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RIEMANN-HILBERT FACTORIZATION OF MATRICES INVARIANT UNDER INVERSION IN A CIRCLE

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ABSTRACT. We consider matrix functions with certain invariance under inversion in the unit circle. If such a function satisfies a positivity assumption on the unit circle, then only zero partial indices appear in its Riemann-Hilbert (Wiener-Hopf) factorization. It implies the unique solvability of a certain class of Riemann-Hilbert boundary value problems. It includes the ones associated with the inverse scattering transform of the focusing/defocusing integrable discrete nonlinear Schrödinger equations.

1. INTRODUCTION

Riemann-Hilbert problems (RHPs), formulated in various ways, are a powerful tool in the study of integrable systems. As is proved in [10], if a matrix function is invariant under Schwarz reflection, its Riemann-Hilbert (Wiener-Hopf) factorization involves only zero partial indices and it implies the unique solvability of the corresponding RHPs formulated in other ways (a singular integral equation and a boundary value problem). The zero partial indices property is a key in the argument in [11]. The main result there is the bijectivity of the scattering and inverse scattering maps. So bijectivity is known for NLS in sufficient detail (see also [4, 12]), but the integrable discrete nonlinear Schrödinger equation (IDNLS) still lacks a satisfactory theory. In order to construct such a theory, we need detailed information about relevant RHPs. In the discrete case, the real axis must be replaced by the unit circle ([1, 7–9]). If a matrix function invariant under the inversion in S^1, namely $z \to 1/\bar{z}$, and satisfies a positivity condition on S^1, then it has only zero partial indices. It implies the unique solvability of a certain class of Riemann-Hilbert boundary value problems, including the ones associated with IDNLS. This fact can be a basis of the bijectivity proof of the scattering/inverse scattering transforms for IDNLS. See also the approach in [5] based on a vanishing lemma.

Factorization of matrices given on S^1 is a topic that can be studied from other directions. See, e.g., [3, 6]. It is known that a positive Hermitian matrix function v on S^1 has an expression $v = w^*w$, where w is holomorphic inside S^1. We give a generalization of this fact to the case of an inversion invariant contour including S^1.

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2. Function spaces

In the present paper a *contour* always means a finite disjoint union of smooth simple closed curves in $\mathbb{C}$. Let $\Sigma \subset \mathbb{C}$ be a contour. Then we have $\Sigma = \bigcup_{j=1}^{J} \Sigma_j$, where each Σ_j is a smooth simple closed curve and $\Sigma_j \cap \Sigma_k = \emptyset\, (j \neq k)$. It is possible to assign an orientation on Σ such that it is the positively oriented boundary of an open set Ω_+. Set $\Omega_- = \mathbb{C} \setminus (\Sigma \cup \Omega_+)$. Then Σ is the negatively oriented boundary of the open set Ω_-. We introduce function spaces following [10, 11]. The L^2 norm of a matrix function $f\colon \Sigma \to \mathbf{M}_n$($\mathbf{M}_n$ is the complex $n \times n$ matrix algebra) is defined by $\|f\|_2 = \left(\int_\Sigma |f|^2 |dz|\right)^{1/2}, |f| = (\mathrm{tr} f^* f)^{1/2}$, where the asterisk means the Hermitian conjugate. We write $L^2(\Sigma)$ for $L^2(\Sigma, \mathbf{M}_n)$. We denote by $H^k(\Sigma)\,(k \geq 1)$ the space of all the matrix functions f such that $f^{(j)} \in L^2(\Sigma)$ for all $j = 0, \ldots, k$ in the distribution sense. Its norm is $\|f\|_{2,k} = \left(\sum_{j=0}^{k} \|f^{(j)}\|_2^2\right)^{1/2}$, and $H^k(\Sigma)$ is a Hilbert space with continuous pointwise multiplication. Sometimes we write $\|f\|_{2,k}$ as $\|f\|_k$ for brevity. A function $f \in H^k(\Sigma)$ is Hölder continuous.

In the present paper, we choose a formulation in which the contour is bounded. In [10] and [11], however, the author assumes that the contour is unbounded. At some places of the present paper, we reduce the proof to the unbounded case. Let $C_\pm$ be the Cauchy operators defined by

$$C_\pm f(z) = \lim_{z' \to z} \frac{1}{2\pi} \int_\Sigma \frac{f(w)\, dw}{w - z'},$$

where the nontangential limit $z' \to z$ is taken from $\Omega_\pm$ respectively. They are bounded from $L^2(\Sigma)$ to $L^2(\Sigma)$ and from $H^k(\Sigma)$ to $H^k(\Sigma)$. It is known that C_+ and $-C_-$ are complementary projections. Moreover, a function in $\mathrm{Ker}\, C_\pm = \mathrm{Range}\, C_\mp$ has a holomorphic extension to $\Omega_\mp$.

