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BIORTHOGONAL RATIONAL FUNCTIONS OF R_{II}-TYPE

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Abstract. In this work, a sequence of orthonormal rational functions leading to recurrence relations of R_{II}-type is constructed. This sequence is proved to be biorthogonal to another sequence of rational functions as well. Two illustrations of such recurrence relations of R_{II}-type, one through the associated linear pencil matrix leading to the -1 little Jacobi polynomials and the other through the bilinear transformation yielding the Bannai-Ito polynomials, which are orthogonal on the real line are exhibited.

1. Introduction

Recurrence relations of the form

$$\mathcal{P}_{n+1}(z)=\rho_n(z-\nu_n)\mathcal{P}_n(z)+\tau_n(z-a_n)(z-b_n)\mathcal{P}_{n-1}(z),\quad n\geq 1, \tag{1.1}$$

with initial conditions $\mathcal{P}_0(z)=1$ and $\mathcal{P}_1(z)=\rho_0(z-\nu_0)$ are studied extensively [15] to define families of biorthogonal functions having explicit representations in terms of basic hypergeometric functions (see [16, 23] for recent works). Further, it was shown [15] that if

$$\mathcal{P}_n(a_n)\neq 0,\quad \mathcal{P}_n(b_n)\neq 0,\quad \tau_n\neq 0, \tag{1.2}$$

then there exists a rational function $\phi_n(z)=\prod_{k=1}^{n}(z-a_k)^{-1}(z-b_k)^{-1}\mathcal{P}_n(z)$ and a linear functional $\mathfrak{M}$ defined on the span $\{z^k\phi_n(z):0\leq k\leq n\}$ such that the relation $\mathfrak{M}(z^k\phi_n(z))=0$ for $0\leq k<n$ holds. Conversely, one can always obtain (1.1) from a sequence of rational functions $\{\phi_n(z)\}_{n=0}^{\infty}$ having poles at $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ and satisfying a three-term recurrence relation. Following [15] (see also [17]), we call (1.1) a recurrence relation of R_{II}-type.

Related to such recurrence relations are important concepts of rational functions satisfying both orthogonality and biorthogonality properties. The theory of rational functions orthogonal on the unit circle is developed parallel to that of polynomials orthogonal on the unit circle and is available in the monograph [7]. A sequence of orthogonormal rational functions is obtained from the Gram-Schmidt orthonormalization process in the linear space of rational functions which, in fact, can be characterized by the poles of the basis elements as well. In this direction, [6, 21], starting from a set of pre-defined poles, the rational functions are characterized by Favard-type theorems as well as in terms of three-term recurrence relations similar to that of orthogonal polynomials on the real line [8, 14], but with rational

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coefficients. The effect of poles on the asymptotics of the Christoffel functions associated with the orthogonal rational functions and their interval of orthogonality is also studied [10]. For recent generalizations in the theory, see [5, 9, 27] and the references therein.

Following [20], two sequences of functions $\{\mathcal{R}_n(z)\}_{n=0}^{\infty}$ and $\{\mathcal{Q}_n(z)\}_{n=0}^{\infty}$ are said to be biorthogonal if they satisfy

$$\mathfrak{N}(\mathcal{R}_n(z)\mathcal{Q}_m(z)) = \kappa_n\delta_{n,m}, \quad \kappa_n \neq 0, \quad n, m \geq 0, \tag{1.3}$$

with respect to a linear functional $\mathfrak{N}$. We observe that in contrast to the usual orthogonality condition, two different sequences are used for the biorthogonality condition. Further, unlike the case for orthogonal polynomials on the real line [8], the polynomial $\mathcal{P}_n(z)$ satisfying (1.1) is the characteristic polynomial of a matrix pencil $\mathcal{G}_n - z\mathcal{H}_n$, where both $\mathcal{G}_n$ and $\mathcal{H}_n$ are tridiagonal matrices [24, 25, 30].

1.1. **Motivation for the problem.** The components of the eigenvectors of the matrix pencil $\mathcal{G}_n - z\mathcal{H}_n$ are rational functions with the numerator polynomials $\mathcal{P}_n(z)$ satisfying (1.1). However, these rational functions are not the ones that were used initially to obtain the matrix pencil. In fact, while the three-term recurrence relation satisfied by $\phi_n(z)$ is used to obtain the matrix pencil, the usual process [2] is to partition the poles to form two new sequences of rational functions

$$p_n^L(z) = \frac{\mathcal{P}_n(z)}{\prod_{k=1}^{n}(z-a_k)}, \quad p_n^R(z) = \frac{\mathcal{P}_n(z)}{\prod_{k=1}^{n}(z-b_k)}, \tag{1.4}$$

which form the components of the left and right eigenvectors of the matrix pencil $\mathcal{G} - z\mathcal{H}$. Even though the right and left eigenvectors of a non-Hermitian matrix are not orthonormal in general, they can be biorthogonalized. In this way, the two sequences $\{p_n^L(z)\}_{n=0}^{\infty}$ and $\{p_n^R(z)\}_{n=0}^{\infty}$ are then used to define two new sequences of rational functions [2, 13] satisfying the biorthogonality relation (1.3). However, we note that two sequences of rational functions that are biorthogonal to each other need not themselves form an orthogonal sequence.

On similar lines, we discuss the other eventuality of considering the generalized eigenvalue problem of the orthogonal rational functions $\{\varphi_n(z)\}_{n=0}^{\infty}$ and also obtaining the biorthogonality of $\{\varphi_n(z)\}_{n=0}^{\infty}$ to another sequence of rational functions, which is motivated by the procedure of proving biorthogonality [30]. Hence the theme of the manuscript is to construct a sequence of orthogonal rational functions $\{\varphi_n(z)\}_{n=0}^{\infty}$ that is also biorthogonal to another sequence of rational functions. It is observed that $\{\varphi_n(z)\}_{n=0}^{\infty}$ leads to recurrence relations of R_{II}-type such that the related matrix pencil has the numerator polynomials $\mathcal{P}_n(z)$ of $\varphi_n(z)$, $n \geq 0$, as the characteristic polynomials and $\varphi_n(z)$, $n \geq 0$, as components of the eigenvectors.

