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COMPLEMENTARY ROMANOVSKI-ROUTH POLYNOMIALS: FROM ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE TO COULOMB WAVE FUNCTIONS

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Abstract. We consider properties and applications of a sequence of polynomials known as complementary Romanovski-Routh polynomials (CRR polynomials for short). These polynomials, which follow from the Romanovski-Routh polynomials or complexified Jacobi polynomials, are known to be useful objects in the studies of the one-dimensional Schrödinger equation and also the wave functions of quarks. One of the main results of this paper is to show how the CRR-polynomials are related to a special class of orthogonal polynomials on the unit circle. As another main result, we have established their connection to a class of functions which are related to a subfamily of Whittaker functions that includes those associated with the Bessel functions and the regular Coulomb wave functions. An electrostatic interpretation for the zeros of CRR-polynomials is also considered.

1. Introduction

The Romanovski-Routh polynomials are defined by the Rodrigues formula

$$R_n^{(\alpha,\beta)}(x) = \frac{1}{\omega^{(\alpha,\beta)}(x)} \frac{d^n}{dx^n}\left[\omega^{(\alpha,\beta)}(x)(1+x^2)^n\right], \quad n \geq 1 \tag{1.1}$$

(see [26]), where $\omega^{(\alpha,\beta)}(x) = (1+x^2)^{\beta-1}(e^{-\operatorname{arccot} x})^{\alpha}$. These polynomials first appeared in Routh [30] and were rediscovered by Romanovski [29] in his work

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regarding probability distributions; see also [25] for further historical facts and connections. They are found to be solutions of the second order differential equations

$$(1.2)\qquad (1+x^2)y'' + (2\beta x+\alpha)y' - n(2\beta+n-1)y = 0, \quad n \geq 1.$$

Since $R_n^{(\alpha,\beta)}(x) = (-2i)^n n!\, P_n^{(\beta-1+\frac{i}{2}\alpha,\, \beta-1-\frac{i}{2}\alpha)}(ix)$, $n \geq 0$, the polynomials $R_n^{(\alpha,\beta)}$ are also known as complexified Jacobi polynomials. This connection formula between the polynomials $R_n^{(\alpha,\beta)}$ and the Jacobi polynomials $P_n^{(a,b)}$ can be directly verified from (1.1) and the Rodrigues formula (see, for example, [17, eq. (4.2.8)]) for the Jacobi polynomials.

The Romanovski-Routh polynomials $R_n^{(\alpha,\beta)}$ are not orthogonal in the usual sense. For instance, for $\beta < 0$ they exhibit an incomplete (finite) set of orthogonality relations (see, for example, [28]):

$$\int_{-\infty}^{\infty} R_n^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) \omega^{(\alpha,\beta)}(x) dx = 0 \quad \text{for} \quad m \neq n$$

only when $m+n-1 < -2\beta$.

In this paper we will focus on a study of the so-called *complementary Romanovski-Routh polynomials* (CRR polynomials for short). These polynomials were defined by Weber [41] by considering a variation of the Rodrigues formula (1.1). From the definition given by Weber [41], the CRR polynomials are

$$(1.3)\qquad Q_n^{(\alpha,\beta)}(x) = \frac{(1+x^2)^n}{\omega^{(\alpha,\beta)}(x)} \frac{d^n}{dx^n} \omega^{(\alpha,\beta)}(x), \quad n \geq 1.$$

The idea of looking at such complementary polynomials started in the work [40] of the same author.

As already observed in [28], one can easily verify that $Q_n^{(\alpha,\beta)}(x) = R_n^{(\alpha,\beta-n)}(x)$, $n \geq 1$. Moreover, as shown also in [28] and [41], the polynomials $Q_n^{(\alpha,\beta)}$ satisfy the three term recurrence formula

$$(1.4)\quad Q_{n+1}^{(\alpha,\beta)}(x) = [\alpha + 2(\beta-n-1)x] Q_n^{(\alpha,\beta)}(x) - n(-2\beta+n+1)(1+x^2) Q_{n-1}^{(\alpha,\beta)}(x),$$

for $n \geq 1$, with $Q_0^{(\alpha,\beta)}(x) = 1$ and $Q_1^{(\alpha,\beta)}(x) = \alpha + 2(\beta-1)x$. The differential equation (1.2) implies also that

$$(1.5)\quad Q_n^{(\alpha,\beta)}(x) = (-2i)^n \Big(\beta - n + i\frac{\alpha}{2}\Big)_n \, {}_2F_1\Big(-n, 2\beta-1-n;\, \beta-n+i\frac{\alpha}{2};\, \frac{1-ix}{2}\Big),$$

with the standard notation of $(\cdot)_n$ and ${}_2F_1$ for the Pochhammer symbol and the hypergeometric function, respectively.

The CRR polynomials find application (see [28]) in the study of wave functions of quarks in accord with QCD (quantum chromodynamics) quark-gluon dynamics. Moreover, these polynomials also play an important role in the studies of (one-dimensional) Schrödinger equations with hyperbolic Scarf potential.

The main objective in the present work is to consider some further properties of these CRR polynomials. However, for convenience, we will use an alternative notation $\mathcal{P}_n(b;x)$ for the CRR polynomials, where $b = \lambda + i\eta$ and

$$(1.6)\qquad \mathcal{P}_n(b;x) = \frac{(-1)^n}{2^n(\lambda)_n} Q_n^{(2\eta,-\lambda+1)}(x) = \frac{(-1)^n}{2^n(\lambda)_n} R_n^{(2\eta,-n-\lambda+1)}(x), \quad n \geq 1.$$

We will also assume that $\lambda = \mathrm{Re}(b) > 0$ and refer to the polynomials $\mathcal{P}_n(b;.)$ as modified CRR polynomials or simply CRR polynomials.[1]

In the following theorem we have gathered some of the basic properties of the CRR polynomials.