3. Formulation of Riemann-Hilbert problems

Assume that $v \in H^k(\Sigma)$ is invertible and that it admits a factorization $v = (b^-)^{-1} b^+$ for invertible $b^\pm \in H^k(\Sigma)$. Since Σ is bounded, $|\det b^-|$ is uniformly away from 0 and $(b^-)^{-1} \in H^k(\Sigma)$. A factorization as above always exists (we have only to choose $b^+ = I$ or $b^- = I$). Set $w^\pm = \pm(b^\pm - I)$, i.e., $b^\pm = I \pm w^\pm$. We call $w = (w^+, w^-)$ *a pair of factorization data* of v. The set of all such pairs is denoted by FD_k. We have

$$FD_k = \left\{ (w^+, w^-) \in \bigoplus^2 H^k(\Sigma);\ I \pm w^\pm \text{ is invertible} \right\}.$$

Now we introduce three mutually equivalent problems: (3.1), the classical RHP in Proposition 3.2, and the singular integral equation (3.2). Each of these formulations has its own advantage. The classical one can be understood intuitively, and (3.2) can be dealt with by using index theory. The problem (3.1) is the bridge between these two and shows the importance of finding a good pair of factors.

Definition 3.1. An element $\mu \in H^j(\Sigma)(j = 0, \ldots, k)$ is said to be a solution of the Riemann-Hilbert problem (RHP) of the pair of factorization data w if

$$\mu b^\pm - h \in \mathrm{Range}\, C_\pm \tag{3.1}$$

for some constant matrix h.

The definition in [10] has been modified here because $\mu(\infty)$ is not defined in the present paper. Notice that $m_\pm := \mu b^\pm \in H^j(\Sigma)$. Since they are in the ranges of the Cauchy operators modulo h, they have a holomorphic extension to $\mathbb{C}\setminus\Sigma$, which we denote by m. We call it the solution of the Riemann-Hilbert problem of v or w.

Proposition 3.2. *If μ is a solution of* (3.1) *for fixed h, then the holomorphic extension m is a solution of a Riemann-Hilbert boundary value problem in the classical sense:*

$$m_+ = m_- v, m(\infty) = \lim_{z\to\infty} m(z) = h.$$

Conversely, if a holomorphic function m satisfies $m(\infty) = h$ and $m_+ = m_- v$, then $\mu = m_+(b^+)^{-1} = m_-(b^-)^{-1}$ is a solution of (3.1).

Proof. We have $m_+(b^+)^{-1} = \mu = m_-(b^-)^{-1}$ and $m_+ = m_-(b^-)^{-1}b^+ = m_- v$. Next, $m(\infty) = h$ follows from $m = h +$ (a Cauchy integral). The converse is now easy. □

For $w = (w^+, w^-)$, set

$$C_w\phi = C_+(\phi w^-) + C_-(\phi w^+).$$

Then C_w is a bounded operator from $H^j(\Sigma)$ to itself for every $j = 0, 1, \ldots, k$.

Proposition 3.3. *An element μ of $L^2(\Sigma)$ is a solution of* (3.1) *if and only if*

$$(I - C_w)\mu = h \tag{3.2}$$

holds. If $I - C_w$ is a bijection, then a solution of (3.1) *exists uniquely.*

Proof. We follow the proof of [10, Prop. 3.3]. Recall that $C_+ - C_- = I$. If μ satisfies (3.2), we have

$$\begin{aligned}\mu b^+ - h &= \mu(I + w^+) - (I - C_w)\mu = \mu w^+ + C_w\mu \\ &= (C_+ - C_-)(\mu w^+) + C_+(\mu w^-) + C_-(\mu w^+) \\ &= C_+(\mu w^+ + \mu w^-) \in \operatorname{Range} C_+,\end{aligned}$$

and similarly $\mu b^- - h = C_-(\mu w^+ + \mu w^-) \in \operatorname{Range} C_-$.