We note that such a system of an orthogonal sequence that is also biorthogonal to another sequence exists in the space of polynomials. For instance, the two polynomials $\mathcal{R}_n(z;\alpha,\beta) = {}_2F_1(-n, \alpha+\beta+1; 2\alpha+1; 1-z)$, $\mathcal{Q}_n(z) = \mathcal{R}_n(z;\alpha,-\beta)$, $n \geq 1$, were proved to be biorthogonal [1] with respect to the weight function $\omega(\theta) = (2-2\cos\theta)^{\alpha}(-e^{i\theta})^{\beta}$, $\theta \in [-\pi,\pi]$, $\operatorname{Re}\alpha > -1/2$. The sequence $\{\mathcal{R}_n(z;\alpha,\beta)\}_{n=0}^{\infty}$ was later proved to be orthogonal with respect to the weight $\hat{\omega}(\theta) = 2^{2\alpha}e^{(\pi-\theta)\operatorname{Im}\beta}\sin^{2\alpha}\theta/2$ if $\alpha \in \mathbb{R}$, $\alpha > -1/2$ and $i\beta \in \mathbb{R}$ [22]. The present problem serves to find an abstract rational analogue of such cases of orthogonal sequences satisfying biorthogonality properties as well, while staying in the same space defined by the sequence of poles.

One of the ways to explore the point of staying in the same space is that there exist sequences of orthogonal polynomials that can be related, in a one-to-one correspondence, with the underlying rational functions $\varphi_n(z)$ that yield the recurrence relations of R_{II}-type. Clearly, showing $\varphi_n(z)$ is also biorthogonal can prove to be an added advantage in many cases. However, this aspect is not discussed in the present work and can certainly be further explored with specific examples of biorthogonal rational functions.

The paper is organized as follows. Section 2 introduces the fundamental spaces and the orthogonal rational functions that lead to recurrence relations of R_{II}-type. In Section 3, the reverse procedure, that is, starting with R_{II} recurrences, and recovering the same orthogonal rational functions via biorthogonality relations is provided. In Section 4, a special form of the recurrence relations of R_{II}-type is considered and it is further shown to be related to two different sequences on orthogonal polynomials on the real line, one through the associated linear pencil matrix and another through a bilinear transformation of the variable.

2. Fundamental spaces and associated rational functions

Let $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=0}^{\infty}$ be two given sequences where $\beta_0 := 0$,

$$\alpha_j, \beta_j \in \mathbb{C}\setminus\{0\}, \quad \alpha_j \neq \infty, \quad j \geq 1. \tag{2.1}$$

Following the notation in [7], we define $u_{2j}(z) := \frac{1}{1-z\bar{\beta}_j}$, $u_{2j+1}(z) := \frac{1}{z-\alpha_{j+1}}$, $j \geq 0$. The basis $\{u_j\}_{j=0}^{n}$, $n \geq 1$, generates the linear spaces $\mathcal{L}_n$=span$\{u_0, u_1, \cdots, u_n\}$ and $\mathcal{L} = \bigcup_{n=0}^{\infty}\mathcal{L}_n$. Equivalently, we also have $\mathcal{L}_n$= span$\{\mathfrak{u}_0, \mathfrak{u}_1, \cdots, \mathfrak{u}_n\}$, where

$$\mathfrak{u}_{2j}(z) = \frac{z^{2j}}{\prod_{k=1}^{j}(z-\alpha_k)\prod_{k=1}^{j}(1-z\bar{\beta}_k)}, \quad \mathfrak{u}_{2j+1}(z) = \frac{z}{z-\alpha_{j+1}}\mathfrak{u}_{2j}(z), \quad j \geq 0.$$

Further, the product spaces $\mathcal{L}_m \cdot \mathcal{L}_n$ and $\mathcal{L}\cdot\mathcal{L}$ consist of functions of the form $h_{m,n}(z) = f_m(z)g_n(z)$ and $h(z) = f(z)g(z)$, respectively, where $f_m(z) \in \mathcal{L}_m$, $g_n(z) \in \mathcal{L}_n$, and $f(z), g(z) \in \mathcal{L}$.

The substar transform $h_*(z)$ of a function $h(z)$ is defined as $h_*(z) = \overline{h(1/\bar{z})}$. Let $\mathfrak{L}$ be a linear functional defined on $\mathcal{L}\cdot\mathcal{L}$ such that

$$\langle f(z), g(z)\rangle := \mathfrak{L}(f(z)g_*(z)) \tag{2.2}$$

is Hermitian and positive-definite, and hence defines an inner product on the space $\mathcal{L}$. We note that $\mathfrak{L}$ is said to be Hermitian if it satisfies $\mathfrak{L}(h_*) = \overline{\mathfrak{L}(\bar{h})}$ for every $h \in \mathcal{L}\cdot\mathcal{L}$ and positive-definite if $\mathfrak{L}(hh_*) > 0$ for every $h \neq 0 \in \mathcal{L}$. Let $\varphi_j(z)$, $j \geq 0$, be the sequence of functions that are orthonormal with respect to $\mathfrak{L}$ and obtained from the Gram-Schmidt process of the basis $\{\mathfrak{u}_j\}_{j=0}^{n}$, $n \geq 1$. That is, $\varphi_j(z)$, $j \geq 0$, satisfy the orthogonality property $\langle\varphi_m(z), \varphi_n(z)\rangle = \mathfrak{L}(\varphi_m(z)\varphi_{n*}(z)) = \delta_{m,n}$, $m, n = 0, 1, \cdots$. Further, it is clear that $\varphi_n(z)$ are rational functions of the form

$$\begin{aligned} \varphi_0(z) = 1, \quad \varphi_{2j+2}(z) &= \frac{r_{2j+2}(z)}{\prod_{k=1}^{j+1}(z-\alpha_k)\prod_{k=1}^{j+1}(1-z\bar{\beta}_k)}, \quad j \geq 0, \\ \varphi_{2j+1}(z) &= \frac{r_{2j+1}(z)}{\prod_{k=1}^{j+1}(z-\alpha_k)\prod_{k=1}^{j}(1-z\bar{\beta}_k)}, \quad j \geq 0, \end{aligned} \tag{2.3}$$

where $r_n(z) \in \Pi_n$, the linear space of polynomials of degree at most n. Moreover, $\mathcal{L}_{2n}$ can now be interpreted as the space of rational functions having poles belonging to the set $\{\alpha_1, 1/\bar{\beta}_1, \cdots, \alpha_n, 1/\bar{\beta}_n\}$ with the order of the pole at α_j or $1/\bar{\beta}_j$ depending on its multiplicity. A similar interpretation for $\mathcal{L}_{2n+1}$ follows.

