

## UNIQUENESS OF MINIMAL ENERGY SOLUTIONS FOR A SEMILINEAR PROBLEM INVOLVING THE FRACTIONAL LAPLACIAN

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ABSTRACT. In this paper we study a semilinear problem for the fractional laplacian that is the counterpart of the Neumann problems in the classical setting. We show uniqueness of minimal energy solutions for small domains.

### 1. INTRODUCTION

In recent years there has been an increasing amount of attention to problems involving nonlocal diffusion operators. These problems are so vast that it is impossible to give a comprehensive list of references. Just to cite a few, see [16, 17, 22, 25, 30, 38] for some physical models, [2, 26, 34] for some applications in finances, [10] for applications in fluid dynamics, [24, 29, 32] for applications in ecology, and [23] for some applications in image processing.

Among these applications, one operator that is of particular importance is the *fractional laplacian* that is defined (up to some normalization constant) as

$$(-\Delta)^s u(x) = C(n, s) \text{ p.v. } \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,$$

where p.v. stands for *in principal value* and  $\Omega \subset \mathbb{R}^n$  is a bounded domain and  $C(n, s)$  is a normalizing constant that will be taken as 1 throughout the text, with the exception of Appendix A.

This operator is classical and has been studied by several authors. See for instance [1, 6, 7, 12–14, 28, 35], etc.

The fractional laplacian defined above is an operator associated to the so-called  $\alpha$ -stable processes. This type of stochastic process has a *stability index*  $H = 1/\alpha$  and is stable for  $0 < \alpha < 2$ . In our case  $\alpha = 2s$ , and for  $s = 1$  the process is the Brownian motion (see for example Theorem 3.15 in [33]).

Recall that this operator is the so-called *regional* (or censored) fractional laplacian that corresponds to Levy processes where the long jumps are restricted to be

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inside  $\Omega$ . See, for instance, [15] for a discussion on this. Moreover, we refer the interested reader to the very important paper [4], where the behavior of the censored process associated to the regional fractional laplacian is discussed. See also [9].

This operator is commonly used as a fractional version of the Neumann laplacian, at least for  $s$  close to 1. In the literature, there exist several forms of the fractional version of the Neumann laplacian. We refer the interested reader again to the article [15] for more on this.

In this paper, we address the following semilinear problem associated to  $(-\Delta)^s$ :

$$(1.1) \quad (-\Delta)^s u + u = \lambda |u|^{q-2} u, \quad \text{in } \Omega.$$

For  $s$  close to 1, problem (1.1) can be seen as a fractional counterpart of the classical Neumann problem

$$(1.2) \quad \begin{cases} -\Delta u + u = \lambda |u|^{q-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega. \end{cases}$$

See Appendix A for a short proof of this fact (Theorem A.9).

The exponent  $q$  in (1.1) considered in this paper is a *subcritical exponent* in the sense of the Sobolev embeddings; that is,

$$1 \leq q < 2_s^* := \frac{2n}{n - 2s}.$$

The problem can be separated into three different types of behavior: sublinear ( $1 \leq q < 2$ ), linear ( $q = 2$ ), and superlinear ( $q > 2$ ).

The linear case is by now well understood as an eigenvalue problem and will not be considered here.

For the sublinear and superlinear cases, the parameter  $\lambda$  is superfluous since if one gets a solution for some particular value of  $\lambda$ , then taking a suitable multiple of the solution the value of  $\lambda$  can be taken to be 1.

It is fairly easy to see that any solution to (1.1) is a critical point of the functional

$$\mathcal{F}(u) := \frac{\frac{1}{2}[u]_{s;\Omega}^2 + \|u\|_{2;\Omega}^2}{\|u\|_{q;\Omega}^2},$$

where

$$[u]_{s;\Omega} := \left( \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dx dy \right)^{\frac{1}{2}}$$

is the so-called *Gagliardo seminorm* of  $u$  and, as usual,  $\|u\|_{r;\Omega}$  denotes the  $L^r(\Omega)$ -norm.

By standard variational methods, one can see that there exist *minimal energy solutions* to (1.1), that is, functions  $u \in H^s(\Omega)$  such that

$$\mathcal{F}(u) = \inf_{v \in H^s(\Omega)} \mathcal{F}(v).$$

Moreover, by a direct application of the Ljusternik–Schnirelmann method, one can construct a sequence  $\lambda_k \uparrow \infty$  of critical energy levels and a sequence of critical points  $\{u_k\}_{k \in \mathbb{N}}$  of  $\mathcal{F}$  associated to  $\{\lambda_k\}_{k \in \mathbb{N}}$ .