Theorem 1.1. *For $b = \lambda + i\eta$, $\lambda > 0$, $\eta \in \mathbb{R}$, the complementary Romanovski-Routh polynomials $\mathcal{P}_n(b;.)$ can be given by the hypergeometric expression*

$$\mathcal{P}_n(b;x) = \frac{(x-i)^n}{2^n}\frac{(2\lambda)_n}{(\lambda)_n}\,{}_2F_1\Big(-n, b;\, b+\bar{b};\, \frac{-2i}{x-i}\Big), \quad n \geq 1.$$

They satisfy the three term recurrence

$$\mathcal{P}_{n+1}(b;x) = (x - c_{n+1}^{(b)})\mathcal{P}_n(b;x) - d_{n+1}^{(b)}(x^2+1)\mathcal{P}_{n-1}(b;x), \quad n \geq 1, \tag{1.7}$$

with $\mathcal{P}_0(b;x) = 1$ *and* $\mathcal{P}_1(b;x) = x - c_1^{(b)}$, *where*

$$c_n^{(b)} = \frac{\eta}{\lambda+n-1} \quad \text{and} \quad d_{n+1}^{(b)} = d_{n+1}^{(\lambda)} = \frac{1}{4}\frac{n(2\lambda+n-1)}{(\lambda+n-1)(\lambda+n)}, \quad n \geq 1. \tag{1.8}$$

Moreover, if $\lambda > 1/2$, then they also satisfy the orthogonality

$$\int_{-\infty}^{\infty} x^m \frac{\mathcal{P}_n(b;x)}{(1+x^2)^n}\,\nu^{(\lambda,\eta)}(x)dx = \gamma_n^{(\lambda)}\delta_{m,n}, \quad m = 0,1,\ldots,n, \tag{1.9}$$

where

$$\nu^{(\lambda,\eta)}(x) = \frac{2^{2\lambda-1}|\Gamma(b)|^2}{\Gamma(2\lambda-1)}\frac{e^{\eta\pi}}{2\pi}\omega^{(2\eta,-\lambda+1)}(x) = \frac{2^{2\lambda-1}|\Gamma(b)|^2}{\Gamma(2\lambda-1)}\frac{e^{\eta\pi}}{2\pi}\frac{(e^{-\operatorname{arccot} x})^{2\eta}}{(1+x^2)^{\lambda}}.$$

Here, $\gamma_0^{(\lambda)} = \int_{-\infty}^{\infty}\nu^{(\lambda,\eta)}(x)dx = 1$ *and* $\gamma_n^{(\lambda)} = (1-\mathcal{L}_n^{(\lambda)})\gamma_{n-1}^{(\lambda)}$, $n \geq 1$, *with*

$$\mathcal{L}_n^{(\lambda)} = \frac{1}{2}\frac{2\lambda+n-2}{\lambda+n-1}, \quad n \geq 1. \tag{1.10}$$

Remark 1.1. Here, we have assumed $\operatorname{arccot}(x)$ to be a continuous function that decreases from π to 0 as x increases from $-\infty$ to ∞.

Part of the statement of Theorem 1.1 directly follows from some of the known results stated above. However, an alternative proof of this theorem that we present in Section 2 is based on some recent results on orthogonal polynomials on the unit circle.

From the three term recurrence (1.7) for $\{\mathcal{P}_n(b;.)\}_{n\geq 0}$ one can easily observe that the leading coefficient of $\mathcal{P}_n(b;.)$ is positive. Precisely, if $\mathcal{P}_n(b;x) = \mathfrak{p}_n^{(b)}x^n +$ lower order terms, then $\mathfrak{p}_0^{(b)} = 1$ and $\mathfrak{p}_n^{(b)} = (1-\ell_n^{(\lambda)})\mathfrak{p}_{n-1}^{(b)}$, $n \geq 1$, where

$$\ell_n^{(\lambda)} = \frac{n-1}{2(\lambda+n-1)}, \quad n \geq 1. \tag{1.11}$$

The sequence $\{d_{n+1}^{(\lambda)}\}_{n\geq 1}$ in (1.7) is a so-called positive chain sequence with $\{\ell_{n+1}^{(\lambda)}\}_{n\geq 0}$, defined above, being its minimal parameter sequence. That is,

$$(1-\ell_n^{(\lambda)})\ell_{n+1}^{(\lambda)} = d_{n+1}^{(\lambda)},\ n \geq 1, \quad \text{with} \quad \ell_1^{(\lambda)} = 0 \ \text{ and } \ 0 < \ell_n^{(\lambda)} < 1,\ n \geq 2.$$

[1]One of the reasons for using $\mathcal{P}_n(b;.)$ instead of $P_n^{(\lambda,\eta)}$ is to avoid confusion with the notation used for Jacobi polynomials. But this notation has some other advantages. For example, in the three term recurrence formula (1.7) for $\{\mathcal{P}_n(b;.)\}_{n\geq 0}$ given below, the sequence of coefficients $\{d_{n+1}^{(b)}\}_{n\geq 1}$ is exactly the same as the sequence of coefficients that appears in the three term recurrence formula for the monic Gegenbauer (i.e., ultraspherical) polynomials $\{\hat{C}_n^{(\lambda)}\}_{n\geq 0}$.

Any sequence $\{g_{n+1}\}_{n\geq 0}$ such that $(1-g_n)g_{n+1} = d_{n+1}^{(\lambda)}$, $n \geq 1$, with $0 \leq g_1 < 1$ and $0 < g_n < 1$ for $n \geq 2$, can be referred to as a parameter sequence of the positive chain sequence $\{d_{n+1}^{(\lambda)}\}_{n\geq 1}$. When $0 < \lambda \leq 1/2$, the sequence $\{\ell_{n+1}^{(\lambda)}\}_{n\geq 0}$ is the only parameter sequence of $\{d_{n+1}^{(\lambda)}\}_{n\geq 1}$.

When $\lambda > 1/2$, the sequence $\{\mathcal{L}_{n+1}^{(\lambda)}\}_{n\geq 0}$ given by (1.10) is also such that

$$(1-\mathcal{L}_n^{(\lambda)})\mathcal{L}_{n+1}^{(\lambda)} = d_{n+1}^{(\lambda)},\ n \geq 1, \quad \text{with} \quad 0 < \mathcal{L}_n^{(\lambda)} < 1,\ n \geq 1. \tag{1.12}$$

Hence, when $\lambda > 1/2$, the sequence $\{\mathcal{L}_{n+1}^{(\lambda)}\}_{n\geq 0}$ is also a parameter sequence of the positive chain sequence $\{d_{n+1}^{(\lambda)}\}_{n\geq 1}$. It turns out that $\{\mathcal{L}_{n+1}^{(\lambda)}\}_{n\geq 0}$ is the so-called maximal parameter sequence of this positive chain sequence. For definitions and for many of the basic results concerning positive chain sequences we refer to [9].

We can verify from (1.2) that the differential equation satisfied by $\mathcal{P}_n(b;.)$ is

$$A(x)\,\mathcal{P}_n''(b;x) - 2B(n,\lambda,\eta;x)\,\mathcal{P}_n'(b;x) + C(n,\lambda)\,\mathcal{P}_n(b;x) = 0, \tag{1.13}$$

where $A(x) = x^2+1$, $B(n,\lambda,\eta;x) = (\lambda+n-1)x - \eta$, $C(n,\lambda) = n(n-1+2\lambda)$.

Many other properties of $\mathcal{P}_n(b;.)$ (i.e., of the polynomials $Q_n^{(\alpha,\beta)}$) have been explored in [41] and [28]. Perhaps one of the most interesting and simplest of these properties is the structure relation

$$\frac{d\,\mathcal{P}_n(b;x)}{dx} = n(1-\ell_n^{(\lambda)})\mathcal{P}_{n-1}(b;x), \quad n \geq 1, \tag{1.14}$$

which can be verified from (1.5). With this property, clearly (1.7) and (1.13) are equivalent statements. The results in Section 4 of this paper are developed as a consequence of the property (1.14).