Conversely, assume (3.1). Then $\mu b^\pm - h \in \operatorname{Ker} C_\mp$. We have

$$\begin{aligned}(I - C_w)\mu &= (C_+ - C_-)\mu - \left[C_+(\mu w^-) + C_-(\mu w^+)\right] \\ &= C_+(\mu b^-) - C_-(\mu b^+) \\ &= C_+(\mu b^- - h) - C_-(\mu b^+ - h) + h = h.\end{aligned}$$

□

4. Factorization and partial indices

We introduce two classes of holomorphic matrix functions following [11]:

$$\begin{aligned}\mathcal{H}^k(\mathbb{C}\setminus\Sigma) &:= \{m; m_\pm - m(\infty) \in \operatorname{Range} C_\pm\}, \\ G\mathcal{H}^k(\mathbb{C}\setminus\Sigma) &:= \left\{m \in \mathcal{H}^k(\mathbb{C}\setminus\Sigma); \det m \text{ vanishes nowhere}\right\},\end{aligned}$$

where $C_\pm \colon H^k(\Sigma) \to H^k(\Sigma)$.

Theorem 4.1. *Any* $v \in H^k(\Sigma)$ *with* $\det v \neq 0$ *admits a Riemann-Hilbert (Wiener-Hopf) factorization* $v = m_-^{-1}\theta m_+$ *relative to* Σ *in* $H^k(\Sigma)$. *Here* $m_\pm$ *are the boundary values of an element* m *of* $G\mathcal{H}^k(\mathbb{C}\setminus\Sigma)$. *The matrix* θ *is*

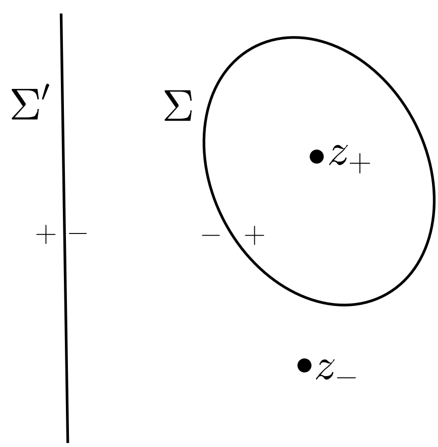
$$\theta = \operatorname{diag}\left[\left(\frac{z-z_+}{z-z_-}\right)^{k_1}, \dots, \left(\frac{z-z_+}{z-z_-}\right)^{k_n}\right], \tag{4.1}$$

where $z_\pm \in \Omega^\pm$ *and* $k_1, \dots, k_n$ *are integers such that* $k_1 \geq \cdots \geq k_n$. *We call* $k_1, \dots, k_n$ *the partial indices of* v. *They are uniquely determined.*

Proof. Fix $z_\pm \in \Omega^\pm$. We embed our contour Σ into $\widehat{\Sigma} \ni \infty$ and reduce the proof to [10, Thm. 9.1] or [11, Thm. 2.1.3]. Let Σ' be a line (a circle in the Riemann sphere) defined by $\operatorname{Re} z = -p$, where p is so large that Σ' is far away from Σ and $z_\pm$. First we assume that Ω_+ is bounded and that Ω_- is unbounded. The orientation of Σ' is from $-p-i\infty$ to $-p+i\infty$. If p is sufficiently large, we have $z_+ \in \Omega_+ \subset \widehat{\Omega}_+, z_- \in \widehat{\Omega}_- \subset \Omega_-$. Set $\widehat{\Sigma} = \Sigma \cup \Sigma'$. It has a compatible orientation in the sense that it is a positively oriented boundary of an open set $\widehat{\Omega}_+$ and is a negatively oriented boundary of an open set $\widehat{\Omega}_-$. Extend $v \in H^k(\Sigma)$ to $\widehat{\Sigma}$ by setting $\hat{v}|_\Sigma = v, \hat{v}|_{\Sigma'} = I$. Then $\hat{v}$ is not an element of $H^k(\widehat{\Sigma})$, but it belongs to $H^k_I(\widehat{\Sigma}) = H^k(\Sigma) \oplus \mathbf{M}_n$ introduced in [10] and [11]. It consists of matrix functions f on $\widehat{\Sigma}$ with the limit $f(\infty)$ such that $f - f(\infty) \in H^k(\widehat{\Sigma})$. The norm is the square root of $|f(\infty)|^2 + \|f - f(\infty)\|_k^2$. Since there is no self-intersection, it is not necessary to introduce $H^k(\widehat{\Sigma}^\pm)$ and $H^k(\Sigma^\pm)$.