Therefore, there exist infinitely many solutions to problem (1.1).

In this paper, we focus on *minimal energy solutions* to (1.1), in particular to the multiplicity problem of such solutions.

To this end, inspired by the results of [19], given a domain  $\Omega$  we consider the family of contracted domains

$$(1.3) \quad \Omega_\varepsilon := \varepsilon \cdot \Omega = \{\varepsilon x : x \in \Omega\} \quad \text{as } \varepsilon \downarrow 0$$

and look for the asymptotic behavior of minimal energy solutions in  $\Omega_\varepsilon$  as  $\varepsilon \downarrow 0$ .

We first show that the asymptotic behavior of every minimal energy solution is the same and, using this asymptotic behavior, we are able to conclude the uniqueness of minimal energy solutions for contracted domains.

Finally, we give an estimate on the contraction parameter in order to obtain the uniqueness result.

To end this introduction, we want to remark that the same ideas can be used to deal with the Neumann problem (1.2). The changes needed are easy and are left to the interested reader.

We want to recall that uniqueness results for problem (1.2) have been considered in the literature before. See for instance [27].

This result is a first step in an investigation which might be pursued in various directions: for instance, taking different Neumann fractional operators, as the one considered in [15], or showing if the known uniqueness results for the classical Neumann laplacian hold in this situation. See the articles [8, 36].

We leave these questions for further investigation.

## 2. PRELIMINARIES

Let  $\Omega \subset \mathbb{R}^n$  be a bounded smooth domain and let  $0 < s < 1$ . The fractional order Sobolev space is defined as

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) : \frac{u(x) - u(y)}{|x - y|^{\frac{n}{2} + s}} \in L^2(\Omega \times \Omega) \right\}.$$

This space is endowed with the norm

$$\|u\|_{s;\Omega} := ([u]_{s;\Omega}^2 + \|u\|_{2;\Omega}^2)^{\frac{1}{2}}.$$

It is well known (see, for instance, [14]) that there exists a critical exponent

$$2_s^* := \begin{cases} \frac{2n}{n-2s} & \text{if } 2s < n, \\ \infty & \text{otherwise} \end{cases}$$

such that for any  $1 \leq q < 2_s^*$  the embedding  $H^s(\Omega) \subset L^q(\Omega)$  is compact.

We define the Sobolev constant as the number

$$S(\Omega) = S_{s,q}(\Omega) := \inf_{u \in H^s(\Omega)} \frac{\|u\|_{s;\Omega}^2}{\|u\|_{q;\Omega}^2}.$$

It is easy to see, as a consequence of the compactness of the embedding, that there exists an extremal for  $S(\Omega)$ , that is, a function  $u \in H^s(\Omega)$  where the above infimum is attained.

Also, any extremal for  $S(\Omega)$  is a minimal energy (weak) solution to (1.1).

The constant  $\lambda$  in (1.1) depends on the normalization of the extremal. For instance, if the extremal  $u$  is normalized as  $\|u\|_{q;\Omega} = 1$ , then  $\lambda = S(\Omega)$ .

Recall that the operator  $(-\Delta)^s$  is a bounded operator between the Sobolev space  $H^s(\Omega)$  and its dual  $(H^s(\Omega))'$  and can be computed by

$$\langle (-\Delta)^s u, v \rangle = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy.$$

Therefore, for a solution to (1.1) we mean a function  $u \in H^s(\Omega)$  such that

$$\frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy + \int_{\Omega} uv dx = \lambda \int_{\Omega} |u|^{q-2} uv dx$$

for every  $v \in H^s(\Omega)$ .

### 3. ASYMPTOTIC BEHAVIOR IN THIN DOMAINS

Throughout this section we fix the exponent  $q \in [1, 2_s^*]$ ,  $q \neq 2$ .

Our objective in this section is to study the asymptotic behavior of the constant  $S(\Omega_\varepsilon)$  as  $\varepsilon \downarrow 0$ , where the contracted domains  $\Omega_\varepsilon$  are given by (1.3).

We begin with a simple estimate.

**Lemma 3.1.** *Under the above notation, we have that*

$$S(\Omega_\varepsilon) \leq |\Omega_\varepsilon|^{1-\frac{2}{q}} = \varepsilon^{n(1-\frac{2}{q})} |\Omega|^{1-\frac{2}{q}}.$$

*Proof.* The lemma follows just by taking  $u = 1$  as a test function in the definition of  $S(\Omega_\varepsilon)$ . □

Now we want to be more precise. We need to show that the asymptotic behavior of  $S(\Omega_\varepsilon)$  is given precisely by  $\varepsilon^{n(1-\frac{2}{q})}$  and also to study the behavior of the associated extremals.

