

## SMALL TIME ASYMPTOTICS FOR BROWNIAN MOTION WITH SINGULAR DRIFT

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(Communicated by David Levin)

ABSTRACT. We establish a small time large deviation principle and a Varadhan type asymptotics for Brownian motion with singular drift on  $\mathbb{R}^d$  with  $d \geq 3$  whose infinitesimal generator is  $\frac{1}{2}\Delta + \mu \cdot \nabla$ , where each  $\mu_i$  of  $\mu = (\mu_1, \dots, \mu_d)$  is a measure in some suitable Kato class.

### 1. INTRODUCTION

For non-divergence form elliptic operator  $\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$  on  $\mathbb{R}^d$  with bounded, symmetric, and uniformly elliptic diffusion matrix  $A(x) = (a_{ij}(x))$  that is uniformly Hölder continuous, Varadhan showed the following small time asymptotics for the heat kernel  $p(t, x, y)$  of  $\mathcal{L}$  in [V1],

$$(1.1) \quad \lim_{t \downarrow 0} t \log p(t, x, y) = -d(x, y)^2/2 \quad \text{for } x, y \in \mathbb{R}^d,$$

where  $d(x, y)$  is the Riemannian metric induced by  $A(x)^{-1}$ . Later, (1.1) is extended by Norris [N] to divergence form elliptic operator

$$\mathcal{L} = \frac{1}{2\rho(x)} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( \rho(x) a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

where  $A(x) = (a_{ij}(x))$  is symmetric locally bounded and locally uniformly elliptic and  $\rho(x)$  is a measurable function that is locally bounded between two positive constants. In fact, the results in [N] are more general, allowing  $\mathbb{R}^d$  being replaced by a Lipschitz manifold. In [AH], Ariyoshi and Hino studied the integral version of (1.1) and showed that for the heat semigroup  $\{P_t; t \geq 0\}$  associated with any symmetric local Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(E; m)$  with  $\sigma$ -finite measure  $m$ ,

$$(1.2) \quad \lim_{t \downarrow 0} t \log P_t(A_1, A_2) = -d(A_1, A_2)^2/2$$

for any Borel subsets  $A_i \subset E$  with  $0 < m(A_i) < \infty$  for  $i = 1, 2$ . Here  $P_t(A_1, A_2) := \int_{A_2} P_t \mathbf{1}_{A_1}(x) m(dx)$  and  $d(A_1, A_2)$  is the intrinsic distance between  $A_1$  and  $A_2$  induced by the local Dirichlet form  $(\mathcal{E}, \mathcal{F})$ . The above result (1.2) has recently been extended in [HM] to lower order perturbations of symmetric strongly local Dirichlet forms. For earlier and other related work, see the references in [Zh, N, HR].

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Received by the editors June 25, 2018, and, in revised form, December 7, 2018.

2010 *Mathematics Subject Classification*. Primary 60H15; Secondary 93E20, 35R60.

*Key words and phrases*. Kato class measure, heat kernel, small time large deviation, small time asymptotics.

This work was partially supported by Simons Foundation Grant 520542 and NNSF of China (11671372, 11431014, 11401557, 11731009, 11721101).

In this paper, we study the pointwise asymptotic property (1.1) for Brownian motion with singular drifts in  $\mathbb{R}^d$  with  $d \geq 3$ ; that is,

$$(1.3) \quad dX_t = dW_t + dA_t \quad \text{with } X_0 = x,$$

where  $A$  is a continuous additive functional of  $X$  having ‘‘Revuz measure  $\mu$ ’’. Informally  $\{X_t, t \geq 0\}$  is a diffusion process in  $\mathbb{R}^d$  with generator  $\frac{1}{2}\Delta + \mu \cdot \nabla$ , where  $\mu = (\mu_1, \dots, \mu_d)$  is a vector-valued signed measure on  $\mathbb{R}^d$  belonging to the Kato class  $\mathbf{K}_{d,1}$  to be introduced below. When  $\mu_i(dx) = b_i(x)dx$  for some function  $b_i$ ,  $X$  is a solution to the stochastic differential equation