By [10, Thm. 9.1] or [11, Thm. 2.1.3], $\hat{v} \in H^k_I(\widehat{\Sigma})$ admits a Riemann-Hilbert factorization

$$\hat{v} = \hat{m}_-^{-1}\theta\hat{m}_+,$$
$$\theta = \operatorname{diag}\left[\left(\frac{z-z_+}{z-z_-}\right)^{k_1}, \dots, \left(\frac{z-z_+}{z-z_-}\right)^{k_n}\right].$$



On Σ', we have $\hat{v} = I = \hat{m}_-^{-1}\theta\hat{m}_+$, which implies $\hat{m}_+ = \theta^{-1}\hat{m}_-$. Set $m = \theta^{-1}\hat{m}$ in $\operatorname{Re} z < -p$ (the positive side of Σ') and $m = \hat{m}$ elsewhere. Then m is holomorphic for $z \notin \Sigma$, and we have a factorization $v = m_-^{-1}\theta m_+$ on Σ. In particular, v and $\hat{v}$ has the same partial indices.

Next, if Ω_+ is unbounded and Ω_- is bounded, we reverse the orientation of Σ'. We get $\hat{m}_- = \theta\hat{m}_+$ on Σ' and set $m = \theta\hat{m}$ in $\operatorname{Re} z < -p$. □

Remark 4.2. The anonymous referee has commented that Theorem 4.1 is in essence the Birkhoff-Grothendieck theorem (any vector bundle on $\mathbb{CP}^1$ is holomorphically equivalent to the direct sum of line bundles $\sum \mathcal{O}(k_j)$).

Theorem 4.3. *The operator* $I - C_w \colon H^k(\Sigma) \to H^k(\Sigma)$ *is Fredholm. Let* $k_1, \dots, k_n$ *be the partial indices of* v. *Then*

$$\dim \operatorname{Ker}(I - C_w) = n \sum_{k_j > 0} k_j,$$

$$\dim \operatorname{Coker}(I - C_w) = -n \sum_{k_j < 0} k_j.$$

Proof. We employ the embedding argument in the proof of Theorem 4.1. We extend $v \in H^k(\Sigma)$ and w to $\widehat{\Sigma}$ by setting $\hat{v}|_\Sigma = v, \hat{v}|_{\Sigma'} = I$ and $\hat{w}|_\Sigma = w, \hat{w}|_{\Sigma'} = (0,0)$. Then $\hat{w}$ is a pair of factorization data of $\hat{v}$. Recall that v and $\hat{v}$ have the same partial indices.

On $H^k_I(\widehat{\Sigma}) = H^k(\Sigma) \oplus H^k_I(\Sigma')$, we have $C_{\hat{w}} = C_w \oplus 0$. By [10, Thm. 9.2] and [11, Thm. 2.1.6], we have $\dim \operatorname{Ker}(I - C_w) = \dim \operatorname{Ker}(I - C_{\hat{w}}) = n \sum_{k_j > 0} k_j$. The assertion about the cokernel is proved in the same way. □

Corollary 4.4. *If the partial indices are all zero, the Riemann-Hilbert problem* (3.1) *has a unique solution, and so does* (3.2).

Proof. Use Proposition 3.3 and Theorem 4.3. □

5. Inversion in the unit circle

For a subset A of $\mathbb{C}$ and a matrix function f, we set $A^\sharp = \{1/\bar{z};\, z \in A\}$ and $f^\sharp(z) = f(1/\bar{z})^*$. It is the inversion in the unit circle $S^1 = \{z;\, |z| = 1\}$. For example, if θ is as in (4.1) and $z_\pm \neq 0$, then we have

$$\theta^\sharp = \operatorname{diag}\left[\left(\frac{\bar{z}_+}{\bar{z}_-} \cdot \frac{z - 1/\bar{z}_+}{z - 1/\bar{z}_-} \right)^{k_1}, \dots, \left(\frac{\bar{z}_+}{\bar{z}_-} \cdot \frac{z - 1/\bar{z}_+}{z - 1/\bar{z}_-} \right)^{k_n} \right]. \tag{5.1}$$