The contents in the remaining sections of this paper are as follows.

- In Section 2, as one of the main results of this paper, we show how the CRR polynomials $\mathcal{P}_n(b;.)$ are related to a special class of orthogonal polynomials on the unit circle. A proof of Theorem 1.1 is also given in this section.
- In Section 3, we discuss an electrostatic interpretation for the zeros of CRR polynomials.
- In Section 4, we turn to the generating function of the monic CRR polynomials, which has the form $e^{xw}\mathcal{N}(b;w)$. It turns out that the functions $\mathcal{N}(b;.)$ are closely related to the subfamily $M_{-i\eta,\lambda-1/2}$ of Whittaker functions [2]. Thus, special cases of the function $\mathcal{N}(b;.)$ are also related to the Bessel functions and the regular Coulomb wave functions. We have referred to this subfamily of Whittaker functions as extended regular Coulomb wave functions.

2. Orthogonal polynomials on the unit circle. Proof of Theorem 1.1.

We now consider the connection the CRR polynomials $\mathcal{P}_n(b;.)$ have with the polynomials $\Phi_n(b;.)$ which are orthogonal on the unit circle with respect to the probability measure

$$d\mu^{(b)}(e^{i\theta}) = \frac{4^{\mathrm{Re}(b)}|\Gamma(b+1)|^2}{\Gamma(b+\bar{b}+1)}\,\frac{1}{2\pi}[e^{\pi-\theta}]^{\mathrm{Im}(b)}[\sin^2(\theta/2)]^{\mathrm{Re}(b)}d\theta. \tag{2.1}$$

Observe that the above measure presents a Fischer-Hartwig type singularity. The monic orthogonal polynomials $\Phi_n(b;.)$ and the associated orthogonality relation which exist for $\lambda > -1/2$ are (see [35])

$$\Phi_n(b;z) = \frac{(2\lambda+1)_n}{(b+1)_n}\, {}_2F_1(-n, b+1;\, b+\bar{b}+1;\, 1-z), \quad n \geq 0, \tag{2.2}$$

and, for $n, m = 0, 1, 2, \ldots,$

$$\int_0^{2\pi} \overline{\Phi_m(b;e^{i\theta})}\, \Phi_n(b;e^{i\theta})\, d\mu^{(b)}(e^{i\theta}) = \frac{(2\lambda+1)_n\, n!}{|(b+1)_n|^2}\delta_{m,n}.$$

For general information on orthogonal polynomials on the unit circle we refer to the monographs of Szegő [38], Simon [31], [32], and Ismail [17].

Define the polynomials $\{\mathcal{R}_n(b;.)\}_{n\geq 0}$ by

$$\mathcal{R}_n(b;\zeta) = \frac{2^n}{(x-i)^n}\mathcal{P}_n(b;x), \quad n \geq 0, \tag{2.3}$$

where $\zeta = (x+i)/(x-i)$. This transformation maps the compactification of the real line $\overline{\mathbb{R}}$ onto the unit circle $\mathbb{T} := \{\zeta = e^{i\theta}\colon\ 0 \leq \theta < 2\pi\}$. With known results about the polynomials $\mathcal{R}_n(b;.)$ on the unit circle, which we will point out as needed, we now look into a proof of Theorem 1.1.

Proof of Theorem 1.1. From (1.13) and (2.3), it is not difficult to see that

$$z(1-z)\frac{d^2\mathcal{R}_n(b;\,z)}{dz^2} - [b+\bar{b}-(-n+b+1)(1-z)]\frac{d\mathcal{R}_n(b;\,z)}{dz} + nb\mathcal{R}_n(b;\,z) = 0,$$

for $n \geq 1$. From this differential equation we can easily identify the polynomials $\mathcal{R}_n(b;\,.)$ to be

$$\mathcal{R}_n(b;\,z) = \frac{(2\lambda)_n}{(\lambda)_n}\, {}_2F_1(-n, b;\, b+\bar{b};\, 1-z), \quad n \geq 0. \tag{2.4}$$

This result, together with (2.3), gives the proof of the hypergeometric expression in Theorem 1.1. However, the hypergeometric expression in Theorem 1.1 can also be obtained from (1.5) using two transformations of Pfaff, (2.2.6) and (2.3.14) in [2].

The three term recurrence in Theorem 1.1 is equivalent to

$$\mathcal{R}_{n+1}(b;\,z) = \Big[(1+i\,c_{n+1}^{(b)})z + (1-i\,c_{n+1}^{(b)})\Big]\,\mathcal{R}_n(b;\,z) - 4d_{n+1}^{(b)}z\mathcal{R}_{n-1}(b;\,z), \tag{2.5}$$

for $n \geq 1$, with $\mathcal{R}_0(b;\,z) = 1$ and $\mathcal{R}_1(b;\,z) = (1+i\,c_1^{(b)})z + (1-i\,c_1^{(b)})$, where the coefficients $c_n^{(b)}$ and $d_{n+1}^{(b)}$ are as in (1.8). But (2.5) holds, as was shown in [10] and [35], which proves (1.7). Once again, the three term recurrence relation in Theorem 1.1 can also be easily derived from the three term recurrence relation (1.4) for the polynomials $Q_n^{(2\eta,-\lambda+1)}$.

Polynomials $\mathcal{R}_n(b;\cdot)$ (see [35]) are also para-orthogonal, due to the following identity:

$$\mathcal{R}_n(b;\,z) = \frac{(b)_n}{(\lambda)_n}\big[z\Phi_{n-1}(b;z) + \frac{(\bar{b})_n}{(b)_n}\Phi_{n-1}^*(b;z)\big], \quad n \geq 1, \tag{2.6}$$

with Φ_n as in (2.2).

Even though the polynomials $\mathcal{R}_n(b;.)$ and their associated CRR polynomials $\mathcal{P}_n(b;.)$ are defined for $\mathrm{Re}(b) = \lambda > 0$, the orthogonal polynomials $\Phi_n(b;.)$ themselves can be considered for $\lambda > -1/2$.

When $\lambda > 1/2$, the polynomials $\mathcal{R}_n(b;.)$ can be written in terms of the reproducing kernel evaluated at $w = 1$. Recall that the reproducing or Christoffel-Darboux (or CD) kernels are [31]

$$K_n(b;z,w) = \frac{\overline{\varphi_{n+1}(b;w)}\,\varphi_{n+1}(b;z) - \overline{\varphi^*_{n+1}(b;w)}\varphi^*_{n+1}(b;z)}{\overline{w}z - 1}, \quad n \geq 0,$$

where $\varphi_n(b;.)$ comprise the orthonormal version of $\Phi_n(b;\cdot)$. For any fixed w, $K_n(b;z,w)$ is a polynomial in z of degree $\leq n$, and, in particular, if $|w| \geq 1$ it is of exact degree n.