**Lemma 3.2.** *Under the above notation, we have that*

$$\lim_{\varepsilon \downarrow 0} \frac{S(\Omega_\varepsilon)}{\varepsilon^{n(1-\frac{2}{q})}} = |\Omega|^{1-\frac{2}{q}}.$$

Moreover if  $u_\varepsilon \in H^s(\Omega_\varepsilon)$  is an extremal for  $S(\Omega_\varepsilon)$ , the rescaled extremals  $\bar{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x)$  normalized such that  $\|\bar{u}_\varepsilon\|_{q;\Omega} = 1$  verify that

$$\bar{u}_\varepsilon \rightarrow |\Omega|^{-\frac{1}{q}} \quad \text{strongly in } H^s(\Omega).$$

*Proof.* First, observe that for  $v \in H^s(\Omega_\varepsilon)$ , if we denote  $\bar{v}(x) = v(\varepsilon x)$ , then  $\bar{v} \in H^s(\Omega)$ . Moreover,  $[v]_{s;\Omega_\varepsilon} = \varepsilon^{\frac{n}{2}-s} [\bar{v}]_{s;\Omega}$  and  $\|v\|_{r;\Omega_\varepsilon} = \varepsilon^{\frac{n}{r}} \|\bar{v}\|_{r;\Omega}$  for  $1 \leq r < 2_s^*$ . Therefore

$$\frac{\|v\|_{s;\Omega_\varepsilon}^2}{\|v\|_{q;\Omega_\varepsilon}^2} = \varepsilon^{n(1-\frac{2}{q})} \frac{\varepsilon^{-2s} [\bar{v}]_{s;\Omega}^2 + \|\bar{v}\|_{2;\Omega}^2}{\|\bar{v}\|_{q;\Omega}^2}.$$

Now, let  $u_\varepsilon \in H^s(\Omega_\varepsilon)$  be an extremal for  $S(\Omega_\varepsilon)$  and let  $\bar{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ . Then,

$$S(\Omega_\varepsilon) = \varepsilon^{n(1-\frac{2}{q})} \frac{\varepsilon^{-2s} [\bar{u}_\varepsilon]_{s;\Omega}^2 + \|\bar{u}_\varepsilon\|_{2;\Omega}^2}{\|\bar{u}_\varepsilon\|_{q;\Omega}^2}.$$

Now, by Lemma 3.1, it follows that

$$(3.1) \quad \frac{\varepsilon^{-2s} [\bar{u}_\varepsilon]_{s;\Omega}^2 + \|\bar{u}_\varepsilon\|_{2;\Omega}^2}{\|\bar{u}_\varepsilon\|_{q;\Omega}^2} \leq |\Omega|^{1-\frac{2}{q}}.$$

Let us now fix the normalization of the extremal  $u_\varepsilon$  such that  $\|\bar{u}_\varepsilon\|_{q;\Omega} = 1$ , and by (3.1), we obtain that  $\bar{u}_\varepsilon$  is bounded in  $H^s(\Omega)$  uniformly on  $\varepsilon > 0$ . So, there exists  $\bar{u} \in H^s(\Omega)$  such that (up to some sequence  $\varepsilon_k \rightarrow 0$ )

$$(3.2) \quad \bar{u}_{\varepsilon_k} \rightharpoonup \bar{u} \quad \text{weakly in } H^s(\Omega),$$

$$(3.3) \quad \bar{u}_{\varepsilon_k} \rightarrow \bar{u} \quad \text{strongly in } L^r(\Omega) \text{ for any } 1 \leq r < 2_s^*.$$

Also, from (3.1) and (3.2), we have that

$$[\bar{u}]_{s;\Omega} \leq \liminf_{\varepsilon \downarrow 0} [\bar{u}_\varepsilon]_{s;\Omega} = 0;$$

therefore  $\bar{u}$  is constant, and, since  $\|\bar{u}_\varepsilon\|_{q;\Omega} = 1$ , from (3.3) we obtain that  $\|\bar{u}\|_{q;\Omega} = 1$ .

All these together imply that  $\bar{u} = |\Omega|^{-\frac{1}{q}}$ .