$$(1.4) \quad dX_t = dW_t + b(X_t)dt \quad \text{with } X_0 = x.$$

**Definition 1.1.** A signed measure  $\nu$  on  $\mathbb{R}^d$  with  $d \geq 3$  is said to be in the Kato class  $\mathbf{K}_{d,k}$  (for  $k = 1, 2$ ) if

$$\limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} \frac{|\nu|(dy)}{|x-y|^{d-k}} = 0,$$

where  $|\nu|$  is the total variation of  $\nu$ . A measurable function  $f$  on  $\mathbb{R}^d$  is said to be in the Kato class  $\mathbf{K}_{d,k}$  (for  $k = 1, 2$ ) if  $|f(x)|dx \in \mathbf{K}_{d,k}$ .

Clearly all bounded measurable functions are in the Kato class  $\mathbf{K}_{d,k}$  for  $k = 1, 2$ . By Hölder’s inequality, it is easy to see that  $L^p(\mathbb{R}^d; dx) \subset \mathbf{K}_{d,1}$  for  $p > d$  and  $L^p(\mathbb{R}^d; dx) \subset \mathbf{K}_{d,2}$  for  $p > d/2$ . But a measure in  $\mathbf{K}_{d,1}$  can be quite singular. It is shown in [BC, Proposition 2.1] that for a Borel measure  $\mu$  on  $\mathbb{R}^d$ , if there are constants  $\kappa > 0$  and  $\gamma > 0$  so that  $\mu(B(x, r)) \leq \kappa r^{d-1+\gamma}$  for all  $x \in \mathbb{R}^d$  and  $r \in (0, 1]$ , then  $\mu \in \mathbf{K}_{d,1}$ . Thus in particular, if  $A \subset \mathbb{R}^d$  is an Alfrors  $\lambda$ -regular set with  $\lambda \in (d - 1, d]$ , then the Hausdorff measure  $\mathcal{H}^\lambda$  restricted to  $A$  is in  $\mathbf{K}_{d,1}$ .

To recall the precise definition of a Brownian motion with a measure-valued drift  $\mu = (\mu_1, \dots, \mu_d)$  with  $\mu_i \in \mathbf{K}_{d,1}$  for  $1 \leq i \leq d$ , fix a non-negative smooth function  $\varphi$  in  $\mathbb{R}^d$  with compact support and  $\int \varphi(x)dx = 1$ . For any positive integer  $n$ , we put  $\varphi_n(x) = 2^{nd}\varphi(2^n x)$ . For  $1 \leq i \leq d$ , define

$$b_i^{(n)}(x) = \int \varphi_n(x - y)\mu_i(dy).$$

Put  $b^{(n)} = (b_1^{(n)}, \dots, b_d^{(n)})$ . The following definition is taken from [BC].

**Definition 1.2.** A Brownian motion with drift  $\mu$  is a family of probability measures  $\{\mathbb{P}_x, x \in \mathbb{R}^d\}$  on  $C([0, \infty), \mathbb{R}^d)$ , the space of continuous functions on  $[0, \infty)$ , such that under each  $\mathbb{P}_x$  the coordinator process  $X$  has the decomposition

$$X_t = x + W_t + A_t,$$

where

- (a)  $A_t = (A_t^{(1)}, \dots, A_t^{(d)}) = \lim_{n \rightarrow \infty} \int_0^t b^{(n)}(X_s)ds$  uniformly in  $t$  over finite intervals, where the convergence is in probability;
- (b) there exists a subsequence  $\{n_k\}$  such that for every  $t > 0$ ,

$$\sup_{k \geq 1} \int_0^t |b^{n_k}(X_s)|ds < \infty \quad a.s.;$$

- (c)  $W_t$  is a standard Brownian motion in  $\mathbb{R}^d$  starting from the origin.