The regularity conditions in the present case can be obtained as follows. The expansion in terms of the basis elements gives

$$\varphi_{2n}(z) = A_0 + \frac{A_1 z}{z-\alpha_1} + \frac{A_2 z^2}{(z-\alpha_1)(1-z\bar{\beta}_1)} + \cdots + \frac{A_{2n} z^{2n}}{\prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n}(1-z\bar{\beta}_i)},$$

so that $r_{2n}(z) = A_0 \prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n}(1-z\bar{\beta}_i) + \cdots + A_{2n}$. Then $A_{2n} \neq 0$ if

$$r_{2n}(\alpha_n) \neq 0 \quad \text{and} \quad r_{2n}(1/\bar{\beta}_n) \neq 0. \tag{2.4}$$

Similarly, for $\varphi_{2n+1}(z)$, we obtain

$$r_{2n+1}(\alpha_{n+1}) \neq 0 \quad \text{and} \quad r_{2n+1}(1/\bar{\beta}_n) \neq 0. \tag{2.5}$$

The regularity conditions (2.4) and (2.5) are required to guarantee that $\varphi_{2n}(z) \in \mathcal{L}_{2n} \setminus \mathcal{L}_{2n-1}$ and $\varphi_{2n+1}(z) \in \mathcal{L}_{2n+1} \setminus \mathcal{L}_{2n}$, respectively. Using the definition (2.2) of the inner product $\langle \cdot, \cdot \rangle$, one can also derive the following results in $\mathcal{L}$.

Lemma 2.1. *Let $\gamma_n \in \mathbb{C} \setminus \{0\}$, $n = 1, 2, \cdots$. The following equality:*

$$\left\langle \frac{1-z\bar{\gamma}_n}{z-\gamma_{n-1}} f, g \right\rangle = \left\langle f, \frac{z-\gamma_n}{1-z\gamma_{n-1}} g \right\rangle, \qquad \left\langle \frac{z-\gamma_{n+1}}{1-z\bar{\gamma}_n} f, g \right\rangle = \left\langle f, \frac{1-z\bar{\gamma}_{n+1}}{z-\gamma_n} g \right\rangle,$$

holds for the rational functions $f := f(z)$ and $g := g(z)$ in $\mathcal{L}$.

We now use Lemma 2.1 to derive recurrence relations for the orthogonal rational functions $\varphi_j(z)$. This derivation is motivated by proof of [6, Theorem 4.1] and in addition to the conditions (2.4) and (2.5). We also assume $r_{2n}(\beta_{n-1}) \neq 0$, $r_{2n}(1/\bar{\alpha}_n) \neq 0$, $r_{2n+1}(\beta_n) \neq 0$, $r_{2n+1}(1/\bar{\alpha}_n) \neq 0$. Here, and in what follows, we consider the sequences $\{\alpha_j\}$ and $\{\beta_j\}$ as defined in (2.1), unless specified otherwise.

Theorem 2.2. *The orthonormal rational functions $\{\boldsymbol{\phi}_n(\lambda)\}_{n=0}^{\infty}$ with $\boldsymbol{\phi}_{-1}(\lambda) := 0$ and $\boldsymbol{\phi}_0(\lambda) := 1$ satisfy the recurrence relations,*

(2.6a)

$$\varphi_{2n+1}(z) = \left[\frac{e_{2n+1}}{z-\alpha_{n+1}} + \frac{d_{2n+1}(z-\beta_n)}{z-\alpha_{n+1}}\right]\varphi_{2n}(z) + c_{2n+1}\frac{1-z\bar{\alpha}_n}{z-\alpha_{n+1}}\varphi_{2n-1}(z),$$

(2.6b)

$$\varphi_{2n+2}(z) = \left[\frac{e_{2n+2}}{1-z\bar{\beta}_{n+1}} + \frac{d_{2n+2}(1-z\bar{\alpha}_{n+1})}{1-z\bar{\beta}_{n+1}}\right]\varphi_{2n+1}(z) + c_{2n+2}\frac{z-\beta_n}{1-z\bar{\beta}_{n+1}}\varphi_{2n}(z)$$

for $n \geq 0$, where the constants $e_j, d_j \in \mathbb{C}$ and $c_j \in \mathbb{C} \setminus \{0\}$, $j \geq 0$.

Proof. Consider the function

$$\mathcal{W}_{2n}(z) = \frac{1-z\bar{\beta}_n}{z-\beta_{n-1}}\varphi_{2n}(z) - \frac{a_{2n}}{z-\beta_{n-1}}\varphi_{2n-1}(z), \quad n \geq 1.$$

Using the rational forms (2.3) we note that

$$a_{2n} = \frac{r_{2n}(\beta_{n-1})}{r_{2n-1}(\beta_{n-1})} \neq 0 \Longrightarrow \mathcal{W}_{2n}(z) \in \mathcal{L}_{2n-1} \setminus \mathcal{L}_{2n-2}.$$

Hence, we can write $\mathcal{W}_{2n}(z) = b_{2n}\varphi_{2n-1}(z) + c_{2n}\varphi_{2n-2}(z) + \sum_{j=0}^{2n-3} \mathfrak{a}_j^{(2n)}\varphi_j(z)$, where $\mathfrak{a}_j^{(2n)} = \langle \mathcal{W}_{2n}(z), \varphi_j(z) \rangle$, $j = 0, 1, \cdots, 2n-3$. However, it can be shown that

$\frac{z-\beta_n}{1-z\bar\beta_{n-1}}\varphi_j \in \mathcal{L}_{2n-2}$ and $\frac{z}{1-z\bar\beta_{n-1}}\varphi_j \in \mathcal{L}_{2n-2}$, $j = 0, 1, \cdots, 2n-3$. Using Lemma 2.1, we conclude $\mathfrak{a}_j^{(2n)} = 0$ for $j = 0, 1, \cdots, 2n-3$ and hence

$$\varphi_{2n}(z) = \left[\frac{a_{2n}}{1-z\bar\beta_n} + b_{2n}\frac{z-\beta_{n-1}}{1-z\bar\beta_n}\right]\varphi_{2n-1}(z) + c_{2n}\frac{z-\beta_{n-1}}{1-z\bar\beta_n}\varphi_{2n-2}(z), \quad n \geq 1.$$

However, we note that both $\{1, z-\beta_{n-1}\}$ and $\{1, 1-z\bar\alpha_n\}$ form a basis for Π_1 and hence writing $a_{2n} + b_{2n}(z-\beta_{n-1}) = e_{2n} + d_{2n}(1-z\bar\alpha_n)$, the recurrence relation (2.6b) follows. To prove $c_{2n} \neq 0$, we multiply both sides of (2.6b) by

$$\frac{1-z\bar\beta_n}{\prod_{i=1}^{n}(1-z\bar\alpha_i)\prod_{i=1}^{n-1}(z-\beta_i)},$$

so that the definition of the inner product (2.2) gives

$$c_{2n}\left\langle \varphi_{2n-2}(z), \frac{z^{2n-2}}{\prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n-2}(1-z\bar\beta_i)}\right\rangle + e_{2n}\left\langle \varphi_{2n-1}(z), \mathfrak{u}_{2n-1}(z)\right\rangle = 0,$$

which proves $c_{2n} \neq 0$, $n \geq 1$.