From these estimates, one easily concludes that

$$\begin{aligned} |\Omega|^{1-\frac{2}{q}} &\leq \liminf_{\varepsilon \downarrow 0} \varepsilon^{-2s} [\bar{u}_\varepsilon]_{s;\Omega}^2 + \|\bar{u}_\varepsilon\|_{2;\Omega}^2 \\ &= \liminf_{\varepsilon \downarrow 0} \frac{S(\Omega_\varepsilon)}{\varepsilon^{n(1-\frac{2}{q})}} \leq \limsup_{\varepsilon \downarrow 0} \frac{S(\Omega_\varepsilon)}{\varepsilon^{n(1-\frac{2}{q})}} \leq |\Omega|^{1-\frac{2}{q}}. \end{aligned}$$

The proof is complete.  $\square$

#### 4. UNIQUENESS OF EXTREMALS FOR SMALL DOMAINS

In this section we show the uniqueness of extremals if the domain is contracted enough.

For that purpose, observe that if  $u_\varepsilon$  is an extremal for  $S(\Omega_\varepsilon)$  and  $\bar{u}_\varepsilon$  is the rescaled extremal normalized as  $\|\bar{u}_\varepsilon\|_{q;\Omega} = 1$ , then  $\bar{u}_\varepsilon$  is a weak solution of the problem

$$(4.1) \quad (-\Delta)^s u + \varepsilon^{2s} u = \varepsilon^{2s} \lambda_\varepsilon |u|^{q-2} u \quad \text{in } \Omega,$$

where  $\lambda_\varepsilon = S(\Omega_\varepsilon) \varepsilon^{-n(1-\frac{2}{q})}$ . Recall also that  $0 \leq \lambda_\varepsilon \leq |\Omega|^{1-\frac{2}{q}}$  (cf. Lemma 3.1).

So, we define the space

$$\mathcal{E} := \{u \in H^s(\Omega) : \|u\|_{q;\Omega} = 1\}.$$

It is easy to see that  $\mathcal{E}$  is a  $C^1$ -manifold.

We then define  $F: \mathcal{E} \times [0, 1) \rightarrow (H^s(\Omega))'$  as

$$\begin{aligned} \langle F(u, \varepsilon), v \rangle &:= \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy \\ &\quad + \varepsilon^{2s} \int_{\Omega} uv dx - \varepsilon^{2s} \lambda_\varepsilon \int_{\Omega} |u|^{q-2} uv dx. \end{aligned}$$

Denote  $u_0 = |\Omega|^{-\frac{1}{q}} \in \mathcal{E}$  and observe that  $F(u_0, 0) = 0$ .

Following the ideas of [19] we want to use the Implicit Function Theorem (IFT) to show the existence of a small number  $\delta > 0$  and a curve  $\phi: [0, \delta) \rightarrow \mathcal{E}$  such that

$$\phi(0) = u_0 \quad \text{and} \quad F(\phi(\varepsilon), \varepsilon) = 0 \quad \text{for every } 0 \leq \varepsilon < \delta,$$

and if  $(u, \varepsilon) \in \mathcal{E} \times [0, \delta)$  is such that  $F(u, \varepsilon) = 0$  and  $u$  is close to  $u_0$ , then  $u = \phi(\varepsilon)$ .

Observe that if we can apply the IFT, then combining this with Lemma 3.2, we automatically obtain the uniqueness of extremals of  $S(\Omega_\varepsilon)$  for  $\varepsilon$  small.

In order to be able to apply the IFT we need to check that  $d_u F|_{(u_0, 0)}$  is invertible (see [3] or [37]). Recall that since  $F$  is defined on a manifold, the derivative is defined on the tangent space of  $\mathcal{E}$  at the point  $u_0$ .

Let us begin with a couple of lemmas.

**Lemma 4.1.** *The tangent space of  $\mathcal{E}$  at  $u_0$ , which we denote by  $T_{u_0}\mathcal{E}$ , is given by*

$$T_{u_0}\mathcal{E} = \left\{ v \in H^s(\Omega) : \int_{\Omega} v dx = 0 \right\}.$$

*Proof.* Let  $v \in T_{u_0}\mathcal{E}$ . Then, there exists a differentiable curve  $\alpha: (-1, 1) \rightarrow \mathcal{E}$  such that  $\alpha(0) = u_0$  and  $\dot{\alpha}(0) = v$ .