For $\lambda > 1/2$,

$$\mathcal{R}_n(b;\, z) = \xi_n^{(b-1)}\, K_n(b-1; z, 1), \quad n \geq 0,$$

where $\xi_0^{(b-1)} = \int_{\mathbb{T}} d\mu^{(b-1)}(\zeta)$ and $\xi_n^{(b-1)} = \xi_0^{(b-1)} \prod_{j=1}^n (1 - \mathcal{L}_j^{(\lambda)})$, $n \geq 1$. Here, $\{\mathcal{L}_n^{(\lambda)}\}_{n\geq 1}$ is the sequence given by (1.10) and, since the measure $\mu^{(b-1)}$ being a probability measure, $\xi_0^{(b-1)} = 1$. A direct consequence of the definition of the CD kernel is the orthogonality condition (see [10])

$$\int_{\mathbb{T}} \zeta^{-k} \mathcal{R}_n(b;\zeta)(1-\zeta^{-1}) d\mu^{(b-1)}(\zeta) = 0, \quad 0 \leq k \leq n-1,$$

which by means of the transformation (2.3) yields (1.9) in Theorem 1.1, with $\nu^{(\lambda,\eta)}(x)dx = -d\mu^{(b-1)}(\zeta)$.

Finally, to obtain the formula for $\gamma_n^{(\lambda)}$ in Theorem 1.1, from (1.7) we have

$$\frac{x^{n-1}\mathcal{P}_{n+1}(b;x)}{(x^2+1)^n} = (x - c_{n+1}^{(b)})\frac{x^{n-1}\mathcal{P}_n(b;x)}{(x^2+1)^n} - d_{n+1}^{(\lambda)} \frac{x^{n-1}\mathcal{P}_{n-1}(b;x)}{(x^2+1)^{n-1}},$$

for $n \geq 1$. Hence, integration with respect to $\nu^{(\lambda,\eta)}$ gives $\gamma_{n+1}^{(\lambda)} = \gamma_n^{(\lambda)} - d_{n+1}^{(\lambda)}\gamma_{n-1}^{(\lambda)}$, $n \geq 1$, which can be written in the alternative form

$$\frac{\gamma_n^{(\lambda)}}{\gamma_{n-1}^{(\lambda)}}\left(1 - \frac{\gamma_{n+1}^{(\lambda)}}{\gamma_n^{(\lambda)}}\right) = d_{n+1}^{(\lambda)}, \quad n \geq 1.$$

Thus from (1.12), what one has to verify is $\gamma_1^{(\lambda)}/\gamma_0^{(\lambda)} = (1 - \mathcal{L}_1^{(\lambda)}) = (2\lambda)^{-1}$. Clearly, $\gamma_0^{(\lambda)} = \int_{-\infty}^{\infty} \nu^{(\lambda,\eta)}(x)dx = \int_{\mathbb{T}} d\mu^{(b-1)}(\zeta) = 1$. However,

$$\begin{aligned} \gamma_1^{(\lambda)} &= \int_{-\infty}^{\infty} x\frac{\mathcal{P}_1(b;x)}{(1+x^2)}\,\nu^{(\lambda,\eta)}(x)dx = \frac{1}{4}\int_{\mathbb{T}} \frac{\zeta+1}{\zeta}\mathcal{R}_1(b;\zeta)d\mu^{(b-1)}(\zeta) \\ &= \frac{1}{4}\left[(1+ic_1^{(b)})\mu_{-1}^{(b-1)} + 2 + (1-ic_1^{(b)})\mu_1^{(b-1)}\right], \end{aligned}$$

where $\mu_n^{(b-1)} = \int_{\mathbb{T}} \zeta^{-n} d\mu^{(b-1)}(\zeta)$. Thus, using the expression for $c_1^{(b)}$ in Theorem 1.1 together with $\mu_1^{(b-1)} = \overline{\mu}_{-1}^{(b-1)} = (-b+1)/\overline{b}$, we find that $\gamma_1^{(\lambda)} = (2\lambda)^{-1}$. This completes the proof of Theorem 1.1. □

Remark 2.1. Both polynomials $\mathcal{R}_n(b;\cdot)$ and the associated orthogonal polynomials $\Phi_n(b;\cdot)$, studied in [35], have been used as examples in a sequence of papers [6–8, 10, 12, 19, 21], not aware of their connection to the CRR polynomials. The results obtained in [19] are focused on the three term recurrence of the type (1.7) and the associated generalized eigenvalue problem (in this respect, see also [18] and [45]). As it follows from [4, p. 304], polynomials $\mathcal{R}_n(b;\cdot)$ and $\Phi_n(b;\cdot)$ were also classified as hypergeometric biorthogonal polynomials. But, again in [4], no such connection to the Romanovski-Routh polynomials was mentioned.

However, the connection between the circular Jacobi polynomials and a subfamily of the CRR polynomials is known (see [43]). We recall that the circular Jacobi polynomials are the subclass of the polynomials $\Phi_n(b;\cdot)$ in which $\mathrm{Im}(b)=\eta=0$.

3. Electrostatic interpretation for the zeros of CRR polynomials

The electrostatic interpretation of the zeros of Jacobi polynomials, found by Stieltjes, has been extended to other families of polynomials; see e.g. [17, 20, 37] and references therein.

From the recurrence relation (1.7) it follows that (see [19]) the zeros of $\mathcal{P}_n(b;x)$ are real, simple, and interlace with the zeros of $\mathcal{P}_{n-1}(b;x)$. Let $\xi_1^{(m)}(b)>\xi_2^{(m)}(b)>\cdots>\xi_m^{(m)}(b)$ be the zeros of $\mathcal{P}_m(b;x)$.

In order to give the electrostatic interpretation of the zeros of $\mathcal{P}_n(b;x)$ we consider the set

$$\Xi^m=\{\mathbf{x}=(x_1,x_2,\ldots,x_m)^T:\infty>x_1>x_2>\cdots>x_m>-\infty\}$$

(Weyl chamber of type A_{m-1}), and the electrostatic field in Ξ^m which obeys the logarithmic potential law and corresponds to a vector $\mathbf{x}\in\Xi^m$ of moving positive unit charges, and an external field created by two fixed negative charges of size $\lambda_m/2$ at i and $-i$, and a background field given by the arctan. More precisely, we consider the logarithmic energy $E=E(x_1,x_2,\ldots,x_m)$,

$$E=\sum_{1\le j<i\le m}\ln\frac{1}{|x_j-x_i|}-\frac{\lambda_m}{2}\sum_{j=1}^m\left[\ln\frac{1}{|x_j-i|}+\ln\frac{1}{|x_j+i|}\right]-\eta\sum_{j=1}^m\arctan(x_j),$$

and study the minimizer of E in Ξ^m, at least for λ_m sufficiently large.