To derive the recurrence relation for $\varphi_{2n+1}(z)$, consider

$$\mathcal{W}_{2n+1}(z) = \frac{z-\alpha_{n+1}}{1-z\bar\alpha_n}\varphi_{2n+1}(z) - \frac{a_{2n+1}}{1-z\bar\alpha_n}\varphi_{2n}(z), \quad n \geq 0$$

for $a_{2n+1} = r_{2n+1}(1/\bar\alpha_n)/r_{2n}(1/\bar\alpha_n) \neq 0$. As in the case for $\varphi_{2n}(z)$, we arrive at

$$\varphi_{2n+1}(z) = \left[\frac{a_{2n+1}}{z-\alpha_{n+1}} + b_{2n+1}\frac{1-z\bar\alpha_n}{z-\alpha_{n+1}}\right]\varphi_{2n}(z) + c_{2n+1}\frac{1-z\bar\alpha_n}{z-\alpha_{n+1}}\varphi_{2n-1}(z)$$

for $n \geq 0$ which can also be written as (2.6a) since $\{1, 1-z\bar\alpha_n\}$ and $\{1, z-\beta_n\}$ both span the linear space Π_1.

To prove $c_{2n+1} \neq 0$, we multiply (2.6a) by

$$\frac{(z-\alpha_{n+1})}{\prod_{i=1}^{n}(1-z\bar\alpha_i)\prod_{i=1}^{n}(z-\beta_i)}.$$

The inner product (2.2) and Lemma 2.1 gives

$$c_{2n+1}\left\langle \varphi_{2n-1}(z), \frac{z^{2n-1}}{\prod_{i=1}^{n-1}(z-\alpha_i)\prod_{i=1}^{n}(1-z\bar\beta_i)}\right\rangle + e_{2n+1}\langle \varphi_{2n}(z), \mathfrak{u}_{2n}(z)\rangle = 0,$$

from which it follows that $c_{2n+1} \neq 0$, $n \geq 1$. □

2.1. $\varphi_j(z)$, $j \geq 0$, **as components of an eigenvector.** The numerator polynomials of orthogonal rational functions satisfy the recurrence relations of R_{II}-type. Indeed, from (2.6a) and (2.6b), it can be shown that

(2.7a) $$r_{2n+1}(z) = \gamma_{2n+1}(z)r_{2n}(z) + c_{2n+1}(1-z\bar\alpha_n)(1-z\bar\beta_n)r_{2n-1}(z),$$

(2.7b) $$r_{2n+2}(z) = \gamma_{2n+2}(z)r_{2n+1}(z) + c_{2n+2}(z-\alpha_{n+1})(z-\beta_n)r_{2n}(z)$$

for $n \geq 0$ where we define $r_0(z) := 1$, $r_1(z) = \gamma_1(z)$,

$$\gamma_{2n+1}(z) := [e_{2n+1} + d_{2n+1}(z-\beta_n)] \quad \text{and}$$
$$\gamma_{2n+2}(z) := [e_{2n+2} + d_{2n+2}(1-z\bar\alpha_{n+1})].$$

We use (2.7a) and (2.7b) to obtain a generalized eigenvalue problem such that the zeros of $r_j(z)$, $j \geq 1$, are the eigenvalues (that is, $r_j(z)$ is the characteristic polynomial) while the corresponding rational functions are the components of the corresponding eigenvector.

Consider the linear pencil matrix $\mathcal{G} - z\mathcal{H}$ given by

$$\begin{pmatrix} -\gamma_1(z) & \frac{c_2}{h_{1,0}}(z-\alpha_1) & 0 & 0 & \cdots \\ -h_{1,0}(z-\beta_0) & -\gamma_2(z) & \frac{c_3\bar{\alpha}_1\bar{\beta}_1}{h_{21}}(z-1/\bar{\beta}_1) & 0 & \cdots \\ 0 & -h_{2,1}(z-1/\bar{\alpha}_1) & -\gamma_3(z) & \frac{c_4}{h_{32}}(z-\alpha_2) & \cdots \\ 0 & 0 & -h_{3,2}(z-\beta_1) & -\gamma_4(\lambda) & \cdots \\ 0 & 0 & 0 & -h_{4,3}(z-1/\bar{\alpha}_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where α_j, β_j, e_j, d_j and c_j are the constants appearing in the recurrence relations (2.7a) and (2.7b) while $\{h_{i,i-1}\}_{i=1}^{\infty}$ is a sequence of arbitrary non-vanishing complex numbers.

Proposition 2.3 ([30]). *Let $\mathcal{H}_j$ and $\mathcal{G}_j$ denote the jth principal minors of $\mathcal{H}$ and $\mathcal{G}$, respectively. Then the zeros of $(-1)^j r_j(z)$, $j \geq 1$, are the eigenvalues of the generalized eigenvalue problem*

$$\mathcal{G}_j \boldsymbol{\varrho}_j = z\mathcal{H}_j \boldsymbol{\varrho}_j, \tag{2.8}$$

where $\{r_j\}$ satisfies (2.7a) *and* (2.7b).

The generalized eigenvalue problem (2.8) has $j-1$ free variables $h_{i,i-1}$ which shows that the matrix pencil associated with the recurrence relations of R_{II}-type is not unique. We now assign appropriate values to these free variables to obtain an eigenvector $\boldsymbol{\varrho}_j$.

Proposition 2.4. *Let the terms of the sequence $\{h_{i,i-1}\}_{i=1}^{\infty}$ be assigned the values*

$$h_{2i,2i-1} = -c_{2i+1}\bar{\alpha}_i, \quad h_{2i-1,2i-2} = c_{2i}, \quad i \geq 1.$$

Then, $\boldsymbol{\varrho}_j = \begin{pmatrix} \varphi_0 & \varphi_1 & \cdots & \varphi_j \end{pmatrix}^T$ is the eigenvector of the generalized eigenvalue problem (2.8) *corresponding to the eigenvalue which is a zero of $r_j(\lambda)$.*

Remark 2.5. The sequence of poles $\{\alpha_1, 1/\bar{\alpha}_2, \alpha_3, 1/\bar{\alpha}_4, \cdots\}$ has also been considered [27, Section 5], in which the eigenvalues are obtained as matrix transformations of five-diagonal matrices. A different distribution of poles $1/\bar{\alpha_1}, \alpha_2, 1/\bar{\alpha}_3, \alpha_4, \cdots$ is also studied [4] to derive rational quadrature formulas. These poles are such that they lie alternatingly inside the unit disk and its complement. In the present case, however, we place no such restrictions on the poles α_j and β_j, $j \geq 1$.

Theorem 2.2 and Proposition 2.4 serve the first step of our construction. That is, we have obtained a sequence of rational functions that is orthogonal with respect to the linear functional $\mathfrak{L}$. These rational functions are also the components of the eigenvector of a matrix pencil whose characteristic polynomials are the numerator polynomials of such rational functions.