But, since  $\alpha(t) \in \mathcal{E}$  for every  $t \in (-1, 1)$  it follows that

$$\int_{\Omega} |\alpha(t)|^q dx = 1 \quad \text{for every } t \in (-1, 1).$$

Differentiating both sides of the equality gives

$$\int_{\Omega} q|\alpha(t)|^{q-2}\alpha(t)\dot{\alpha}(t) dx = 0.$$

So, if we evaluate at  $t = 0$  and recall that  $u_0$  is constant, we obtain that

$$(4.2) \quad \int_{\Omega} v dx = 0.$$

On the other hand, if  $v \in H^s(\Omega)$  verifies (4.2), we construct the curve  $\alpha: (-1, 1) \rightarrow \mathcal{E}$  as

$$\alpha(t) = \frac{u_0 + tv}{\|u_0 + tv\|_q}.$$

Straightforward computations show that  $\alpha(0) = u_0$  and  $\dot{\alpha}(0) = v$ . □

Now, we denote  $\mathcal{A} = (\text{span}\{1\})^\perp = \{f \in (H^s(\Omega))': \langle f, 1 \rangle = 0\}$ .

**Lemma 4.2.** *We have that*

$$d_u F|_{(u_0,0)}: T_{u_0}\mathcal{E} \rightarrow \mathcal{A}.$$

Moreover, the following expression holds:

$$\langle d_u F|_{(u_0,0)}(u), v \rangle = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy.$$

*Proof.* To prove the lemma, first observe that

$$\langle F(u, 0), v \rangle = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy$$

for every  $u \in \mathcal{E}$ .

From this expression the lemma follows. □

It remains to be seen that  $d_u F|_{(u_0,0)}$  has a continuous inverse.

**Lemma 4.3.** *The derivative  $d_u F|_{(u_0,0)}: T_{u_0}\mathcal{E} \rightarrow \mathcal{A}$  has a continuous inverse.*

*Proof.* First observe that  $T_{u_0}\mathcal{E}$  is a Hilbert space with inner product given by

$$(u, v) = \langle d_u F|_{(u_0,0)}(u), v \rangle = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy.$$

So, the lemma follows from the Riesz representation theorem. □

Combining Lemmas 4.1, 4.2, and 4.3 we are in position to apply the IFT and conclude the main result of the section.

**Theorem 4.4.** *Given  $\Omega \subset \mathbb{R}^n$  smooth and of finite measure and  $1 < q < 2_s^*$ , there exists  $\delta > 0$  such that  $S(\Omega_\varepsilon)$  has a unique extremal for  $0 < \varepsilon < \delta$ .*

*Proof.* At this point the proof is a direct consequence of the IFT and the remarks made at the beginning of the section. □

## 5. ESTIMATES FOR THE CONTRACTION PARAMETER

Let us first define

$$\varepsilon_0 = \sup\{\delta > 0: \exists! \text{ normalized extremal for } S(\Omega_\varepsilon) \forall \varepsilon < \delta\}.$$

From the results of the previous section, we know that  $\varepsilon_0 > 0$ . We now want to find a lower bound for  $\varepsilon_0$ .

**Lemma 5.1.** *There exists  $u_0 \in H^s(\Omega_{\varepsilon_0})$ , an extremal for  $S(\Omega_{\varepsilon_0})$ , such that the rescaled function  $\bar{u}_0(x) = u_0(\varepsilon_0 x)$  normalized as  $\|\bar{u}_0\|_{q;\Omega} = 1$  verifies that  $d_u F|_{(\bar{u}_0, \varepsilon_0)}$  is not invertible.*

*Proof.* Assume the opposite.

We first claim that there is a unique extremal for  $S(\Omega_{\varepsilon_0})$ . Otherwise, if  $u_0 \neq u_1$  are extremals such that the rescaled functions  $\bar{u}_i(x) = u_i(\varepsilon_0 x)$ ,  $i = 0, 1$ , normalized as  $\|\bar{u}_i\|_{q;\Omega} = 1$ ,  $i = 0, 1$ , verify that  $d_u F|_{(\bar{u}_i, \varepsilon_0)}$  is invertible for  $i = 0, 1$ . But then, by the IFT, there exist  $\delta > 0$  and two curves  $\phi_i: (\varepsilon_0 - \delta, \varepsilon_0 + \delta) \rightarrow \mathcal{E}$  such that

$$F(\phi_i(\varepsilon), \varepsilon) = 0 \quad \text{for every } \varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 + \delta), \quad i = 0, 1.$$

But this contradicts the uniqueness of extremals for  $\varepsilon < \varepsilon_0$ .

Now, let  $\varepsilon_k > \varepsilon_0$  be such that  $\varepsilon_k \rightarrow \varepsilon_0$  as  $k \rightarrow \infty$  and let  $u_k$  be an extremal for  $S(\Omega_{\varepsilon_k})$ . We normalized these extremals so that the rescaled functions  $\bar{u}_k(x) = u_k(\varepsilon_k x)$  verify  $\|\bar{u}_k\|_{q;\Omega} = 1$ .

