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INTEGRAL POWER SUMS OF FOURIER COEFFICIENTS OF SYMMETRIC SQUARE L-FUNCTIONS

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Abstract. Let $f(z)$ be a holomorphic Hecke eigenform of even weight k for $\mathrm{SL}(2,\mathbb{Z})$, and denote $L(s,\mathrm{sym}^2 f)$ be the corresponding symmetric square L-function associated to f. Suppose that $\lambda_{\mathrm{sym}^2 f}(n)$ is the nth normalized Fourier coefficient of $L(s,\mathrm{sym}^2 f)$. In this paper, we investigate the sum $\sum_{n\le x}\lambda^j_{\mathrm{sym}^2 f}(n)$ for $j=2,3,4$, and get some new results which improve the previous results.

1. Introduction

Let $H_k(\Gamma)$ denote the set of normalized primitive holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Then $f(z)\in H_k(\Gamma)$ has a Fourier expansion at the cusp ∞ given by

$$f(z)=\sum_{n=1}^{\infty}\lambda_f(n)n^{\frac{k-1}{2}}e(nz), \tag{1.1}$$

where we normalize $f(z)$ so that $\lambda_f(1)=1$. From the theory of Hecke operators, $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n)=\sum_{d|(m,n)}\lambda_f\left(\frac{mn}{d^2}\right), \tag{1.2}$$

where $m,n\ge 1$ are any integers.

According to Deligne [2], for any prime number p, there are two complex numbers $\alpha(p)$ and $\beta(p)$ such that

$$\lambda_f(p)=\alpha(p)+\beta(p) \quad\text{and}\quad |\alpha(p)|=|\beta(p)|=\alpha(p)\beta(p)=1. \tag{1.3}$$

He also proved the Ramanujan–Petersson conjecture

$$|\lambda_f(n)|\le d(n), \tag{1.4}$$

where $d(n)$ is the divisor function.

For any $f\in H_k(\Gamma)$, the Hecke L-function is defined by

$$L(s,f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s} \tag{1.5}$$

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for $\mathrm{Re}(s) > 1$. The symmetric square L-function $L(s, \mathrm{sym}^2 f)$ is defined by

$$
\begin{aligned}
L(s, \mathrm{sym}^2 f) &= \prod_p \left(1 - \frac{\alpha^2(p)}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\beta^2(p)}{p^s}\right)^{-1} \\
&:= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^2 f}(n)}{n^s}
\end{aligned}
\tag{1.6}
$$

for $\mathrm{Re}(s) > 1$. We know that $\lambda_{\mathrm{sym}^2 f}(n)$ is a real, multiplicative function.

The sum of Fourier coefficients of a symmetric square L-function is an interesting subject in number theory. In 2005, Fomenko [3] studied the sum

$$S(x) = \sum_{n \le x} \lambda_{\mathrm{sym}^2 f}(n)$$

and obtained that

$$S(x) \ll x^{1/2} \log^2 x.$$

In this paper, we consider the sum $\sum_{n \le x} \lambda^j_{\mathrm{sym}^2 f}(n)$ for $j = 2, 3, 4$. We use the method mentioned in [17] to get some new results which improve the previous results. From Lemmas 2.2, 2.3, and 2.4, we can also consider the sum $\sum_{n \le x} \lambda^j_f(n^2)$ and get some similar results.

For $j = 2$, Fomenko [4] studied the mean square estimate for the coefficients of the symmetric square L-function and proved that

$$\sum_{n \le x} \lambda^2_{\mathrm{sym}^2 f}(n) = Cx + O(x^{\gamma}), \tag{1.7}$$

where $\gamma < 1$ is a constant. After that, Tang [24] proved that

$$\sum_{n \le x} \lambda^2_{\mathrm{sym}^2 f}(n) = Cx + O(x^{\frac{10}{13}} \log^9 x). \tag{1.8}$$

For the sum $\sum_{n \le x} \lambda^2_f(n^2)$, the first result is due to Lao–Sankaranarayanan [15], they proved that

$$\sum_{n \le x} \lambda^2_f(n^2) = C_1 x + O(x^{\frac{9}{11}+\varepsilon}). \tag{1.9}$$

After that, they [16] improved the error term to $O(x^{\frac{53}{69}+\varepsilon})$, and the best result is due to Lao-Mckee–Ye [14] with the error term $O(x^{\frac{139}{181}+\varepsilon})$.