3. A biorthogonality relation for the rational functions

In the present section, we use the recurrence relations (2.7a) and (2.7b) obtained in Section 2 to define biorthogonality relations involving the orthogonal rational functions $\{\varphi_j\}$. To start with, we introduce the rational functions $\mathcal{O}_0(z) = 1$ and

$$\begin{aligned} \mathcal{O}_{2n+1}(z) &= \frac{r_{2n+1}(z)}{\prod_{j=1}^{n+1}(z-\alpha_j)\prod_{j=1}^{n}(1-z\bar{\alpha}_j)\prod_{j=0}^{n}(z-\beta_j)\prod_{j=1}^{n}(1-z\bar{\beta}_j)}, \\ \mathcal{O}_{2n+2}(z) &= \frac{r_{2n+2}(z)}{\prod_{j=1}^{n+1}(z-\alpha_j)\prod_{j=1}^{n+1}(1-z\bar{\alpha}_j)\prod_{j=0}^{n}(z-\beta_j)\prod_{j=1}^{n+1}(1-z\bar{\beta}_j)} \end{aligned} \tag{3.1}$$

for $n \geq 0$. Here r_j, $j \geq 0$, satisfies (2.7a) and (2.7b) so that the sequence $\{\mathcal{O}_j(z)\}$ satisfies

$$(z-\alpha_{n+1})(z-\beta_n)\mathcal{O}_{2n+1}(z) = [e_{2n+1} + d_{2n+1}(z-\beta_n]\mathcal{O}_{2n}(z) + c_{2n+1}\mathcal{O}_{2n-1}(z),$$
$$(1-z\bar{\alpha}_n)(1-z\bar{\beta}_n)\mathcal{O}_{2n}(z) = [e_{2n} + d_{2n}(1-z\bar{\alpha}_n)]\mathcal{O}_{2n-1}(z) + c_{2n}\mathcal{O}_{2n-2}(z)$$

for $n \geq 1$. Then, similar to Theorem 3.5 and its following corollary of Ismail and Masson in [15], we have the following.

Theorem 3.1. *Consider the rational functions given by* (3.1). *Then there exists a linear functional* $\mathfrak{N}$ *on the span of rational functions* $\{z\mathcal{O}_n(z)\}$ *such that the orthogonality relation*

$$\mathfrak{N}(z^k\mathcal{O}_n(z)) = 0, \quad k = 0, 1, \cdots, n-1,$$

holds. Further, if $\mathfrak{N}(1) = m_0$, $\mathfrak{N}(z^n\mathcal{O}_n(z)) = m_n$, $n \geq 1$, *then*

$$\begin{aligned} \bar{\alpha}_n\bar{\beta}_n m_{2n} + d_{2n}\bar{\alpha}_n m_{2n-1} - c_{2n}m_{2n-2} &= 0, \quad n \geq 1 \\ m_{2n+1} - d_{2n+1}m_{2n} - c_{2n+1}m_{2n-1} &= 0, \quad n \geq 1. \end{aligned} \tag{3.2}$$

We also need the following relations among the leading coefficients $r_j(z)$, $j \geq 1$. If $r_j = \kappa_j z^j +$ lower order terms, then from (2.7a) and (2.7b),

$$\begin{aligned} \kappa_{2n} + d_{2n}\bar{\alpha}_n\kappa_{2n-1} - c_{2n}\kappa_{2n-2} &= 0, \quad n \geq 1, \\ \kappa_{2n+1} - d_{2n+1}\kappa_{2n} - \bar{\alpha}_n\bar{\beta}_n c_{2n+1}\kappa_{2n-1} &= 0, \quad n \geq 1. \end{aligned} \tag{3.3}$$

It is clear that each of the recurrence relations (3.2) and (3.3) involve two arbitrary initial values. We choose m_0 and m_1 such that $m_1 \neq d_1 m_0$. Since $\kappa_0 = 1$ and $\kappa_1 = d_1$, this implies $\kappa_0 m_1 - \kappa_1 m_0 \neq 0$.

Consider another sequence of rational functions $\{\tilde{\varphi}_j(z)\}_{j=0}^{\infty}$ where $\tilde{\varphi}_0(z) := 1$,

$$\begin{aligned} \tilde{\varphi}_{2n+1}(z) &= \frac{r_{2n+1}(z)}{\prod_{j=1}^{n}(1-z\bar{\alpha}_j)\prod_{j=0}^{n}(z-\beta_j)} \quad \text{and} \\ \tilde{\varphi}_{2n+2}(z) &= \frac{r_{2n+2}(z)}{\prod_{j=1}^{n+1}(1-z\bar{\alpha}_j)\prod_{j=0}^{n}(z-\beta_j)} \end{aligned} \tag{3.4}$$

for $n \geq 0$. Here $\{r_j(z)\}$ satisfies (2.7a) and (2.7b). Let $\tilde{\mathcal{J}}_m(z) = \chi_m^{-1}\tilde{\varphi}_m(z)$, where

$$\chi_{2m} = \bar{\alpha}_1(\bar{\beta}_1)^{-1}\cdots\bar{\alpha}_m(\bar{\beta}_m)^{-1} \quad \text{and} \quad \chi_{2m+1} = \bar{\alpha}_1(\bar{\beta}_1)^{-1}\cdots\bar{\alpha}_m(\bar{\beta}_m)^{-1}\bar{\alpha}_{m+1}.$$

Define

$$\tilde{\psi}_{2j}(z) := \frac{c_{2j+1}(\bar{\beta}_j)^2}{\bar{\alpha}_{j+1}}\tilde{\mathcal{J}}_{2j-1}(z) - \frac{d_{2j+1}}{\bar{\alpha}_{j+1}}\tilde{\mathcal{J}}_{2j}(z) + \tilde{\mathcal{J}}_{2n+1}(z), \quad n \geq 1,$$

$$\tilde{\psi}_{2j+1}(z) := \frac{c_{2j+2}\bar{\beta}_{j+1}}{\bar{\alpha}_{j+1}}\tilde{\mathcal{J}}_{2j}(z) - d_{2j+2}\bar{\alpha}_{j+1}\bar{\beta}_{j+1}\tilde{\mathcal{J}}_{2j+1}(z) + \bar{\alpha}_{j+1}\tilde{\mathcal{J}}_{2j+2}(z), \quad n \geq 0,$$

with $\tilde{\psi}_0(z) := 1$. The following theorem gives the biorthogonality relations for $\varphi(z)$ constructed in the previous section.