We want to see that  $\{\bar{u}_k\}_{k \in \mathbb{N}}$  converges to  $\bar{u}_0$  that is the rescaled function of the unique extremal  $u_0$  for  $S(\Omega_{\varepsilon_0})$ .

In fact, it is immediate to see that  $\sup_{k \in \mathbb{N}} \|\bar{u}_k\|_{s;\Omega} < \infty$ , and so, up to a subsequence, there exists  $\bar{w} \in H^s(\Omega)$  such that

$$\begin{aligned} \bar{u}_k &\rightharpoonup \bar{w} \quad \text{weakly in } H^s(\Omega) \\ \bar{u}_k &\rightarrow \bar{w} \quad \text{strongly in } L^r(\Omega), \text{ for every } 1 \leq r < 2_s^*. \end{aligned}$$

From these convergence results it follows that

$$1 = \|\bar{u}_k\|_{q;\Omega} \rightarrow \|\bar{w}\|_{q;\Omega}$$

and

$$\varepsilon_0^{-2s} [\bar{w}]_{s,2;\Omega}^2 + \|\bar{w}\|_{2;\Omega}^2 \leq \liminf_{k \rightarrow \infty} \varepsilon_k^{-2s} [\bar{u}_k]_{s,2;\Omega}^2 + \|\bar{u}_k\|_{2;\Omega}^2 = \liminf_{k \rightarrow \infty} \frac{S(\Omega_{\varepsilon_k})}{\lambda_{\varepsilon_k}}.$$

Now, let  $u_0 \in H^s(\Omega_{\varepsilon_0})$  be the unique extremal for  $S(\Omega_{\varepsilon_0})$  normalized such that the rescaled function  $\bar{u}_0$  satisfies  $\|\bar{u}_0\|_{q;\Omega} = 1$ .

Then,

$$\limsup_{k \rightarrow \infty} \frac{S(\Omega_{\varepsilon_k})}{\lambda_{\varepsilon_k}} \leq \limsup_{k \rightarrow \infty} \varepsilon_k^{-2s} [\bar{u}_0]_{s,2;\Omega}^2 + \|\bar{u}_0\|_{2;\Omega}^2 = \varepsilon_0^{-2s} [\bar{u}_0]_{s,2;\Omega}^2 + \|\bar{u}_0\|_{2;\Omega}^2 = \frac{S(\Omega_{\varepsilon_0})}{\lambda_{\varepsilon_0}}.$$

These two inequalities combined imply that  $w(x) = \bar{w}(\varepsilon_0^{-1}x)$  is an extremal for  $S(\Omega_{\varepsilon_0})$  and so  $w = u_0$ .

Now, since we are assuming that  $d_u F|_{(u_0, \varepsilon_0)}$  is invertible, we can apply the IFT as in the proof of Theorem 4.4 to conclude that for some  $\delta > 0$  there is a unique extremal for  $S(\Omega_\varepsilon)$  for  $\varepsilon < \varepsilon_0 + \delta$ . But this contradicts the definition of  $\varepsilon_0$ .  $\square$

*Remark 5.2.* By a simple application of Fredholm's alternative, it follows that  $d_u F|_{(u_0, \varepsilon_0)}$  is not invertible if and only if it has a nontrivial kernel.

The following Poincaré-type inequality plays an important role in the bound of  $\varepsilon_0$ .

**Lemma 5.3.** *Let  $\Omega \subset \mathbb{R}^n$  be open, smooth, and of finite measure. Let  $0 < s < 1$  and  $1 \leq q < 2_s^*$ . Then, there exists  $c > 0$  that depends on  $q, s$ , and  $\Omega$ , such that*

$$c\|w\|_{q;\Omega}^2 \leq \frac{1}{2}[w]_{s;\Omega}^2$$

for every  $w \in H^s(\Omega)$  such that  $\int_{\Omega} w \, dx = 0$ .

*Proof.* The proof follows by a standard compactness argument and is omitted.  $\square$

We are ready to prove the main result of the section.

**Theorem 5.4.** *Under the notation and assumptions of the section, we have that*

$$\varepsilon_0 \geq \left( \frac{c}{(q-1)|\Omega|^{1-\frac{2}{q}}} \right)^{\frac{1}{2s}},$$

where  $c > 0$  is the constant in the Poincaré-type inequality of Lemma 5.3.