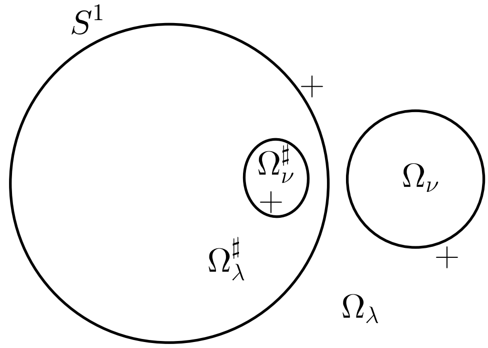
Theorem 5.1. *Let* $\Sigma \supset S^1$ *be a contour invariant under inversion in* S^1. *If* $v \in H^k(\Sigma)$ *with* $\det v \neq 0$ *satisfies*

$$v = v^\sharp \text{ on } \Sigma \setminus S^1 \quad \text{and} \quad \operatorname{Re} v = (v + v^*)/2 > 0 \quad \text{on } S^1,$$

then it has only zero partial indices, and a solution of (3.1) *exists uniquely when* h *is fixed.*

Proof. Let $w = (w^+, w^-)$ be an arbitrary pair of factorization data of v and set $b^\pm = I \pm w^\pm$. It is enough to prove the bijectivity of $I - C_w$. Let $v = m_-^{-1}\theta m_+$ be the factorization of v on Σ as in Theorem 4.1. By inversion, we have $v^\sharp = m_+^\sharp \theta^\sharp (m_-^{-1})^\sharp$ on $\Sigma^\sharp = \Sigma$. Since $1/\bar{z}_\pm \in \Omega^\mp$, $m_+^\sharp \in \operatorname{Range} C_-$, $(m_-^{-1})^\sharp \in \operatorname{Range} C_+$, the expression (5.1) of $\theta^\sharp$ implies that the partial indices of $v^\sharp$ are $-k_n, \dots, -k_1$. If $w^\sharp$ is a pair of factorization data of $v^\sharp$, we have by Theorem 4.3

$$\dim \operatorname{Coker}(I - C_w) = \dim \operatorname{Ker}(I - C_{w^\sharp}).$$



Since $v^\sharp$ also satisfies the assumptions of the theorem, it is enough to prove that $\mathrm{Ker}(I - C_w) = 0$.

Let Ω_ν be a component of $\mathbb{C} \setminus \Sigma$ outside S^1. In the figure, the orientation of Σ is indicated by placing plus signs on the positive sides of the curves. We may assume that S^1 has the clockwise orientation following the convention of [1]. Assume $\mu \in \mathrm{Ker}(I - C_w)$. Then by Proposition 3.3, we have $m_\pm := \mu b^\pm \in \mathrm{Range}\, C_\pm$, and they have a holomorphic extension, which we denote by m. Let $m_{\nu 1}, m_{\nu 2}$ be the boundary values of $m|_{\Omega_\nu}, m|_{\Omega_\nu^\sharp}$ respectively. If $\Omega_\nu \subset \Omega_\pm$, then $\Omega_\nu^\sharp \subset \Omega_\mp$. So if $m_{\nu 1}$ is the boundary value from (a part of) $\Omega_\pm$, then $m_{\nu 2}$ is the boundary value from (a part of) $\Omega_\mp$. Cauchy's theorem implies that

$$\int_{\partial\Omega_\nu} m_{\nu 1} m_{\nu 2}^\sharp = 0.$$

Here notice that $\partial\Omega_\nu$ is positively oriented with respect to Ω_ν. We calculate the sum with respect to all the components Ω_ν outside S^1, including the one whose boundary contains S^1 (like Ω_λ in the figure). We have

$$\sum_\nu \int_{\partial\Omega_\nu} m_{\nu 1} m_{\nu 2}^\sharp = 0. \tag{5.2}$$

Now we show that

$$\sum_\nu \int_{\partial\Omega_\nu} m_{\nu 1} m_{\nu 2}^\sharp = \int_{S^1} m_- v m_-^\sharp. \tag{5.3}$$

Let Σ_j be a component of Σ outside S^1. Then it is a part of the common boundary of some components $\Omega_\alpha \subset \Omega^+$ and $\Omega_\beta \subset \Omega^-$. Let m_1^+, m_1^- be the boundary values of m on Σ_j from $\Omega_\alpha, \Omega_\beta$ respectively (hence $m_1^+ = m_1^- v$) and let m_2^+, m_2^- be the boundary values of m on $\Sigma_j^\sharp$ from $\Omega_\beta^\sharp, \Omega_\alpha^\sharp$ respectively (hence $m_2^+ = m_2^- v$). In the left-hand side of (5.3), the integral along S^1 appears only once as $\int_{S^1} m_+ m_-^\sharp = \int_{S^1} m_- v m_-^\sharp$. The integrals along Σ_j appear twice in the left-hand side of (5.3), once as $\int_{\Sigma_j} m_{1+} m_{2-}^\sharp = \int_{\Sigma_j} m_{1-} v m_{2-}^\sharp$ and once again as $\int_{-\Sigma_j} m_{1-} m_{2+}^\sharp = \int_{-\Sigma_j} m_{1-} v^\sharp m_{2-}^\sharp$. Here the orientation of Σ_j is induced from that of Σ (the positively oriented boundary of $\Omega_\alpha \subset \Omega_+$), and $-\Sigma_j$ has the reversed orientation (determined by $\Omega_\beta \subset \Omega_-$). Since $v = v^\sharp$ on Σ_j, these integrals cancel each other out, and (5.3) has been proved. By (5.2) and (5.3), we have