For $j = 4$, the result is due to Lao [13] who proved that

$$\sum_{n \le x} \lambda^4_{\mathrm{sym}^2 f}(n) = x P_2(\log x) + O(x^{\frac{79}{81}+\varepsilon}) \tag{1.10}$$

and

$$\sum_{n \le x} \lambda^4_f(n^2) = x \widetilde{P}_2(\log x) + O(x^{\frac{79}{81}+\varepsilon}), \tag{1.11}$$

where $P_2(t)$ and $\widetilde{P}_2(t)$ are polynomials in t of degree 2.

Our results are the following theorems.

Theorem 1.1. *Let $f \in H_k(\Gamma)$, and let $\lambda_f(n)$ (resp., $\lambda_{\mathrm{sym}^2 f}(n)$) be the nth normalized Fourier coefficients associated with f (resp., the symmetric square L-function associated with f). Then for any $\epsilon > 0$, we have*

$$\sum_{n \le x} \lambda^2_{\mathrm{sym}^2 f}(n) = c_1 x + O(x^{\frac{13}{17}+\varepsilon}) \tag{1.12}$$

and

$$\sum_{n \le x} \lambda^2_f(n^2) = \widetilde{c}_1 x + O(x^{\frac{13}{17}+\varepsilon}), \tag{1.13}$$

where $c_1, \widetilde{c}_1 > 0$ are some constants.

For comparison, we have $9/11 = 0.818\ldots > 10/13 = 0.769\ldots > 53/69 = 0.768\ldots > 139/181 = 0.767\ldots > 13/17 = 0.764\ldots$.

Theorem 1.2. *Let $f \in H_k(\Gamma)$, and let $\lambda_f(n)$ (resp., $\lambda_{\mathrm{sym}^2 f}(n)$) be the nth normalized Fourier coefficients associated with f (resp., the symmetric square L-function associated with f). Then for any $\epsilon > 0$, we have*

$$\sum_{n \le x} \lambda^3_{\mathrm{sym}^2 f}(n) = c_2 x + O(x^{\frac{248}{269}+\varepsilon}) \tag{1.14}$$

and

$$\sum_{n \le x} \lambda^3_f(n^2) = \widetilde{c}_2 x + O(x^{\frac{248}{269}+\varepsilon}), \tag{1.15}$$

where $c_2, \widetilde{c}_2 > 0$ are some constants.

Theorem 1.3. *Let $f \in H_k(\Gamma)$, and let $\lambda_f(n)$ (resp., $\lambda_{\mathrm{sym}^2 f}(n)$) be the nth normalized Fourier coefficients associated with f (resp., the symmetric square L-function associated with f). Then for any $\epsilon > 0$, we have*

$$\sum_{n \le x} \lambda^4_{\mathrm{sym}^2 f}(n) = x P_2(\log x) + O(x^{\frac{3223}{3307}+\varepsilon}) \tag{1.16}$$

and

$$\sum_{n \le x} \lambda^4_f(n^2) = x \widetilde{P}_2(\log x) + O(x^{\frac{3223}{3307}+\varepsilon}), \tag{1.17}$$

where $P_2(t)$ and $\widetilde{P}_2(t)$ are polynomials in t of degree 2.

For comparison, we have $79/81 = 0.975\ldots > 3223/3307 = 0.974\ldots$.

2. Preliminary

In this section we will briefly show some fundamental lemmas, which play an important role in the proof of the theorems.

For $0 \le j \le 4$, the jth symmetric power L-function attached to $f \in H_k(\Gamma)$ is defined by

$$\begin{aligned} L(s, \mathrm{sym}^j f) &= \prod_p \prod_{m=0}^{j} \left(1 - \frac{\alpha^{j-2m}(p)}{p^s}\right)^{-1} \\ &:= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \end{aligned} \tag{2.1}$$

for $\mathrm{Re}(s) > 1$. We know that $\lambda_{\mathrm{sym}^j f}(n)$ is a multiplicative function. Then we have

$$L(s, \mathrm{sym}^j f) = \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots\right). \tag{2.2}$$

From (2.1) and (2.2), we get

$$\lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \alpha^{j-2m}(p). \tag{2.3}$$

From (1.3), we have

$$|\lambda_{\mathrm{sym}^j f}(n)| \le d_{j+1}(n), \tag{2.4}$$

where $d_k(n)$ is the nth coefficient of the Dirichlet series $\zeta^k(s)$.