Theorem 3.1. *Let $\lambda_m=\lambda+m-1$. Then there exists a unique minimizer $\mathbf{x}^*=(x_1^*,x_2^*,\ldots,x_m^*)^T$ of $E(x_1,x_2,\ldots,x_m)$ within Ξ^m. Moreover, $x_j^*=\xi_j^{(m)}(b)$, $j=1,2,\ldots,m$, are the zeros of $\mathcal{P}_m(b;x)$.*

Proof. The problem of minimizing E is equivalent to maximizing the positive function

$$F=F(x_1,x_2,\ldots,x_m)=e^{-2E(x_1,x_2,\ldots,x_m)}.$$

Clearly,

$$F=\prod_{1\le j<i\le m}(x_j-x_i)^2\;\prod_{j=1}^m(1+x_j^2)^{-\lambda_m}\;\prod_{j=1}^m e^{2\eta\arctan x_j}.$$

With $\lambda_m>m-1$, the function F is continuous, positive, and bounded in Ξ^m and takes the value **zero** along the borders of Ξ^m. Thus, there is at least one critical point of F within Ξ^m, and if this critical point is unique, then it is a maximum.

Following [13], let us write

$$F=f_m(\hat{x}\backslash x_k)\;\prod_{j\ne k}(x_k-x_j)^2\;\;(1+x_k^2)^{-\lambda_m}\;e^{2\eta\arctan x_k},$$

where $f_m(\hat{x}\backslash x_k)$ is the part of F that is independent of x_k. With $r_k(x) = r(x)/(x-x_k)$, where $r(x)=\prod_{j=1}^m (x-x_j)$, we can also write

$$F = f_m(\hat{x}\backslash x_k)\; r_k^2(x_k)\; (1+x_k^2)^{-\lambda_m}\; e^{2\eta \arctan x_k},$$

and our problem is now to find a polynomial $r(x)$ of degree m that maximizes F.

Differentiating F with respect to x_k gives

$$\begin{aligned}\frac{\partial F}{\partial x_k} =& f_m(\hat{x}\backslash x_k)\Big[2r_k(x_k) r_k'(x_k)\,(1+x_k^2)^{-\lambda_m}\, e^{2\eta\arctan x_k} \\ &\qquad - 2\lambda_m x_k (1+x_k^2)^{-\lambda_m-1}\, r_k^2(x_k)\, e^{2\eta\arctan x_k} \\ &\qquad\qquad + \frac{2\eta}{1+x_k^2} e^{2\eta\arctan x_k}\, r_k^2(x_k)\,(1+x_k^2)^{-\lambda_m}\Big] \\ =& f_m(\hat{x}\backslash x_k)\; r_k(x_k)(1+x_k^2)^{-\lambda_m-1} e^{2\eta\arctan x_k} \\ &\qquad \times \Big[2(1+x_k^2) r_k'(x_k) - 2(\lambda_m x_k - \eta) r_k(x_k)\Big].\end{aligned}$$

Using $r'(x_k) = r_k(x_k)$ and $r''(x_k) = 2r_k'(x_k)$, we then have

$$\begin{aligned}\frac{\partial F}{\partial x_k} =& f_m(\hat{x}\backslash x_k)\; r_k(x_k)(1+x_k^2)^{-\lambda_m-1} e^{2\eta\arctan x_k} \\ &\qquad \times \big[(1+x_k^2) r''(x_k) - 2(\lambda_m x_k - \eta) r'(x_k)\big].\end{aligned}$$

The expression $s(x) = (1+x^2)r''(x) - 2(\lambda_m x - \eta)r'(x)$ is a polynomial of degree m. We must choose $r(x)$ such that at the critical point of F the polynomial $s(x)$ also vanishes at the zeros of r. That is, $s(x) = \text{const}\times r(x)$.

Thus, if we take $\lambda_m = \lambda+m-1$, where $\lambda>0$, then from the differential equation (1.13) we find that $r(x) = \text{const}\times P_m(b;x)$ and that the elements of any critical point $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ are zeros of $\mathcal{P}_m(b;x)$. The uniqueness of $\mathcal{P}_m(b;x)$ (and of its zeros) shows that this critical point is unique and hence provides a global minimum of E in Ξ^m. ☐

4. Generating functions

Generating functions have been known to be an important tool in the theory of special functions. The following generating function for the CRR polynomials $Q_n^{(2\eta,-\lambda+1)}$ is given in [41, Thm. 1.9]:

$$\frac{(x^2+1)^\lambda e^{2\eta\,\mathrm{arccot}(x)}}{\big[1+[x+w(x^2+1)]^2\big]^\lambda e^{2\eta\,\mathrm{arccot}[x+w(x^2+1)]}} = \sum_{n=0}^\infty Q_n^{(2\eta,-\lambda+1)}(x)\frac{w^n}{n!}.$$

Using (1.6) we get:

Theorem 4.1. *For $b=\lambda+i\eta$ and $\lambda>0$,*

$$\frac{e^{2\eta\,\mathrm{arccot}(x)}}{[(xw-1)^2+w^2]^\lambda e^{2\eta\,\mathrm{arccot}[x-w(x^2+1)]}} = \sum_{n=0}^\infty (2\lambda)_n \widehat{\mathcal{P}}_n(b;x)\frac{w^n}{n!},$$

where $\widehat{\mathcal{P}}_n(b;.)$ are the monic CRR polynomials given by

$$\widehat{\mathcal{P}}_n(b;x) = \frac{1}{\mathfrak{p}_n^{(b)}}\mathcal{P}_n(b;x) = \frac{2^n(\lambda)_n}{(2\lambda)_n}\mathcal{P}_n(b;x), \quad n\ge 1. \tag{4.1}$$

4.1. **Generating function as an Appell sequence.** In order to see the importance of the new generating function that we obtain for $\{\widehat{\mathcal{P}}_n(b;x)\}_{n\geq 0}$, we consider the functions $\mathcal{M}(b;w)$ and $\mathcal{N}(b;w)$ given by

$$\mathcal{M}(b;w) = \mathfrak{C}(b)\, w^{\lambda}\, \mathcal{N}(b;w) \quad \text{and} \quad \mathcal{N}(b;w) = e^{-iw}\, {}_1F_1(b;\, b+\overline{b};\, 2iw), \tag{4.2}$$

where

$$\mathfrak{C}(b) = 2^{\lambda-1} e^{\pi\eta/2} \frac{|\Gamma(b)|}{\Gamma(2\lambda)} \tag{4.3}$$

and, as we have assumed so far, $b = \lambda + i\eta$ and $\lambda > 0$. Here, the notation ${}_1F_1$ denotes Kummer's confluent hypergeometric function.