It is established in [BC] that when each  $\mu_i$  is in the Kato class  $\mathbf{K}_{d,1}$ , Brownian motion with drift  $\mu = (\mu_1, \dots, \mu_d)$ , denoted by  $\{X_t, t \geq 0\}$ , exists and is unique in law for every starting point  $x \in \mathbb{R}^d$ . Moreover, it is shown there that  $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d\}$  forms a conservative Feller process on  $\mathbb{R}^d$ . Later, it is obtained in [KS] that  $X$  has a jointly continuous transition density function  $q(t, x, y)$  which admits the following Gaussian type estimate:

$$(1.5) \quad C_1 e^{-C_2 t} t^{-d/2} e^{-C_3 |x-y|^2/t} \leq q(t, x, y) \leq C_4 e^{C_5 t} t^{-d/2} e^{-C_6 |x-y|^2/t}$$

for all  $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ , where  $C_i, 1 \leq i \leq 6$ , are some positive constants.

In this paper, we are concerned with the precise Varadhan-type small time asymptotics of the transition density function  $q(t, x, y)$  of  $X$ :

$$(1.6) \quad \lim_{t \rightarrow 0} t \log q(t, x, y) = -\frac{|x-y|^2}{2}$$

for every  $x, y \in \mathbb{R}^d$ . Note that since the drift measure  $\mu$  is merely assumed to be in Kato class  $\mathbf{K}_{d,1}$ , the law of  $X$  can be singular with respect to the law of Brownian motion. Moreover, it is not clear if Brownian motion with drift  $\mu$  can be constructed via sectorial non-symmetric Dirichlet form (see [BC, p. 793]). One cannot even apply the result from [HM] to get the integral version of small time asymptotics result (1.2) for our Brownian motion with drift  $\mu$ . Hence a new approach is needed.

For each  $x \in \mathbb{R}^d$ , denote by  $\nu_\varepsilon^x$  the law of  $\{X_{\varepsilon t}, 0 \leq t \leq 1\}$  on  $C([0, 1], \mathbb{R}^d)$  under  $\mathbb{P}_x$ . It is known (see [V1, V2]) that the Varadhan small time asymptotics (1.6) is closely related to the small time large deviation principle of the family  $\{\nu_\varepsilon^x, \varepsilon > 0\}$ . In fact, in this paper, we will establish a small time large deviation principle for  $\{\nu_\varepsilon^x; \varepsilon > 0\}$  for every  $x \in \mathbb{R}^d$  by showing that  $\{\nu_\varepsilon^x; \varepsilon > 0\}$  is exponentially equivalent to the corresponding laws of Brownian motion. This is done in Section 2. The upper bound small time asymptotics of (1.6) is obtained by the construction of the heat kernel of  $X$  and its pointwise upper bound estimate. This is carried out in Proposition 3.3. To establish the lower bound small time asymptotics of (1.6) of the transition density  $q(t, x, y)$ , we will use a crucial estimate (Proposition 2.2) of the Laplace transform of the drift  $A$  to derive a lower bound of the probability that the process  $X$  belongs to a small ball. This is the content of Proposition 3.4 of Section 3.

## 2. SMALL TIME LARGE DEVIATIONS

In this section, we will establish a small time large deviation principle for Brownian motion with measure-valued drift  $\mu$ . For a signed measure  $\mu$  on  $\mathbb{R}^d$ , define

$$(2.1) \quad N_t^\alpha(\mu) := \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} s^{-(d+1)/2} \exp\left(-\alpha \frac{|x-y|^2}{s}\right) |\mu|(dy) ds,$$

where  $|\mu|$  denotes the measure of total variation. Recall (see e.g. [KS]) that  $\mu \in \mathbf{K}_{d,1}$  if and only if  $\lim_{t \rightarrow 0} N_t^\alpha(\mu) = 0$  for any  $\alpha > 0$ . For the transition kernel  $q(t, x, y)$  and a signed measure  $\mu$  on  $\mathbb{R}^d$ , we also introduce

$$(2.2) \quad \Lambda_t(\mu) := \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \frac{q(s, x, y)}{\sqrt{s}} |\mu|(dy) ds.$$

As a consequence of the two-sided Gaussian bounds of the transition density (1.5), it is easy to see that  $\mu$  is in the Kato class  $\mathbf{K}_{d,1}$  if and only if  $\lim_{t \rightarrow 0} \Lambda_t(\mu) = 0$ . For a function  $f$  in  $\mathbf{K}_{d,1}$ , we write  $\Lambda_t(f)$  for  $\Lambda_t(fdx)$ .