The Rankin–Selberg L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j f$ $(0 \le i \le j \le 4)$ is defined by

$$\begin{aligned} L(s, \mathrm{sym}^i f \times \mathrm{sym}^j f) &= \prod_p \prod_{m=0}^{i} \prod_{n=0}^{j} \left(1 - \frac{\alpha^{i-2m}(p)\alpha^{j-2n}(p)}{p^s}\right)^{-1} \\ &:= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)}{n^s} \end{aligned} \tag{2.5}$$

for $\mathrm{Re}(s) > 1$. We know that $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)$ is a multiplicative function. Then we have

$$L(s, \mathrm{sym}^i f \times \mathrm{sym}^j f) = \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots\right). \tag{2.6}$$

From (2.5) and (2.6), we get

$$\begin{aligned} \lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p) &= \sum_{m=0}^{i} \sum_{n=0}^{j} \alpha^{i-2m}(p)\alpha^{j-2n}(p) \\ &= \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j f}(p). \end{aligned} \tag{2.7}$$

From (1.3), we have

$$|\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)| \le d_{(i+1)(j+1)}(n). \tag{2.8}$$

In particular, we make the convention that

$$\begin{cases} L(s, \mathrm{sym}^0 f) = \zeta(s), \\ L(s, \mathrm{sym}^1 f) = L(s, f), \\ L(s, \mathrm{sym}^0 f \times \mathrm{sym}^j f) = L(s, \mathrm{sym}^j f). \end{cases}$$

Remark 2.1. The famous works of Gelbart and Jacquet [5], Kim [10] and Kim and Shahidi [11, 12] show that $L(s; \mathrm{sym}^j f)$ $(1 \le j \le 4)$ is a general L-function, which has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann zeta-type of degree $j+1$. From the works of Jacquet and Shalika [8, 9], Shahidi [22, 23], and Rudnick and Sarnak [21], the Rankin–Selberg L-function $L(s, \mathrm{sym}^i f \times \mathrm{sym}^j f)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ $(1 \le i, j \le 4$, $i \ne j$, and except possibly for simple poles at $s = 0, 1$ for $i = j)$ and satisfies a certain functional equation of Riemann zeta-type of degree $(i+1)(j+1)$.

Lemma 2.2. *Let $f \in H_k(\Gamma)$, and let $\lambda_f(n)$(resp., $\lambda_{\mathrm{sym}^2 f}(n)$) be the nth normalized Fourier coefficients associated with f(resp., the symmetric square L-function associated with f). We introduce*

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{\mathrm{sym}^2 f}(n)}{n^s} \text{ and } \widetilde{L}_1(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_f(n^2)}{n^s}$$

for $\mathrm{Re}(s) > 1$. *Then we have*

$$L_1(s) = \zeta(s) L(s, \mathrm{sym}^2 f) L(s, \mathrm{sym}^4 f) U_1(s)$$

and

$$\widetilde{L}_1(s) = \zeta(s) L(s, \mathrm{sym}^2 f) L(s, \mathrm{sym}^4 f) \widetilde{U}_1(s),$$

where $U_1(s)$, $\widetilde{U}_1(s)$ are Dirichlet series which converge uniformly and absolutely in the half plane $\mathrm{Re}(s) > 1/2$.

Proof. See [24, Lemma 2.1] and [16, Lemma 2.1, 2.2]. □

Lemma 2.3. *Let $f \in H_k(\Gamma)$, and let $\lambda_f(n)$(resp., $\lambda_{\mathrm{sym}^2 f}(n)$) be the nth normalized Fourier coefficients associated with f(resp., the symmetric square L-function associated with f). We introduce*

$$L_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^3_{\mathrm{sym}^2 f}(n)}{n^s} \text{ and } \widetilde{L}_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^3_f(n^2)}{n^s}$$