$$\int_{S^1} m_- v m_-^* = \int_{S^1} m_- v m_-^\sharp = 0.$$

Inversion (or Hermitian conjugation) gives $\int_{S^1} m_- v^* m_-^* = 0$. Adding these two equations, we get

$$\int_{S^1} m_-(v + v^*) m_-^* = 0.$$

By the positivity of $\operatorname{Re} v$, we have $m_- = 0$ on S^1, which implies that $m_+ = m_- v = 0$ there. We get $m = 0$ at least in the components of $\Omega_\pm$ whose boundaries include S^1, like Ω_λ and $\Omega_\lambda^\sharp$ in the figure. Then the boundary value m_+ or m_- from such a component vanishes along any other part of the boundary. Since v is invertible, the boundary value from the other side also vanishes, and we have $m = 0$ in that side. We can repeat this process as many times as necessary (e.g., concentric circles) and finally we get $m_\pm = 0$ and $\mu = 0$ everywhere on Σ. □

Corollary 5.2. *Let $\Sigma \supset S^1$ be a contour invariant under inversion in S^1. If $v \in H^k(\Sigma)$ with* $\det v \neq 0$ *satisfies*

$$v = v^\sharp \text{ on } \Sigma \quad \text{and} \quad v > 0 \text{ on } S^1,$$

then $v = (m_+)^\sharp m_+$ for some $m \in G\mathcal{H}^k(\mathbb{C} \setminus \Sigma)$.

Proof. By the preceding theorem, v has only zero partial indices, and we have $v = n_- n_+$ on Σ for some $n = n(z) \in G\mathcal{H}^k(\mathbb{C} \setminus \Sigma)$. Here we have replaced n_-^{-1} by n_-. It is equivalent to replacing n by its inverse in Ω_-. We have $v^\sharp = n_+^\sharp n_-^\sharp$. Since $v = v^\sharp$, [3, p. 11] implies that there exists a constant matrix C such that $n_- = n_+^\sharp C$ and $n_+ = C^{-1} n_-^\sharp$. Therefore we have $v = n_+^\sharp C n_+$ on Σ. In particular, we have $v = n_+^* C n_+$ on S^1. Since v is Hermitian and positive on S^1, so is C. There exists a positive Hermitian matrix R such that $R^2 = C$. We have $v = n_+^\sharp R^2 n_+ = (Rn)_+^\sharp (Rn)_+$ everywhere on Σ. □

Example 5.3. Let Σ be the unit circle S^1, and set

$$v(z) := \begin{bmatrix} 1 - |r(z)|^2 & -z^{2n}\bar{r}(z) \\ z^{-2n} r(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 - |r'(z)|^2 & -\bar{r'}(z) \\ r'(z) & 1 \end{bmatrix} \ (z \in S^1).$$

Here $r(z)$ is a sufficiently smooth function on S^1, $r'(z) = z^{-2n} r(z)$, and n is an integer. If $|r(z)| < 1$, then $\operatorname{Re} v = \operatorname{diag}[1 - |r(z)|^2, 1] > 0$ and Theorem 5.1 applies. The matrix $v(z)$ is modeled on the one corresponding to the defocusing integrable discrete nonlinear Schrödinger equation (IDNLS). See [1,7,8]. But in the present paper, it is not necessary to assume that $r(z)$ is obtained by the scattering transform. It can be prescribed without reference to a potential, and we do not have to assume $r(-z) = -r(z)$ ([1, (3.2.76)]), a property of the reflection coefficient. The present author hopes this example and Theorem 6.1 below give a basis for establishing the bijectivity of the scattering/inverse scattering transforms (cf. [4, 11, 12]).