Theorem 3.2. *The sequences of rational functions* $\{\varphi_j(z)\}$ *and* $\{\tilde{\psi}_j(z)\}$ *satisfy the following biorthogonality relations:*

$$\mathfrak{N}(\varphi_{2n}(z)\cdot\tilde{\psi}_m(z)) = \frac{c_2c_3\cdots c_{2n+1}(m_1\kappa_0 - m_0\kappa_1)}{\chi_{2n+1}}\delta_{2n,m}, \tag{3.5a}$$

$$\mathfrak{N}(\varphi_{2n+1}(z)\cdot\tilde{\psi}_m(z)) = \frac{c_2c_3\cdots c_{2n+2}(m_1\kappa_0 - m_0\kappa_1)}{\chi_{2n+2}}\delta_{2n+1,m}, \tag{3.5b}$$

where $m_j = \mathfrak{N}(z^jO_j(z))$ *and* κ_j *is the leading coefficient of* $r_j(z)$.

Proof. For simplicity, we write $\varphi_j := \varphi_j(z)$ and similar notation will follow for others. We divide the proof into the following cases. First, let $m < 2n$ and m has even value, say $m = 2j$. Then

$$\mathfrak{N}(\varphi_{2n}\cdot\tilde{\psi}_m) = \frac{c_{2j+1}\bar{\beta}_j}{\bar{\alpha}_{j+1}}\mathfrak{N}(\varphi_{2n}\cdot\tilde{J}_{2j-1}) - \frac{d_{2j+1}}{\bar{\alpha}_{j+1}}\mathfrak{N}(\varphi_{2n}\cdot\tilde{J}_{2j}) + \mathfrak{N}(\varphi_{2n}\cdot\tilde{J}_{2j+1}).$$

We evaluate the first term. We have

$$\begin{aligned}
&\mathfrak{N}(\varphi_{2n}\cdot\tilde{J}_{2j-1})\\
&\quad= \frac{1}{\chi_{2j-1}}\mathfrak{N}\left(\frac{r_{2n}}{\prod_{k=1}^{n}(z-\alpha_k)\prod_{k=1}^{n}(1-z\bar{\beta}_k)}\cdot\frac{r_{2j-1}}{\prod_{k=1}^{j-1}(1-z\bar{\alpha}_k)\prod_{k=0}^{j-1}(z-\beta_k)}\right)\\
&\quad= \frac{1}{\chi_{2j-1}}\mathfrak{N}(\mathcal{O}_{2n}\cdot r_{2j-1}(1-z\bar{\alpha}_j)\cdots(1-z\bar{\alpha}_n)(z-\beta_j)\cdots(z-\beta_{n-1}))\\
&\quad= \frac{(-\bar{\alpha}_j)\cdots(-\bar{\alpha}_n)\kappa_{2j-1}}{\chi_{2j-1}}m_{2n}.
\end{aligned}$$

A similar evaluation of the remaining two terms yields

$$\begin{aligned}
\mathfrak{N}(\varphi_{2n}\cdot\tilde{J}_{2j}) &= \frac{(-\bar{\alpha}_{j+1})\cdots(-\bar{\alpha}_n)\kappa_{2j}}{\chi_{2j}}m_{2n},\\
\mathfrak{N}(\varphi_{2n}\cdot\tilde{J}_{2j+1}) &= \frac{(-\bar{\alpha}_{j+1})\cdots(-\bar{\alpha}_n)\kappa_{2j+1}}{\chi_{2j+1}}m_{2n}.
\end{aligned}$$

Using the relations (3.3), we obtain $\mathfrak{N}(\varphi_{2n}(z)\cdot\tilde{\psi}_m(z)) = 0$ for $m = 2j < 2n$.

In the second case, let $m > 2n$ and m has odd value, say $m = 2j+1$. Then

$$\begin{aligned}
&\mathfrak{N}(\varphi_{2n}\cdot\tilde{\psi}_m)\\
&= \frac{c_{2j+2}\bar{\beta}_{j+1}}{\bar{\alpha}_{j+1}}\mathfrak{N}(\varphi_{2n}\cdot\tilde{\mathcal{J}}_{2j}) - d_{2j+2}\bar{\alpha}_{j+1}\bar{\beta}_{j+1}\mathfrak{N}(\varphi_{2n}\cdot\tilde{\mathcal{J}}_{2j+1}) + \bar{\alpha}_{j+1}\mathfrak{N}(\varphi_{2n}\cdot\tilde{\mathcal{J}}_{2j+2}),
\end{aligned}$$

so that, as in the case of $\tilde{\psi}_{2j}(z)$, we have

$$\begin{aligned}
\mathfrak{N}(\varphi_{2n}\cdot\tilde{\mathcal{J}}_{2j+2}) &= \frac{\kappa_{2n}m_{2j+2}}{\chi_{2j+2}}, \quad \mathfrak{N}(\varphi_{2n}(z)\cdot\tilde{\mathcal{J}}_{2j}(z)) = \frac{\kappa_{2n}m_{2j}}{\chi_{2j}},\\
\mathfrak{N}(\varphi_{2n}(z)\cdot\tilde{\mathcal{J}}_{2j+1}(z)) &= \frac{\kappa_{2n}m_{2j+1}}{\chi_{2j+1}}.
\end{aligned}$$

Hence, using (3.2) we have $\mathfrak{N}(\varphi_{2n}(z)\cdot\tilde{\psi}_m(z)) = 0$ for $m = 2j+1 > 2n$.

In the third case, we prove the biorthogonality relations (3.5a) and (3.5b). For $m = 2n$, we obtain

$$\mathfrak{N}(\varphi_{2n}(z)\cdot\tilde{\psi}_{2n}(z)) = \frac{1}{\chi_{2n+1}}(\kappa_{2n}m_{2n+1} - d_{2n+1}\kappa_{2n}m_{2n} - c_{2n+1}\bar{\beta}_n\bar{\alpha}_n\kappa_{2n-1}m_{2n}).$$

From (3.2), we find that $m_{2n+1}\kappa_{2n} - d_{2n+1}\kappa_{2n}m_{2n} = c_{2n+1}m_{2n-1}\kappa_{2n}$, so that

$$\mathfrak{M}(\varphi_{2n}(z)\cdot\tilde{\psi}_{2n}(z)) = \frac{c_{2n+1}}{\chi_{2n+1}}(\kappa_{2n}m_{2n-1} - \bar{\alpha}_n\bar{\beta}_n\kappa_{2n-1}m_{2n}).$$

To simplify the numerator in the right hand side above, we note from (3.2) and (3.3) that the following relations:

(3.6)
$$\begin{aligned}\kappa_{2n}m_{2n-1} - \bar{\alpha}_n\bar{\beta}_n\kappa_{2n-1}m_{2n} &= c_{2n}(m_{2n-1}\kappa_{2n-2} - m_{2n-2}\kappa_{2n-1}),\\ \kappa_{2n-2}m_{2n-1} - \kappa_{2n-1}m_{2n-2} &= c_{2n-1}(m_{2n-3}\kappa_{2n-2} - \bar{\alpha}_{n-1}\bar{\beta}_{n-1}m_{2n-2}\kappa_{2n-3}),\end{aligned}$$

hold which further imply that

$$\kappa_{2n}m_{2n-1} - \bar{\alpha}_n\bar{\beta}_n\kappa_{2n-1}m_{2n} = c_{2n}c_{2n-1}\cdots c_2(m_1\kappa_0 - m_0\kappa_1) \neq 0.$$

The proof of (3.5b) follows the exact techniques and the line of argument as in the proof of (3.5a). Indeed, proceeding as above we obtain, for $m = 2n+1$,

$$\mathfrak{N}(\varphi_{2n+1}(z)\cdot\tilde{\psi}_{2n+1}(z)) = \frac{c_{2n+2}(\kappa_{2n}m_{2n+2} - \kappa_{2n+1}m_{2n})}{\chi_{2n+2}}.$$

Simplifying the numerator in the right hand side above, we note from (3.6) that

$$m_{2n+1}\kappa_{2n} - \kappa_{2n+1}m_{2n} = c_{2n+1}c_{2n}\cdots c_2(\kappa_0 m_1 - m_0\kappa_1) \neq 0.$$

The proof of the biorthogonality relations (3.5a) and (3.5b) for the remaining cases, that is, $m > 2n$, $m = 2j$ and $m < 2n$, $m = 2j+1$, can be obtained with similar arguments, thus completing the proof. □

Remark 3.3. The technique of using the leading coefficients κ_n and the normalization constants m_n to prove biorthogonality, as is evident in the present section, is available in the literature, for example, in [30]. However, one of the primary objectives of the present section is to prove biorthogonality between functions that belong to space of rational functions defined by the poles $\{\alpha_1, 1/\bar{\beta}_1, \alpha_2, 1/\bar{\beta}_2, \cdots\}$. This is a comparison to the results available in [2, 13, 30], that would have led to prove biorthogonality between rational functions belonging to two separate subspaces defined by the poles $\{\alpha_1, \alpha_2, \cdots\}$ and $\{1/\bar{\beta}_1, 1/\bar{\beta}_2, \cdots\}$, respectively.

4. Recovering two orthogonal systems

In this section, we recover two sequences of orthogonal polynomials from a special form of the R_{II} recurrence relations (2.7a) and (2.7b). The first one is obtained through the associated linear pencil matrix while the second is obtained through a suitable transformation.

Consider the recurrence relations of the form

$$\begin{aligned} r_{2n+1}(z) &= (\sigma_{2n}z + i\bar{\sigma}_{2n})r_{2n}(z) - d_{2n}^2(1+iz)^2 r_{2n-1}(z),\\ r_{2n+2}(z) &= (\sigma_{2n+1}z + i\bar{\sigma}_{2n+1})r_{2n+1}(z) - d_{2n+1}^2(z+i)^2 r_{2n}(z)\end{aligned} \tag{4.1}$$

for $n \geq 0$. Here $\sigma_0 > 0$, $\sigma_j \in \mathbb{C}\setminus\{0\}$ and $d_j > 0 \in \mathbb{R}$ for $j \geq 1$. We also assume that σ_j, $j \geq 1$, has non-negative real and imaginary parts. Then it can be easily verified that the zeros $r_j(z)$ are the eigenvalues of the generalized eigenvalue problem $\mathcal{G}_j + z\mathcal{H}_j$, where $\mathcal{G}_j$ and $\mathcal{H}_j$ are the jth order principal sub-matrices of the matrices $\mathcal{G}$ and $\mathcal{H}$, respectively, where

$$\mathcal{G} = \begin{pmatrix} i\bar{\sigma}_0 & id_1 & 0 & 0 & \cdots \\ id_1 & i\bar{\sigma}_1 & d_2 & 0 & \cdots \\ 0 & d_2 & i\bar{\sigma}_2 & id_3 & \cdots \\ 0 & 0 & id_3 & i\bar{\sigma}_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \sigma_0 & d_1 & 0 & 0 & \cdots \\ d_1 & \sigma_1 & id_2 & 0 & \cdots \\ 0 & id_2 & \sigma_2 & d_3 & \cdots \\ 0 & 0 & d_3 & \sigma_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that $\mathcal{G} = i\bar{\mathcal{H}}$ and both $\mathcal{G}$ and $\mathcal{H}$ are neither Hermitian nor symmetric matrices. However, consider the Hermitian parts

$$\hat{\mathcal{G}} = \left(\begin{array}{c|cc|cc|c} \text{Im } \sigma_0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & \text{Im } \sigma_1 & d_2 & 0 & 0 & \cdots \\ 0 & d_2 & \text{Im } \sigma_2 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & \text{Im } \sigma_3 & d_4 & \cdots \\ 0 & 0 & 0 & d_4 & \text{Im } \sigma_4 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \quad \text{and}$$

$$\hat{\mathcal{H}} = \left(\begin{array}{cc|cc|cc|c} \text{Re } \sigma_0 & d_1 & 0 & 0 & 0 & 0 & \cdots \\ d_1 & \text{Re } \sigma_1 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & \text{Re } \sigma_2 & d_3 & 0 & 0 & \cdots \\ 0 & 0 & d_3 & \text{Re } \sigma_3 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & \text{Re } \sigma_4 & d_5 & \cdots \\ 0 & 0 & 0 & 0 & d_5 & \text{Re } \sigma_5 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

of the matrices $\mathcal{G}$ and $\mathcal{H}$, respectively. We note that both $\hat{\mathcal{G}}$ and $\hat{\mathcal{H}}$ are symmetric block diagonal matrices and hence have real eigenvalues. Further, under the conditions

$$d_{2n}^2 < \text{Im } \sigma_{2n-1}\text{Im } \sigma_{2n} \quad \text{and} \quad d_{2n+1}^2 < \text{Re } \sigma_{2n}\text{Re } \sigma_{2n+1},$$

$\hat{\mathcal{G}}$ is a positive semi-definite (since $\text{Im } \sigma_0 = 0$) matrix while $\hat{\mathcal{H}}$ is a positive-definite matrix. In case $d_{2n}^2 = \text{Im } \sigma_{2n-1}\text{Im } \sigma_{2n}$ and $d_{2n+1}^2 = \text{Re } \sigma_{2n}\text{Re } \sigma_{2n+1}$, the eigenvalues of $\hat{\mathcal{H}}$ are $\pm(\text{Re } \sigma_{2n} + \text{Re } \sigma_{2n+1})^{1/2}$, $n \geq 0$, while the eigenvalues of $\hat{\mathcal{G}}$ are $\pm(\text{Im } \sigma_{2n} + \text{Im } \sigma_{2n-1})^{1/2}$, $n \geq 0$. This implies that in case of equality, both $\hat{\mathcal{G}}$ and $\hat{\mathcal{H}}$ are indefinite matrices.