for $\mathrm{Re}(s) > 1$. *Then we have*

$$L_2(s) = \zeta(s) L(s, \mathrm{sym}^2 f)^2 L(s, \mathrm{sym}^4 f) L(s, \mathrm{sym}^2 f \times \mathrm{sym}^4 f) U_2(s) \tag{2.9}$$

and

$$\widetilde{L}_2(s) = \zeta(s) L(s, \mathrm{sym}^2 f)^2 L(s, \mathrm{sym}^4 f) L(s, \mathrm{sym}^2 f \times \mathrm{sym}^4 f) \widetilde{U}_2(s), \tag{2.10}$$

where $U_2(s)$, $\widetilde{U}_2(s)$ are Dirichlet series which converge uniformly and absolutely in the half plane $\mathrm{Re}(s) > 1/2$.

Proof. For $\mathrm{Re}(s) > 1$, we can write $\zeta(s) L(s, \mathrm{sym}^2 f)^2 L(s, \mathrm{sym}^4 f) L(s, \mathrm{sym}^2 f \times \mathrm{sym}^4 f)$ as a Euler product

$$\begin{aligned} &\zeta(s) L(s, \mathrm{sym}^2 f)^2 L(s, \mathrm{sym}^4 f) L(s, \mathrm{sym}^2 f \times \mathrm{sym}^4 f) \\ &:= \prod_p \left(1 + \frac{b(p)}{p^s} + \cdots + \frac{b(p^k)}{p^{ks}} + \cdots \right). \end{aligned} \tag{2.11}$$

From (2.2) and (2.6), we obtain

$$b(p) = 1 + 2\lambda_{\mathrm{sym}^2 f}(p) + \lambda_{\mathrm{sym}^4 f}(p) + \lambda_{\mathrm{sym}^2 f \times \mathrm{sym}^4 f}(p). \tag{2.12}$$

From (2.3) and (2.7), it is easy to check that

$$b(p) = \lambda^3_{\mathrm{sym}^2 f}(p). \tag{2.13}$$

On the other hand, from (2.4), we know that

$$L_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^3_{\mathrm{sym}^2 f}(n)}{n^s} \tag{2.14}$$

is absolutely convergent in the half plane $\mathrm{Re}(s) > 1$. Noting that $\lambda^3_{\mathrm{sym}^2 f}(n)$ is a multiplicative function, we have that for $\mathrm{Re}(s) > 1$,

$$L_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^3_{\mathrm{sym}^2 f}(n)}{n^s} = \prod_p \left(1 + \frac{\lambda^3_{\mathrm{sym}^2 f}(p)}{p^s} + \cdots + \frac{\lambda^3_{\mathrm{sym}^2 f}(p^k)}{p^{ks}} + \cdots\right). \tag{2.15}$$

Therefore, from (2.11), (2.12), and (2.15), we get that for $\mathrm{Re}(s) > 1$,

$$\begin{aligned} L_2(s) =& \zeta(s) L(s, \mathrm{sym}^2 f)^2 L(s, \mathrm{sym}^4 f) L(s, \mathrm{sym}^2 f \times \mathrm{sym}^4 f) \\ &\times \prod_p \left(1 + \frac{\lambda^3_{\mathrm{sym}^2 f}(p^2) - b(p^2)}{p^2 s} + \cdots\right) \\ :=& \zeta(s) L(s, \mathrm{sym}^2 f)^2 L(s, \mathrm{sym}^4 f) L(s, \mathrm{sym}^2 f \times \mathrm{sym}^4 f) U_2(s). \end{aligned}$$

From (2.4) and (2.8), it is obvious that $U_2(s)$ converges uniformly and absolutely in the half plane $\mathrm{Re}(s) \geq 1/2 + \varepsilon$ for any $\varepsilon > 0$. This completes the proof of equation (2.9).