6. RHP modeled on the focusing IDNLS

In this section, we consider a problem modeled on the focusing IDNLS ([1]). Let z_j $(j = 1, 2, \dots, J)$ be distinct points outside S^1. We consider the Riemann-Hilbert boundary value problem

$$M_+(z) = M_-(z)V(z) \quad \text{on } S^1, \tag{6.1}$$

$$V(z) = \begin{bmatrix} 1 + |r(z)|^2 & z^{2n}\bar{r}(z) \\ z^{-2n}r(z) & 1 \end{bmatrix}, \tag{6.2}$$

$$\mathrm{Res}(M(z); z_j) = \lim_{z \to z_j} M(z) \begin{bmatrix} 0 & 0 \\ z_j^{-2n}c_j & 0 \end{bmatrix}, \tag{6.3}$$

$$\mathrm{Res}(M(z); \bar{z}_j^{-1}) = \lim_{z \to \bar{z}_j^{-1}} M(z) \begin{bmatrix} 0 & \bar{z}_j^{-2n-2}\bar{c}_j \\ 0 & 0 \end{bmatrix}, \tag{6.4}$$

$$M(z) \to I \quad \text{as } z \to \infty. \tag{6.5}$$

Here n is an integer, $r(z)$ is a sufficiently smooth function on the unit circle, and c_j is an arbitrary complex number. Moreover M_+ and M_- are the boundary values from the *outside* and *inside* of the unit circle respectively. The unit circle is oriented clockwise following the convention in [1]. In the study of the focusing IDNLS, one encounters quartets of the form $\{\pm z_j, \pm 1/\bar{z}_j\}$, but in the present paper we generalize the situation and consider pairs of the form $\{z_j, 1/\bar{z}_j\}$. Moreover we do not assume $r(-z) = -r(z)$.

Following [9], we reduce this problem to one without poles.

Let $C[z_j]$ be a sufficiently small circle centered at z_j for each j. Assume that it is oriented clockwise. By inversion in S^1, we get $C[z_j]^\sharp$, which is oriented *counter*clockwise. This simple closed curve encloses $1/\bar{z}_j$.

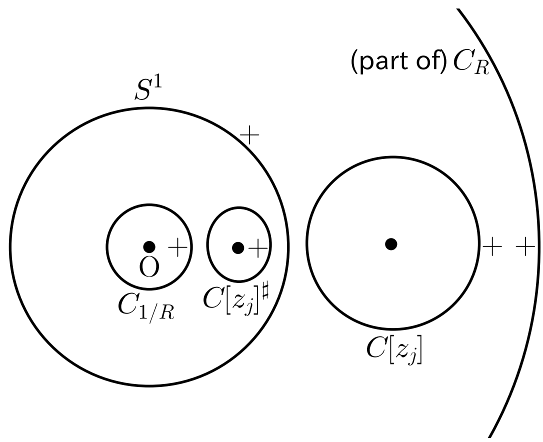
Set

$$m(z) = \begin{cases} M(z) \begin{bmatrix} 1 & 0 \\ -\dfrac{z_j^{-2n}c_j}{z - z_j} & 1 \end{bmatrix} & \text{inside } C[z_j], \\ M(z) \begin{bmatrix} 1 & -\dfrac{\bar{z}_j^{-2n-2}\bar{c}_j}{z - \bar{z}_j^{-1}} \\ 0 & 1 \end{bmatrix} & \text{inside } C[z_j]^\sharp, \end{cases}$$

and $m(z) = M(z)$ elsewhere. Then $m(z)$ is holomorphic near $z_j, \bar{z}_j^{-1}$.

Set $\Sigma = S^1 \cup \bigcup_{j=1}^J C[z_j] \bigcup_{j=1}^J C[z_j]^\sharp$. We introduce a matrix $v(z)$ on Σ by

$$v(z) = \begin{cases} V(z) & \text{on } S^1, \\ \begin{bmatrix} 1 & 0 \\ \dfrac{z_j^{-2n}c_j}{z - z_j} & 1 \end{bmatrix} & \text{on } C[z_j], \\ \begin{bmatrix} 1 & -\dfrac{\bar{z}_j^{-2n-2}\bar{c}_j}{z - \bar{z}_j^{-1}} \\ 0 & 1 \end{bmatrix} & \text{on } C[z_j]^\sharp. \end{cases}$$



Then the RHP (6.1)-(6.5) is equivalent to the following RHP without poles:

$$m_+(z) = m_-(z)v(z) \quad \text{on } \Sigma \quad \text{and} \quad m(z) \to I\,(z \to \infty). \tag{6.6}$$

Theorem 6.1. *The classical Riemann-Hilbert problem* (6.6) *has a unique solution and so does* (6.1)-(6.5). *Moreover v has only zero partial indices.*

Proof. The jump matrix $v(z)$ does not satisfy the assumption of Theorem 5.1 but can be converted to such a one by conjugation, i.e., by introducing a new unknown matrix $m'(z)$.