Clearly, from the ${}_1F_1$ hypergeometric representation

$$\mathcal{M}(b;w) = (i2)^{-\lambda}\mathfrak{C}(b)\, M_{-i\eta,\lambda-1/2}(2iw), \quad \lambda > 0, \tag{4.4}$$

where $M_{-i\eta,\lambda-1/2}$ is a subclass of the Whittaker functions (see, for example, [2, p. 195]). The function $\mathcal{M}(b;w)$ becomes familiar if one considers the alternative notation

$$\mathcal{M}(\overline{b};w) = F_{\lambda-1}(\eta,w) \quad \text{and} \quad \mathfrak{C}(\overline{b}) = C_{\lambda-1}(\eta). \tag{4.5}$$

When λ takes positive integer values, the resulting functions $F_L(\eta,w)$, $L = 0,1,\ldots,$ are the so-called regular Coulomb wave functions. Precisely, with our definitions of $\mathcal{M}(b;w)$ and $\mathcal{N}(b;w)$ the regular Coulomb wave functions can be given by

$$F_L(\eta,w) = \mathcal{M}(L+1-i\eta;w) = C_L(\eta)w^{L+1}\mathcal{N}(L+1-i\eta;w), \quad L = 0,1,2,\ldots. \tag{4.6}$$

It is known that the regular Coulomb wave functions satisfy the differential equation

$$F_L''(\eta,w) + \left[1 - \frac{2\eta}{w} - \frac{L(L+1)}{w^2}\right] F_L(\eta,w) = 0, \tag{4.7}$$

where $F_L''(\eta,w) = d^2F_L(\eta,w)/dw^2$. Moreover, it is also known that

$$F_{L+1}(\eta,w) = \frac{(2L+1)}{L\,|L+1+i\eta|}\left[\frac{L(L+1)}{w} + \eta\right] F_L(\eta,w) - \frac{(L+1)|L+i\eta|}{L\,|L+1+i\eta|}\, F_{L-1}(\eta,w),$$

which holds for $L = 1,2,3,\ldots$. This three term recurrence relation associated with the Coulomb wave functions was first given by Powel in [27].

As stated in [15], the Coulomb wave functions are of great importance in the study of nuclear interactions. They arise when Schrödinger's equation for a charged particle in the Coulomb field of a fixed charge is separated in polar coordinates. For some of the earliest studies on these functions we cite [33] and references therein.

Numerical evaluation of regular Coulomb wave functions has been the subject of many contributions, including [1, 14, 15, 22, 33]. Except for [22, 33], they are mainly based on the use of the above three term recurrence relation. Moreover, the derivation of some of the basic properties of the zeros of these regular Coulomb wave functions and also the evaluation of these zeros have been based on an eigenvalue problem that follows from this three term recurrence relation (see [16, 24]).

By examining the differential equations, the three term recurrence relation, and also the associated eigenvalue problems satisfied by the regular Coulomb wave functions, it is evident that the function $F_\lambda(\eta,w) = \mathcal{M}(\overline{b}+1;w)$, obtained by extending the integer parameter L to the real parameter λ, will preserve many of the properties satisfied by the Coulomb wave function $F_L(\eta,w)$, in particular with regard to the zeros. This is clearly true in the case $\eta = 0$, and the resulting functions lead to the Bessel functions. With such knowledge, the author of [5] looks at some Turán type inequalities associated with these extended regular Coulomb wave functions $F_\lambda(\eta, w)$ and obtains also some information regarding the zeros of these functions. With a different objective, the authors of [36] study the orthogonal polynomials that follow from the extended three term recurrence relation, which they call orthogonal polynomials associated with Coulomb wave functions. It seems the idea behind [36] has also been explored previously in [42]. For some other contributions regarding the functions $F_\lambda(\eta, w)$ for non-integer values of λ, and even complex values of the parameter λ, we cite, for example, [11, 23] and references therein.

In view of the above observations, in the present manuscript we will refer to the functions $\mathcal{M}(b; w) = F_{\lambda-1}(-\eta, w)$ as *extended regular Coulomb wave* functions (ERCW functions for short).

From the recurrence for the regular Coulomb wave functions there follows

$$\mathcal{M}(b+2; w) = \frac{(2\lambda+1)}{\lambda\,|b+1|}\Big[\frac{\lambda(\lambda+1)}{w} - \eta\Big]\mathcal{M}(b+1; w) - \frac{(\lambda+1)|b|}{\lambda\,|b+1|}\,\mathcal{M}(b; w), \tag{4.8}$$

which holds for $\lambda > 0$. This result can be verified from (4.4) and from well-known contiguous relations (see [34, p. 27]) satisfied by Whittaker functions.

The following theorem gives the role played by the ERCW function $\mathcal{M}(b; w)$ as part of a generating function for the monic CRR polynomials $\widehat{\mathcal{P}}_n(b; x)$.

Theorem 4.2. *Let $b = \lambda + i\eta$, where $\lambda > 0$. Then the sequence $\{\widehat{\mathcal{P}}_n(b; .)\}_{n\geq 0}$ of monic complementary Romanovski-Routh polynomials is an Appell sequence and satisfies*

$$\frac{1}{\mathfrak{C}(b)} e^{xw} w^{-\lambda}\mathcal{M}(b; w) = e^{xw}\,\mathcal{N}(b; w) = \sum_{n=0}^{\infty} \widehat{\mathcal{P}}_n(b; x)\,\frac{w^n}{n!}. \tag{4.9}$$

Proof. The property (1.14) of the CRR polynomials $\mathcal{P}_n(b; x)$ shows us that the monic polynomial sequence $\{\widehat{\mathcal{P}}_n(b; x)\}_{n\geq 0}$ is an Appell [3] sequence. Thus, their generating function is of the form $e^{xw}F(w)$. We need to prove $F(w) = \mathcal{N}(b; w)$.

By setting $F(w) = e^{-iw}H(w)$, where $H(w) = \sum_{j=0}^{\infty} h_j^{(b)} w^j$, one needs to prove that (4.9) holds if $H(w) = {}_1F_1(b;\, b+\overline{b};\, 2iw)$; that is,

$$h_j^{(b)} = \frac{(b)_j\,(2i)^j}{(2\lambda)_j\, j!}, \quad j \geq 0. \tag{4.10}$$

We have

$$e^{xw}\, F(w) = e^{(x-i)w} \sum_{j=0}^{\infty} h_j^{(b)} w^j = \Big(\sum_{k=0}^{\infty} \frac{(x-i)^k}{k!}\, w^k\Big)\Big(\sum_{j=0}^{\infty} h_j^{(b)} w^j\Big).$$

Hence,

$$e^{xw}\, F(w) = \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n} h_{n-l}^{(b)} \frac{(x-i)^l}{l!}\Big) w^n.$$

From the hypergeometric expression for $\mathcal{P}_n(b; .)$ in Theorem 1.1 we can easily verify that if

$$\sum_{l=0}^{n} h_{n-l}^{(b)} \frac{(x-i)^l}{l!} = \frac{1}{n!}\widehat{P}_n(b; x), \quad n \geq 0,$$

then $h_j^{(b)}$ are as in (4.10). This completes the proof of the theorem. □

Remark 4.1. The function $H(w)$ considered in the proof of Theorem 4.2 is an ${}_1F_1$ confluent hypergeometric function, and hence it is an entire function. Thus, the function $F(w) = \mathcal{N}(b; w)$ and also, for each x, the generating function $e^{xw}\mathcal{N}(b; w)$ are entire functions. Thus, the right hand side of (4.9) is absolutely convergent for all w on the complex plane.