Let  $X$  be the Brownian motion with measure drift  $\mu$  stated in Section 1 and let  $\mathbb{P}_x$  be the distribution of  $X$  starting from  $x$ . For  $x \in \mathbb{R}^d$  and  $\varepsilon > 0$ , denote by  $\nu_\varepsilon^x$  the distribution of  $\{X_{\varepsilon t}; t \in [0, 1]\}$  under  $\mathbb{P}_x$  on the path space  $C([0, 1], \mathbb{R}^d)$ . We have the following large deviation result.

**Theorem 2.1.** *For each fixed  $x \in \mathbb{R}^d$ ,  $\{\nu_\varepsilon^x, \varepsilon > 0\}$  satisfies a large deviation principle with rate function*

$$I(f) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{f}(t)|^2 dt & \text{if } f \text{ is absolutely continuous,} \\ \infty & \text{otherwise,} \end{cases}$$

where  $\dot{f}$  stands for the derivative of  $t \rightarrow f(t)$ . Namely,

(i) for any closed subset  $C \subset C([0, 1], \mathbb{R}^d)$ ,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \nu_\varepsilon^x(C) \leq - \inf_{f \in C(x)} I(f),$$

where  $C(x) = \{f \in C : f(0) = x\}$ ;

(ii) for any open subset  $G \subset C([0, 1], \mathbb{R}^d)$ ,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \nu_\varepsilon^x(G) \geq - \inf_{f \in G(x)} I(f),$$

where  $G(x) = \{f \in G : f(0) = x\}$ .

To prove Theorem 2.1, we need the following crucial estimate for the Laplace transform of an additive functional.

**Proposition 2.2.** *Let  $b(\cdot)$  be a positive function in the Kato class  $\mathbf{K}_{d,1}$ . Then, for any  $\lambda > 0$ ,*

$$(2.3) \quad \mathbb{E}_x \left[ e^{\lambda \int_0^t b(X_s) ds} \right] \leq \left( 1 + \lambda \sqrt{t} \Lambda_t(b) \right) \exp(\lambda^2 t \Lambda_t(b)^2) \leq 2 \exp(2\lambda^2 t \Lambda_t(b)^2),$$

where  $\Lambda_t(b)$  is defined as in (2.2).

*Proof.* We claim that for any non-negative integer  $n \geq 0$ ,

$$(2.4) \quad \mathbb{E}_x \left[ \left( \int_0^t b(X_s) ds \right)^n \right] \leq n! \alpha_n \left( \sqrt{t} \Lambda_t(b) \right)^n,$$

where  $\alpha_0 = \alpha_1 = 1$ ,  $\alpha_n = \prod_{k=2}^n \left( 1 - \frac{1}{k} \right)^{(k-1)/2} \left( \frac{1}{k} \right)^{1/2}$  for  $n \geq 2$ . Indeed, for  $n \geq 1$ , we have by the Markov property of  $X$ ,

$$\begin{aligned} & \mathbb{E}_x \left[ \left( \int_0^t b(X_s) ds \right)^n \right] \\ (2.5) \quad &= n! \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \mathbb{E}_x [b(X_{s_1}) b(X_{s_2}) \cdots b(X_{s_n})] \\ &= n! \int_{\{0 \leq s_1 < s_2 < \cdots < s_n \leq t\}} ds_1 \cdots ds_n \int_{(\mathbb{R}^d)^{\otimes n}} b(y_1) \cdots b(y_n) \\ & \quad \times q(s_1, x, y_1) q(s_2 - s_1, y_1, y_2) \cdots q(s_n - s_{n-1}, y_{n-1}, y_n) dy_n \cdots dy_1. \end{aligned}$$