From (1.2) and (1.3), we have

$$\lambda_f(p^2) = \sum_{m=0}^{2} \alpha^{2-2m}(p) = \lambda_{\mathrm{sym}^2 f}(p). \tag{2.16}$$

Therefore, from (2.12), we get

$$\lambda_f^3(p^2) = b(p). \tag{2.17}$$

Then we can use a similar argument to prove equation (2.10). □

Lemma 2.4. *Let $f \in H_k(\Gamma)$, and let $\lambda_f(n)$ (resp., $\lambda_{\mathrm{sym}^2 f}(n)$) be the nth normalized Fourier coefficients associated with f (resp., the symmetric square L-function associated with f). We introduce*

$$L_3(s) = \sum_{n=1}^{\infty} \frac{\lambda^4_{\mathrm{sym}^2 f}(n)}{n^s} \text{ and } \widetilde{L}_3(s) = \sum_{n=1}^{\infty} \frac{\lambda^4_f(n^2)}{n^s}$$

for $\mathrm{Re}(s) > 1$. *Then we have*

$$\begin{aligned} L_3(s) =& \zeta(s)^2 L(s, \mathrm{sym}^2 f)^3 L(s, \mathrm{sym}^4 f)^3 L(s, \mathrm{sym}^2 f \times \mathrm{sym}^4 f)^2 \\ &\times L(s, \mathrm{sym}^4 f \times \mathrm{sym}^4 f) U_3(s) \end{aligned}$$

and

$$\begin{aligned} \widetilde{L}_3(s) =& \zeta(s)^2 L(s, \mathrm{sym}^2 f)^3 L(s, \mathrm{sym}^4 f)^3 L(s, \mathrm{sym}^2 f \times \mathrm{sym}^4 f)^2 \\ &\times L(s, \mathrm{sym}^4 f \times \mathrm{sym}^4 f) \widetilde{U}_3(s), \end{aligned}$$

where $U_3(s)$, $\widetilde{U}_3(s)$ are Dirichlet series which converge uniformly and absolutely in the half plane $\mathrm{Re}(s) > 1/2$.

Proof. From (2.1) and (2.5), we can easily compute that

$$L(s, \mathrm{sym}^2 f \times \mathrm{sym}^2 f) = \zeta(s) L(s, \mathrm{sym}^2 f) L(s, \mathrm{sym}^4 f).$$

Then Lemma 2.4 follows from [13, Lemma 2.1, 2.2]. □

For the Riemann Zeta function, we have the following either individual or average subconvexity bounds.

Lemma 2.5. *For any $\varepsilon > 0$, we have that*

$$\int_0^T \left|\zeta\left(\frac{5}{7}+it\right)\right|^{12} dt \ll T^{1+\varepsilon} \tag{2.18}$$

uniformly for $T \geq 1$, and

$$|\zeta(\sigma+it)| \ll (1+|t|)^{\max\left\{\frac{13}{42}(1-\sigma),0\right\}+\varepsilon} \tag{2.19}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1+\varepsilon$ and $|t| \geq 1$.

Proof. See [6, Theorem 8.4 and (8.87)] and [1, Theorem 5]. □

For the symmetric square L-function $L(s, \mathrm{sym}^2 f)$, we have the following subconvexity bound.

Lemma 2.6. *For any $\varepsilon > 0$, we have that*

$$|L(\sigma+it, \mathrm{sym}^2 f)| \ll (1+|t|)^{\max\left\{\frac{5}{4}(1-\sigma),0\right\}+\varepsilon} \tag{2.20}$$

holds uniformly for $\frac{1}{2} \leq \sigma \leq 1+\varepsilon$ and $|t| \geq 1$.

Proof. See [19, Corollary 1.2]. □

From Remark 2.1, we are cognizant that $L(s, \mathrm{sym}^j f)$ $(1 \leq j \leq 4)$ and $L(s, \mathrm{sym}^i f \times \mathrm{sym}^j f)$ $(1 \leq i \leq j \leq 4)$ are general L-functions. The next result follows from [20, Theorem 4] and [18, Proposition 1].

Lemma 2.7. *Suppose that $\mathfrak{L}(s)$ is a general L-function of degree m. Then, for any $\varepsilon > 0$, we have*

$$\int_T^{2T} |L(\sigma+it)|^2 dt \ll T^{\max\{m(1-\sigma),1\}+\varepsilon}, \tag{2.21}$$

uniformly for $1/2 \leq \sigma \leq 1$ and $T \geq 1$, and

$$|L(\sigma+it)| \ll (1+|t|)^{(m/2)(1-\sigma)+\varepsilon}, \tag{2.22}$$

uniformly for $1/2 \leq \sigma \leq 1+\varepsilon$ and $|t| \geq 1$.