By letting $b = \lambda + i\eta = 1$ in Theorem 4.2, it is not difficult to verify the following.

Corollary 4.2.1.

$$e^{xw} w^{-1} \sin(w) = e^{xw}\, \mathcal{N}(1; w) = \sum_{n=0}^{\infty} \widehat{\mathcal{P}}_n(1; x) \frac{w^n}{n!}.$$

The ECWF function $\mathcal{M}(b; w)$, when $\mathrm{Im}(b) = \eta = 0$, is related to the Bessel function of order $\lambda - 1/2$. To be precise,

$$\frac{2\,\Gamma(2\lambda)}{\Gamma(\lambda)} (2w)^{-\lambda} \mathcal{M}(\lambda; w) = \mathcal{N}(\lambda; w) = \Gamma(\lambda + 1/2) \Big(\frac{w}{2}\Big)^{-\lambda+1/2} J_{\lambda-1/2}(w).$$

For more information on Bessel functions see, for example, [1], [2], and [39]. We can now state the following.

Corollary 4.2.2. *For $\alpha > -1/2$ the following expansion formula holds with respect to the Bessel function $J_\alpha(w)$:*

$$e^{xw} J_\alpha(w) = \frac{1}{\Gamma(\alpha+1)} \Big(\frac{w}{2}\Big)^{\alpha} \sum_{n=0}^{\infty} \widehat{\mathcal{P}}_n(\alpha + 1/2; x) \frac{w^n}{n!}.$$

With the connection (4.6) with the regular Coulomb wave functions we can state:

Corollary 4.2.3. *The following expansion formula holds with respect to the regular Coulomb wave function $F_L(\eta, w)$:*

$$e^{xw} \mathcal{M}(L + 1 - i\eta; w) = e^{xw} F_L(\eta, w) = C_L(\eta)\, w^{L+1} \sum_{n=0}^{\infty} \widehat{\mathcal{P}}_n(L + 1 - i\eta; x) \frac{w^n}{n!},$$

for $L = 0, 1, 2, \ldots,$ where the so-called Gamow-Sommerfeld factor $C_L(\eta)$ is as in (4.3) *and* (4.5).

Letting $x = 0$ in Corollary 4.2.3 gives the series expansion

$$F_L(\eta, w) = C_L(\eta)\, w^{L+1} \sum_{k=L+1}^{\infty} A_k^L(\eta)\, w^{k-L-1},$$

where $(k-L-1)!\, A_k^L(\eta) = \widehat{\mathcal{P}}_{k-L-1}(L+1-i\eta; 0)$. This expansion formula, together with the three term recurrence relation (1.7) satisfied by $\{\widehat{\mathcal{P}}_n(L+1-i\eta; 0)\}_{n\geq 0}$, is exactly the expansion formula from [1, p. 538]. This expansion result for $F_L(\eta, w)$ first appeared in Yost, Wheeler, and Breit [44]. Now with the hypergeometric expression in Theorem 1.1 for $\mathcal{P}_n(b; x)$ we can give the following closed form expression for $A_k^L(\eta)$:

$$A_{k+L+1}^L(\eta) = \frac{(-i)^k}{k!}\, {}_2F_1(-k, L + 1 - i\eta; 2L + 2; 2), \quad k = 0, 1, 2, \ldots.$$

One can also verify that the result corresponding to $x = -\eta/(L+1)$ in Corollary 4.2.3 is the expansion formula presented in [22].

We now give some further expansion formulas associated with the functions $\mathcal{N}(b; w)$ and the corresponding ERCW functions $\mathcal{M}(b; w)$.

Theorem 4.3. *Let $b = \lambda + i\eta$, where $\lambda > 0$. Let the real sequences* $\{\mathfrak{a}_n\}_{n\geq 0} = \{\mathfrak{a}_n^{(\lambda,\eta)}\}_{n\geq 0}$ *and* $\{\mathfrak{b}_n\}_{n\geq 0} = \{\mathfrak{b}_n^{(\lambda,\eta)}\}_{n\geq 0}$ *be given by*

$$\begin{bmatrix} \mathfrak{a}_{n+1} \\ \mathfrak{b}_{n+1} \end{bmatrix} = \frac{2}{2\lambda+n} \begin{bmatrix} -\eta & -(\lambda+n) \\ (\lambda+n) & -\eta \end{bmatrix} \begin{bmatrix} \mathfrak{a}_n \\ \mathfrak{b}_n \end{bmatrix}, \quad n \geq 0,$$

with $\mathfrak{a}_0 = 1$ *and* $\mathfrak{b}_0 = 0$. *Then*

$$\begin{aligned} &\frac{1}{\mathfrak{C}(b)} w^{-\lambda} \cos(w)\, \mathcal{M}(b;w) = \cos(w)\, \mathcal{N}(b;w) = \sum_{n=0}^{\infty} \mathfrak{a}_n \frac{w^n}{n!}, \\ &\frac{1}{\mathfrak{C}(b)} w^{-\lambda} \sin(w)\, \mathcal{M}(b;w) = \sin(w)\, \mathcal{N}(b;w) = w \sum_{n=0}^{\infty} \frac{1}{n+1} \mathfrak{b}_{n+1} \frac{w^n}{n!}. \end{aligned} \tag{4.11}$$

Moreover, if $\mathfrak{c}_n(w) = \mathfrak{a}_n \cos(w) + \frac{1}{n+1}\mathfrak{b}_{n+1}\, w \sin(w)$, $n \geq 0$, *then*

$$\mathcal{N}(b;w) = \sum_{n=0}^{\infty} \mathfrak{c}_n(w) \frac{w^n}{n!}.$$

Proof. In (4.9), letting $x = i$ and $x = -i$ and then, respectively, summing and subtracting the resulting equations, we get

$$\begin{aligned} 2\cos(w)\, \mathcal{N}(b;w) &= \sum_{n=0}^{n} \left[\widehat{\mathcal{P}}_n(b;i) + \widehat{\mathcal{P}}_n(b;-i)\right] \frac{w^n}{n!}, \\ i2\sin(w)\, \mathcal{N}(b;w) &= \sum_{n=0}^{n} \left[\widehat{\mathcal{P}}_n(b;i) - \widehat{\mathcal{P}}_n(b;-i)\right] \frac{w^n}{n!}. \end{aligned}$$