*Proof.* By Lemma 5.1, there exists  $u_0$  an extremal for  $S(\Omega_{\varepsilon_0})$  such that the rescaled function  $\bar{u}_0$  normalized such that  $\|\bar{u}_0\|_{q;\Omega} = 1$  verifies that  $d_u F|_{(\bar{u}_0, \varepsilon_0)}$  is not invertible.

Moreover, by Remark 5.2,  $d_u F|_{(\bar{u}_0, \varepsilon_0)}$  has a nontrivial kernel.

Let  $0 \neq z \in \ker(d_u F|_{(\bar{u}_0, \varepsilon_0)})$ . Then  $z$  is a nontrivial weak solution to the problem

$$(5.1) \quad \begin{cases} (-\Delta)^s z + \varepsilon_0^{2s} z = \varepsilon_0^{2s} \lambda_{\varepsilon_0} (q-1) |\bar{u}_0|^{q-2} z & \text{in } \Omega, \\ \int_{\Omega} z \, dx = 0. \end{cases}$$

Using  $z$  as a test function in the weak formulation of (5.1) gives

$$(5.2) \quad \frac{1}{2}[z]_{s,2;\Omega}^2 + \varepsilon_0^{2s} \|z\|_{2;\Omega}^2 = \varepsilon_0^{2s} \lambda_{\varepsilon_0} (q-1) \int_{\Omega} |\bar{u}_0|^{q-2} z^2 \, dx.$$

Now, we use the Poincaré-type inequality of Lemmas 5.3 and 3.1 and Hölder’s inequality to deduce from (5.2) that

$$c\|z\|_{q;\Omega}^2 \leq \varepsilon_0^{2s} |\Omega|^{1-\frac{2}{q}} (q-1) \|\bar{u}_0\|_{q;\Omega}^{q-2} \|z\|_{q;\Omega}^2.$$

Finally, recalling that  $\|\bar{u}_0\|_{q;\Omega} = 1$  and that  $z \neq 0$  we arrive at the desired result.  $\square$

#### APPENDIX A. FROM NONLOCAL TO LOCAL. EQUATION (1.1) AS $s \uparrow 1$

In this appendix, we will provide a proof of the fact that minimal energy solutions to (1.1) converge to minimal energy solutions to (1.2) as  $s \uparrow 1$ .

It is in this case where the normalization constant  $C(n, s)$  in the definition of the fractional laplacian  $(-\Delta)^s$  comes into play. It is known (see for instance [14]) that  $C(n, s)$  behaves as  $(1-s)$  as  $s \uparrow 1$ , so we take  $C(n, s) = 1-s$  in this appendix.

To this end, we recall the very important and famous result of [5].

**Theorem A.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with Lipschitz boundary and let  $u \in L^2(\Omega)$ . Then, there exists a constant  $K$  depending only on  $n$  such that*

$$(A.1) \quad \lim_{s \uparrow 1} (1-s)[u]_{s;\Omega}^2 = K\|\nabla u\|_{2;\Omega}^2.$$



Moreover, the constant  $K$  is given by

$$K = \frac{1}{2} \int_{\mathbb{S}^{n-1}} z_n^2 dS_z.$$

*Remark A.2.* The results in [5] are more general. In fact there the result is proved for  $W^{s,p}(\Omega)$  for any  $1 < p < \infty$ . Here we are using the case  $p = 2$ . We recall that in this case the same result can be produced by means of the Fourier transform. See [14].

*Remark A.3.* This result is understood in the sense that if  $u \in H^1(\Omega)$ , then (A.1) holds, and on the other hand, if

$$\liminf_{s \uparrow 1} (1-s)[u]_{s;\Omega}^2 < \infty,$$

then  $u \in H^1(\Omega)$  and (A.1) holds.

We also need a second result borrowed from [5].

**Theorem A.4.** *Let  $s_k \uparrow 1$  be a sequence and let  $u_k \in H^{s_k}(\Omega)$  be such that*

$$\sup_{k \in \mathbb{N}} (1-s_k)[u_k]_{s_k;\Omega}^2 < \infty \quad \text{and} \quad \sup_{k \in \mathbb{N}} \|u_k\|_{2;\Omega} < \infty.$$

*Then, there exist a subsequence  $\{u_{k_j}\}_{j \in \mathbb{N}} \subset \{u_k\}_{k \in \mathbb{N}}$  and a function  $u \in H^1(\Omega)$  such that*

$$\|u_{k_j} - u\|_{t;\Omega} \rightarrow 0$$

*for every  $t < 1$ .*

*Moreover,*

$$K \|\nabla u\|_{2;\Omega}^2 \leq \liminf_{k \rightarrow \infty} (1-s_k)[u]_{s_k;\Omega}^2.$$

With the help of these two results it is easy to prove the following theorem.