However, the sum $\mathcal{J} = \hat{\mathcal{H}} + \mathcal{D}\hat{\mathcal{G}}$, where $\mathcal{D}$ is a diagonal matrix with -1 on its diagonal yields a Jacobi matrix which leads to the monic three-term recurrence relation

$$\mathcal{P}_{n+1}(x) = (x - \text{Re } \sigma_n + \text{Im } \sigma_n)\mathcal{P}_n(x) - d_n^2\mathcal{P}_{n-1}(x), \quad n \geq 0. \tag{4.2}$$

Then $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the real line [8]. We now choose specific values for σ_j and d_j so that $\mathcal{P}_n(x)$ leads to the -1 little Jacobi polynomials which are q-Jacobi polynomials with the limiting case $q = -1$.

Let $\sigma_j = 1 + (-1)^j\rho_j$, $j \geq 0$, where the sequence $\{\rho_j\}_{j=0}^{\infty}$ is constructed as follows. Given a real sequence $\{\alpha_j\}_{j=0}^{\infty}$ with $-1 < \alpha_j < 1$, let

$$\text{Re}\ \sigma_j - \text{Im}\ \sigma_j = a_j - a_{j-1} \Longrightarrow (1 + (-1)^j\text{Re}\ \rho_j) - (1 + (-1)^j\text{Im}\ \rho_j) = a_j - a_{j-1}$$

for $j \geq 0$, where $a_{-1} = -1$. In the simplest case, we choose

$$\text{Re}\ \rho_{2j} = a_{2j}, \quad \text{Im}\ \rho_{2j} = a_{2j-1}, \quad \text{Re}\ \rho_{2j+1} = a_{2j}, \quad \text{and} \quad \text{Im}\ \rho_{2j+1} = a_{2j+1}.$$

This implies $\sigma_{2j} = (1 + a_{2j}) + i(1 + a_{2j-1})$ and $\sigma_{2j+1} = (1 - a_{2j}) + i(1 - a_{2j+1})$. Further, $d_{2j}^2 = \text{Im}\ \sigma_{2j-1}\text{Im}\ \sigma_{2j} = 1 - a_{2j-1}^2$ and $d_{2j+1}^2 = \text{Re}\ \sigma_{2j}\text{Re}\ \sigma_{2j+1} = 1 - a_{2j}^2$. With these values of σ_n and d_n^2, (4.2) becomes the recurrence relation [12, (3.8)], which was used to obtain the little -1 Jacobi polynomials $P_n^{(-1)}(x; \alpha; \beta)$ using Christoffel transformations. They are orthogonal on the interval $[-2, 2]$ with respect to the weight function $\omega(x) = (x+2)(4-x^2)^{\zeta}|x|^{2\eta+1}$. Here $2\zeta+1 = \alpha$ and $2\eta+1 = \beta$. For details of this analysis including the particular values of a_n, $n \geq 0$, we refer to [12, 28].

Now, we proceed to show that the R_{II} recurrence relations (4.1) under suitable transformation can lead to another sequence of polynomials orthogonal on the real line. With the expressions for ρ_j and d_j^2 obtained above, we consider the following form of the recurrence relations (4.1):

$$r_{2n+1}(z; x) = [(x - \bar{\rho}_{2n})z + i(x - \rho_{2n})]r_{2n}(z; x) - d_{2n}^2(1 + iz)^2 r_{2n-1}(z; x),$$
$$r_{2n+2}(z; x) = [(x + \bar{\rho}_{2n+1})z + i(x + \rho_{2n+1})]r_{2n+1}(z; x) - d_{2n+1}^2(z + i)^2 r_{2n}(z; x)$$

for $n \geq 0$, which under the transformation of the variable z as

$$z = \frac{1 - i\lambda}{\lambda - i} = \frac{\lambda + i}{1 + i\lambda}, \tag{4.3}$$

is transformed into

$$\begin{aligned} q_{2n+1}(x; \lambda) &= [x - (a_{2n} - \lambda a_{2n-1})]q_{2n}(x; \lambda) - d_{2n}^2\lambda^2 q_{2n-1}(x; \lambda), \\ q_{2n+2}(x; \lambda) &= [x - (\lambda a_{2n+1} - a_{2n})]q_{2n+1}(x; \lambda) - d_{2n+1}^2 q_{2n}(x; \lambda) \end{aligned} \tag{4.4}$$

for $n \geq 0$. Here $q_n(x; \lambda) = 2^{-n}(\lambda - i)^n r_n(x; \lambda)$ is a polynomial in the variable x while $r_n(z; x)$ is a polynomial in the variable z. We note that (4.4) are the recurrence relations [12, eq. (7.3)].

The recurrence relations (4.4) are written in the form [29]

$$W_{n+1}(x) + (\theta_0 - A_n - C_n)W_n(x) + A_{n-1}C_n W_{n-1}(x) = xW_n(x), \quad n \geq 0,$$

where the parameter $\theta_0 = \lambda + 1$ is such that $W_n(\theta_0) = 1$ and A_n, C_n are uniquely defined in terms of a_n and λ. With specific values of a_n, $W_n(x)$ are the terminating monic Bannai-Ito polynomials orthogonal on a finite number of points x_i, $i = 0, 1, \cdots, N$ given by

$$x_i = \begin{cases} \theta_0 + 2hi, & i \text{ even}; \\ \theta_0 - 2h(i + 1 - s), & i \text{ odd}, \end{cases}$$

where $x_0 = \theta_0$ and θ_0, h and s are arbitrary parameters [29]. However, as $N \to \infty$, the Bannai-Ito polynomials coincide with the big -1 Jacobi polynomials [29].

The little and big -1 Jacobi polynomials, their q analogues, and the Bannai-Ito polynomials are well explored in the literature [3, 11, 18, 19, 26] particularly their relations to other orthogonal polynomial sequences via the Christoffel transformation [12, 28, 29]. However, one aspect as illustrated above is that they can also be related to R_{II} recurrence relations of the form (2.7a) and (2.7b), either through the linear pencil matrix (little -1 Jacobi polynomials) or through a simple transformation (big -1 Jacobi polynomials).

Particularly, in case of Bannai-Ito polynomials, the map (4.3) amounts to the transformation of the initial point x_0 of orthogonality of these polynomials. Hence, the sequence of orthogonal rational functions in Theorem 2.2 has these transformed Bannai-Ito polynomials as numerators with poles at i and $-i$. The same sequence can also be proved to be biorthogonal using Theorem 3.2.

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