Hence, we set

$$2\mathfrak{a}_n = \left[\widehat{\mathcal{P}}_n(b;i) + \widehat{\mathcal{P}}_n(b;-i)\right] \quad \text{and} \quad i2\mathfrak{b}_n = \left[\widehat{\mathcal{P}}_n(b;i) - \widehat{\mathcal{P}}_n(b;-i)\right], \quad n \geq 0.$$

Clearly, $\mathfrak{a}_0 = 1$ and $\mathfrak{b}_0 = 0$. To obtain the recurrence formula for $\{\mathfrak{a}_n\}_{n\geq 0}$ and $\{\mathfrak{b}_n\}_{n\geq 0}$, we observe from (1.7) and (4.1) that $\widehat{\mathcal{P}}_n(b;i) = \overline{\widehat{\mathcal{P}}_n(b;-i)} = \frac{2^n i^n (b)_n}{(2\lambda)_n}$, $n \geq 0$. Hence,

$$2\mathfrak{a}_n = \frac{2^n}{(2\lambda)_n}\left[(i)^n(b)_n + (-i)^n(\bar{b})_n\right] \quad \text{and} \quad i2\mathfrak{b}_n = \frac{2^n}{(2\lambda)_n}\left[(i)^n(b)_n - (-i)^n(\bar{b})_n\right],$$

for $n \geq 1$. Hence, all one needs to verify is that if

$$2\tilde{\mathfrak{a}}_n = \left[(i)^n(b)_n + (-i)^n(\bar{b})_n\right] \quad \text{and} \quad i2\tilde{\mathfrak{b}}_n = \left[(i)^n(b)_n - (-i)^n(\bar{b})_n\right],$$

then there hold

$$\tilde{\mathfrak{a}}_{n+1} = -\eta\tilde{\mathfrak{a}}_n - (\lambda+n)\tilde{\mathfrak{b}}_n \quad \text{and} \quad \tilde{\mathfrak{b}}_{n+1} = -\eta\tilde{\mathfrak{b}}_n + (\lambda+n)\tilde{\mathfrak{a}}_n, \quad \text{for} \quad n \geq 0.$$

This is easily verified with the substitutions $\eta = [(b+n)-(\bar{b}+n)]/(2i)$ and $(\lambda+n) = [(b+n)+(\bar{b}+n)]/2$. This completes the proof of the formulas in (4.11).

Now to prove the latter part of the theorem we just multiply the first formula in (4.11) by $\cos(w)$ and the second formula in (4.11) by $\sin(w)$ and add the resulting formulas. ∎

In the case of the Coulomb wave function Theorem 4.3 becomes:

Corollary 4.3.1. *For $L \geq 0$, let the real sequences $\{\mathfrak{a}_n\}_{n\geq 0}$ and $\{\mathfrak{b}_n\}_{n\geq 0}$ be given by $\mathfrak{a}_0 = 1$, $\mathfrak{b}_0 = 0$ and*

$$\begin{bmatrix} \mathfrak{a}_{n+1} \\ \mathfrak{b}_{n+1} \end{bmatrix} = \frac{2}{2L+n+2} \begin{bmatrix} \eta & -(L+n+1) \\ (L+n+1) & \eta \end{bmatrix} \begin{bmatrix} \mathfrak{a}_n \\ \mathfrak{b}_n \end{bmatrix}, \quad n \geq 0.$$

Then

$$\begin{aligned} \cos(w)\, F_L(\eta, w) &= C_L(\eta)\, w^{L+1} \sum_{n=0}^{\infty} \mathfrak{a}_n \frac{w^n}{n!}, \\ \sin(w)\, F_L(\eta, w) &= C_L(\eta)\, w^{L+2} \sum_{n=0}^{\infty} \frac{1}{n+1} \mathfrak{b}_{n+1} \frac{w^n}{n!}, \end{aligned} \tag{4.12}$$

where C_L is as in Corollary 4.2.3. Moreover, if $\{\mathfrak{c}_n(w)\}_{n\geq 0}$ is such that

$$\mathfrak{c}_n(w) = \mathfrak{a}_n \cos(w) + \frac{\mathfrak{b}_{n+1}}{n+1}\, w \sin(w), \quad n \geq 0,$$

then

$$F_L(\eta, w) = C_L(\eta)\, w^{L+1} \sum_{n=0}^{\infty} \mathfrak{c}_n(w) \frac{w^n}{n!}.$$

When $\eta = 0$, the formulas for $\mathfrak{a}_n$ and $\mathfrak{b}_n$ in Theorem 4.3 are much simpler. It is easily verified that

$$\mathfrak{a}_1 = \mathfrak{b}_0 = 0, \quad \mathfrak{b}_1 = \frac{2}{2\lambda} \lambda \mathfrak{a}_0 = 1.$$

Hence, by taking $\mathfrak{a}_n = \mathfrak{a}_n^{(\lambda-1/2,0)}$ and $\mathfrak{b}_n = \mathfrak{b}_n^{(\lambda-1/2,0)}$ we can state:

Corollary 4.3.2. *For $\alpha > -1/2$, let*

$$\mathfrak{a}_{2n} = (-1)^n \frac{2^{2n}(\alpha+1/2)_{2n}}{(2\alpha+1)_{2n}}, \quad \mathfrak{b}_{2n+1} = (-1)^n \frac{2^{2n}(\alpha+3/2)_{2n}}{(2\alpha+2)_{2n}}, \quad n \geq 0.$$

Then

$$\begin{aligned} \cos(w)\, J_\alpha(w) &= \frac{1}{\Gamma(\alpha+1)} \left(\frac{w}{2}\right)^{\alpha} \sum_{n=0}^{\infty} \mathfrak{a}_{2n} \frac{w^{2n}}{(2n)!}, \\ \sin(w)\, J_\alpha(w) &= \frac{w}{\Gamma(\alpha+1)} \left(\frac{w}{2}\right)^{\alpha} \sum_{n=0}^{\infty} \frac{1}{2n+1} \mathfrak{b}_{2n+1} \frac{w^{2n}}{(2n)!}. \end{aligned} \tag{4.13}$$

Moreover, if $\mathfrak{c}_{2n}(w) = \mathfrak{a}_{2n} \cos(w) + \frac{1}{2n+1} \mathfrak{b}_{2n+1}\, w \sin(w)$, $n \geq 0$, then

$$J_\alpha(w) = \frac{1}{\Gamma(\alpha+1)} \left(\frac{w}{2}\right)^{\alpha} \sum_{n=0}^{\infty} \mathfrak{c}_{2n}(w) \frac{w^{2n}}{(2n)!}.$$

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