**Theorem A.5.** *Let  $1 \leq q < 2^* = \frac{2n}{n-2}$  and denote  $B_q := \{u \in L^q(\Omega) : \|u\|_{q;\Omega} = 1\}$ . Define the functionals  $J_{s,q}, J_q : B_q \rightarrow [0, \infty]$  as*

$$J_{s,q}(u) := \begin{cases} \frac{(1-s)}{2} [u]_{s;\Omega}^2 + \|u\|_{2;\Omega}^2 & \text{if } u \in H^s(\Omega), \\ \infty & \text{otherwise;} \end{cases}$$

$$J_q(u) := \begin{cases} \frac{K}{2} \|\nabla u\|_{2;\Omega}^2 + \|u\|_{2;\Omega}^2 & \text{if } u \in H^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

*Then  $J_{s,q}$   $\Gamma$ -converges to  $J_q$  as  $s \uparrow 1$ .*

*Remark A.6.* Observe that since  $2_s^* \downarrow 2^*$  as  $s \uparrow 1$ , we always have that  $q < 2_s^*$ . Hence,  $H^s(\Omega) \Subset L^q(\Omega)$  by the Rellich-Kondrashov theorem (see [14]).

*Remark A.7.* This type of  $\Gamma$ -convergence result for nonlocal energies as they approach local ones was, to our knowledge, first observed in [31] and then was also used in a more general context in [18]. See also [20, 21] for an application to some optimal design problem for nonlocal energies.

*Proof.* In order to show the  $\Gamma$ -convergence, we need to show:

- The lim inf inequality. That is, given  $u \in B_q$ ,  $s_k \uparrow 1$ , and any sequence  $u_k \in B_q$  such that  $\|u_k - u\|_{q;\Omega} \rightarrow 0$ ,

$$J_s(u) \leq \liminf_{k \rightarrow \infty} J_{s_k,q}(u_k).$$

- The limsup inequality. That is, for any  $u \in B_q$  and any sequence  $s_k \uparrow 1$ , there exists a *recovery sequence*  $v_k \in B_q$  such that

$$J_s(u) \geq \limsup_{k \rightarrow \infty} J_{s_k, q}(v_k).$$

Now, observe that the limsup inequality follows directly from Theorem A.1 taking the constant sequence  $v_k = u$  as a recovery sequence.

Finally, the liminf inequality follows directly from Theorem A.4.  $\square$

The importance of  $\Gamma$ -convergence is given by the next theorem, whose proof can be found, for instance, in [11].

**Theorem A.8.** *Let  $(X, d)$  be a complete metric space and let  $F_k, F: X \rightarrow [0, \infty]$  be a sequence of functions such that  $F_k$   $\Gamma$ -converges to  $F$ . Assume moreover that there exists  $x_k \in X$  such that  $F_k(x_k) = \inf_X F_k$  for every  $k \in \mathbb{N}$  and that the sequence  $\{x_k\}_{k \in \mathbb{N}}$  is precompact in  $X$ . Then every accumulation point of the sequence  $\{x_k\}_{k \in \mathbb{N}}$  is a minimum of  $F$  and*

$$\min_X F = \liminf_{k \rightarrow \infty} \min_X F_k.$$

In order to apply Theorem A.5 it remains to show that if  $u_k \in H^{s_k}(\Omega)$  is a minimizer for  $J_{s_k, q}$ , then the sequence is precompact in  $L^q(\Omega)$ .

But this follows from Lemma 3.1, Theorem A.4, and the compact inclusion  $H^s(\Omega) \Subset L^q(\Omega)$  for any  $s \in (0, 1)$  as observed in Remark A.6.

Combining all of these results we arrive at the following theorem.

**Theorem A.9.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with Lipschitz boundary, and for any  $s \in (0, 1)$ , let  $u_s \in H^s(\Omega)$  be a minimal energy solution to (1.1) with  $1 \leq q < 2^* = \frac{2n}{n-2}$  and normalized as  $\|u_s\|_{q; \Omega} = 1$ .*

*Then the set  $\{u_s\}_{0 < s < 1}$  is precompact in  $L^q(\Omega)$ , and any accumulation point  $u = \lim_{k \rightarrow \infty} u_{s_k}$  in  $L^q(\Omega)$  with  $s_k \uparrow 1$  is a minimal energy solution to (1.2).*

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