsilon)} T g - g = \sum_{\substack{R, R' \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n} \\ R' \neq R}} a_{R'} \frac{\langle T b_{R'}^{(\theta,\varepsilon)}, b_R^{(\theta,\varepsilon)} \rangle}{\langle T b_R^{(\theta,\varepsilon)}, b_R^{(\theta,\varepsilon)} \rangle} b_R^{(\theta,\varepsilon)}. \tag{5.23}$$

Secondly, by Lemma 4.3, we have the estimate

$$|a_{R'}| \leq \frac{\|g\|_{V_N}}{\|b_{R'}^{(\theta,\varepsilon)}\|_{V_N}} \leq 2^{2(n+1)} \|g\|_{V_N}, \qquad R' \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n};$$

thus, by (5.23) and (5.16), we obtain

$$\|U^{(\theta,\varepsilon)} T g - g\|_{V_N} \leq \frac{\eta_0 2^{8(n+1)}}{\delta - \eta_0 2^{2n}} \|g\|_{V_N}. \tag{5.24}$$

Next, let $I : Y \to V_N$ denote the operator given by $Iy = y$. Observe that by (5.6) we have $\frac{\eta_0 2^{8(n+1)}}{\delta - \eta_0 2^{2n}} < 1$, hence we obtain from (5.24) that

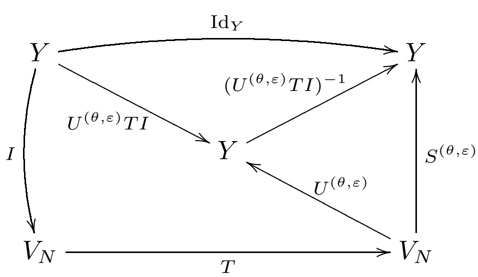
$$\|(U^{(\theta,\varepsilon)} T I)^{-1} g\|_{V_N} \leq \frac{1}{1 - \frac{\eta_0 2^{8(n+1)}}{\delta - \eta_0 2^{2n}}} \|g\|_{V_N}. \tag{5.25}$$

Now, define $S^{(\theta,\varepsilon)} : V_N \to Y$ by putting $S^{(\theta,\varepsilon)} = (U^{(\theta,\varepsilon)} T I)^{-1} U^{(\theta,\varepsilon)}$, and note that (5.22), (5.25), and (5.6) gives us

$$\|S^{(\theta,\varepsilon)}\| \leq \frac{1}{\delta - \eta_0 (2^{2n} + 2^{8(n+1)})} \leq \frac{1+\eta}{\delta}.$$

Moreover, the following diagram commutes:

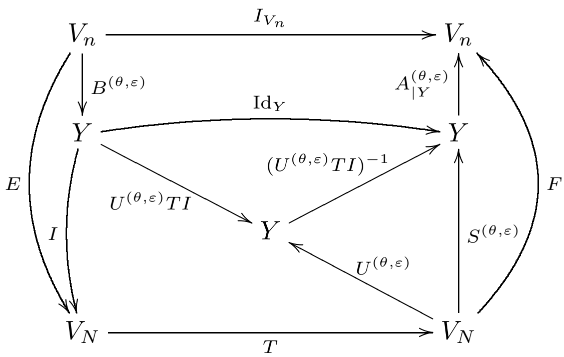
(5.26)



$$\|I\| \|S^{(\theta,\varepsilon)}\| \leq \frac{1+\eta}{\delta}.$$

Merging the diagrams (5.20) and (5.26) yields

(5.27)



$$\|E\|\|F\| \leq \frac{1+\eta}{\delta}.$$

Finally, by reviewing the construction of our block basis $b_R^{(\theta)}$, $R \in \mathcal{D}_{\leq n} \otimes \mathcal{D}_{\leq n}$ (see (5.9) and (5.10)), the definition of our basic operators $B^{(\theta,\varepsilon)}$ and $A^{(\theta,\varepsilon)}$ and the constants defined in (5.6), we conclude that (5.2) is an appropriate choice for N. □

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