Let C_R and $C_{1/R}$ be the circles $|z| = R$ and $|z| = 1/R$ respectively, where $R > 0$ is sufficiently large. We give them both counterclockwise orientation. We introduce

$$A = A(z) = \begin{bmatrix} \prod_{k=1}^J z_k & 0 \\ 0 & z \end{bmatrix},$$

$$B_j = B_j(z) = \begin{bmatrix} \prod_{k=1}^J z_k & 0 \\ -\left(\prod_{k\neq j} z_k\right) z_j^{-2n} c_j & z \end{bmatrix} \; (j = 1, \ldots, J),$$

$$C = C(z) = \begin{bmatrix} \prod_{k=1}^J \bar{z}_k^{-1} & 0 \\ 0 & z \end{bmatrix}.$$

We define $m' = m'(z)$ by the following set of rules: (1) $m' = m$ inside $C_{1/R}$ and outside C_R. (2) $m' = mA$ if z is between S^1 and C_R and is outside $C[z_j]$ for all j. (3) $m' = mB_j$ inside $C[z_j]$. (4) $m' = mC$ between $C_{1/R}$ and S^1 except on $\bigcup_j C[z_j]^\sharp$. Then the normalization condition at ∞ remains the same. Now we calculate the jump matrix $v' = v'(z)$ for m': $m'_+ = m'_- v'$ on Σ.

We have $v' = A$ on C_R and $v' = C^{-1}$ on $C_{1/R}$. Since $A^\sharp = C^{-1}$, we have $v'^\sharp(z) = v(z)$ for $z \in C_R \cup C_{1/R}$.

On $C[z_j]$, we have $v' = B_j^{-1} v A$. We evaluate $v'^{-1} = A^{-1} v^{-1} B_j$ first, because A^{-1} is easier than B_j^{-1}. Then we get

$$v' = (v'^{-1})^{-1} = \begin{bmatrix} 1 & 0 \\ \dfrac{\left(\prod_{k\neq j} z_k\right) z_j^{-2n} c_j}{z - z_j} & 1 \end{bmatrix}$$

on $C[z_j]$. Next on $C[z_j]^\sharp$, we have

$$v' = C^{-1}vC = \begin{bmatrix} 1 & -\dfrac{z\left(\prod_{k\neq j}\bar{z}_k\right)\bar{z}_j^{-2n-1}\bar{c}_j}{z-\bar{z}_j^{-1}} \\ 0 & 1 \end{bmatrix}.$$

Therefore $v'^\sharp(z) = v'(z)$ holds for $z \in C[z_j] \cup C[z_j]^\sharp$.

On S^1, we have $v' = C^{-1}vA = A^\sharp vA = A^*vA$. Since v is a positive Hermitian matrix, so are v' and $\operatorname{Re} v'$.

By Theorem 5.1, the matrix v' has only zero partial indices. By Proposition 3.2 and Corollary 4.4, the classical RHP $m'_+ = m'_- v', m' \to I(z \to \infty)$ has a unique solution, and so does (6.6).

We have $v' = PvQ$, where $\{P, Q\} \subset \left\{A, B_j^{-1}, C, C^{-1}\right\}$. Let $v' = \tilde{m}_-^{-1}\tilde{m}_+$ be its factorization. We have $v = P^{-1}\tilde{m}_-^{-1}\tilde{m}_+Q^{-1}$. This factorization of v means all the partial indices are zero. □

Remark 6.2. In Theorem 6.1 above, $\{(z_j, 1/\bar{z}_j), c_j;\, j = 1, \dots, J\}$ and $r(z)$ are not true scattering data. True ones have two additional characteristics: poles appear in quartets of the form $(\pm z_j, \pm 1/\bar{z}_j)$, and the reflection coefficients satisfy $r(-z) = -r(z)$. According to Theorem 6.1, we can solve the associated RHP uniquely even for this kind of formal or generalized 'scattering data' and apply the potential reconstruction formula, but the 'potential' obtained this way is not necessarily a potential.

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