3. Proof of Theorem 1.1

Now we prove Theorem 1.1. The proof of other two theorems is similar, so we omit it. By the Perron formula (see [7, Proposition 5.54]), we have

$$\sum_{n \leq x} \lambda^2_{\mathrm{sym}^2 f}(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} L_1(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.1}$$

where $b = 1+\varepsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later.

Then we move the line of integration to $\mathrm{Re}(s) = 5/7$. By Cauchy's residue theorem, we have

$$
\begin{aligned}
\sum_{n\le x}\lambda^2_{\mathrm{sym}^2 f}(n) =& \frac{1}{2\pi i}\left\{\int_{5/7-iT}^{5/7+iT}+\int_{5/7+iT}^{b+iT}+\int_{b-iT}^{5/7-iT}\right\}L_1(s)\frac{x^s}{s}ds \\
&+\mathrm{Res}_{s=1}L_1(s)\frac{x^s}{s}+O\left(\frac{x^{1+\varepsilon}}{T}\right) \\
:=& J_1+J_2+J_3+c_1x+O\left(\frac{x^{1+\varepsilon}}{T}\right).
\end{aligned}
\tag{3.2}
$$

For J_1, by Lemma 2.2, we use Hölder's inequality to get

$$J_1 \ll x^{5/7+\varepsilon}\sup_{1\le T_1\le T} I_1(T_1)^{1/12}I_2(T_1)^{5/12}I_3(T_1)^{1/2}T_1^{-1},$$

where

$$I_1(T_1)=\int_{T_1}^{2T_1}\left|\zeta\left(\frac{5}{7}+it\right)\right|^{12}dt,$$

$$I_2(T_1)\ll\int_{T_1}^{2T_1}\left|L\left(\frac{5}{7}+it,\mathrm{sym}^2 f\right)\right|^{12/5}dt,$$

and

$$I_3(T_1)\ll\int_{T_1}^{2T_1}\left|L\left(\frac{5}{7}+it,\mathrm{sym}^4 f\right)\right|^{2}dt.$$

Then, by Lemmas 2.5, 2.6, and 2.7, we have

$$I_1(T_1)\ll T_1^{1+\varepsilon},\quad I_3(T_1)\ll T_1^{10/7+\varepsilon}$$

and

$$I_2(T_1)\ll T_1^{2/5\times 5/4\times(1-5/7)+3\times(1-5/7)+\varepsilon}\ll T_1^{1+\varepsilon}.$$

Thus, we have

$$
\begin{aligned}
J_1 &\ll x^{5/7+\varepsilon}\sup_{1\le T_1\le T} I_1(T_1)^{1/12}I_2(T_1)^{5/12}I_3(T_1)^{1/2}T_1^{-1} \\
&\ll x^{5/7+\varepsilon}T^{3/14+\varepsilon}.
\end{aligned}
\tag{3.3}
$$

For J_2 and J_3, by Lemmas 2.5, 2.6, and 2.7, we get

$$
\begin{aligned}
J_2+J_3 &\ll \max_{5/7\le\sigma\le b} x^{\sigma}T^{(13/42+5/4+5/2)(1-\sigma)+\varepsilon}T^{-1} \\
&\ll \frac{x^{1+\varepsilon}}{T}+x^{5/7+\varepsilon}T^{47/294+\varepsilon}.
\end{aligned}
\tag{3.4}
$$

From (3.2), (3.3), and (3.4), we have

$$\sum_{n\le x}\lambda^2_{\mathrm{sym}^2 f}(n)=c_1x+O\left(\frac{x^{1+\varepsilon}}{T}\right)+O(x^{5/7+\varepsilon}T^{3/14+\varepsilon}). \tag{3.5}$$

Taking $T=x^{4/17}$ in (3.5), we have

$$\sum_{n\le x}\lambda^2_{\mathrm{sym}^2 f}(n)=c_1x+O\left(x^{13/17+\varepsilon}\right). \tag{3.6}$$

This completes the proof of Theorem 1.1.

Remark 3.1. In this paper, J_1 is treated in a similar way to I_1 in the proof of Theorem 1.2 in Lao and Sankaranarayanan [16] but with a different shift of line of integration to $\sigma = 5/7$ rather than $1/2$. This is crucial, and the choice of $\sigma = 1/2$ cannot produce a result as strong as in our theorem.

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