When let  $n = 1$  be,

$$\begin{aligned}
 \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ \int_0^t b(X_s) ds \right] &= \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y) q(s, x, y) dy ds \\
 &\leq \sqrt{t} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y) \frac{q(s, x, y)}{\sqrt{s}} dy \\
 (2.6) \qquad \qquad \qquad &= \sqrt{t} \Lambda_t(b).
 \end{aligned}$$

So (2.4) holds for  $n = 0$  and  $n = 1$  as claimed. When  $n = 2$ , by (2.5) and (2.6),

$$\begin{aligned}
 &\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ \left( \int_0^t b(X_s) ds \right)^2 \right] \\
 &= 2 \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y_1) q(s_1, x, y_1) \left( \int_{s_1}^t \int_{\mathbb{R}^d} b(y_2) q(s_2 - s_1, y_1, y_2) ds_2 dy_2 \right) ds_1 dy_1 \\
 &= 2 \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y_1) q(s_1, x, y_1) \left( \int_0^{t-s_1} \int_{\mathbb{R}^d} b(y_2) q(r, y_1, y_2) dr dy_2 \right) ds_1 dy_1 \\
 &\leq 2 \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y_1) q(s_1, x, y_1) \sqrt{t - s_1} \Lambda_t(b) ds_1 dy_1 \\
 &= 2 \Lambda_t(b) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \sqrt{(t - s_1) s_1} b(y_1) \frac{q(s_1, x, y_1)}{\sqrt{s_1}} ds_1 dy_1 \\
 &\leq 2! \cdot \frac{1}{2} \cdot \left( \sqrt{t} \Lambda_t(b) \right)^2,
 \end{aligned}$$

as  $\max_{s_1 \in [0, t]} \sqrt{s_1(t - s_1)} = t/2$ . This shows that (2.4) holds for  $n = 2$ . Now assuming (2.4) holds for  $n = k \geq 2$ , we next show that it holds for  $n = k + 1$ . By (2.4) for  $n = k$ ,

$$\begin{aligned}
 &\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ \left( \int_0^t b(X_s) ds \right)^{k+1} \right] \\
 &\leq (k + 1) \cdot k! \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y_1) q(s_1, x, y_1) \left( \int_{\{0 < s_2 - s_1 < \dots < s_k - s_1 \leq t - s_1\}} ds_s \cdots ds_n \right. \\
 &\qquad \left. \int_{(\mathbb{R}^d)^{\otimes k}} b(y_2) \cdots b(y_k) q(s_2 - s_1, y_1, y_2) \cdots q(s_n - s_{n-1}, y_{n-1}, y_k) dy_k \right) ds_1 dy_1 \\
 &\leq (k + 1)! \alpha_k \Lambda_t(b)^k \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y_1) \frac{q(s_1, x, y_1)}{\sqrt{s_1}} s_1^{1/2} (t - s_1)^{k/2} ds_1 dy_1 \\
 &\leq (k + 1)! \alpha_k \left( \frac{k}{k + 1} \right)^{k/2} \left( \frac{1}{k + 1} \right)^{1/2} t^{(k+1)/2} \Lambda_t(b)^{k+1} \\
 &= (k + 1)! \alpha_{k+1} \left( \sqrt{t} \Lambda_t(b) \right)^{k+1},
 \end{aligned}$$

as  $\max_{s_1 \in [0, s_1]} s_1^{1/2} (t - s_1)^{k/2} = \left( \frac{k}{k+1} \right)^{k/2} \left( \frac{1}{k+1} \right)^{1/2} t^{(k+1)/2}$ . This shows that (2.4) holds for  $n = k + 1$ . By induction, we have established that (2.4) holds for all  $n \geq 1$ .

Observe that  $\alpha_n \leq \frac{1}{\sqrt{n!}}$ . We have by (2.4) that for  $\lambda > 0$ ,

$$(2.7) \quad \mathbb{E}_x \left[ e^{\lambda \int_0^t b(X_s) ds} \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_x \left[ \left( \lambda \int_0^t b(X_s) ds \right)^n \right] \leq \sum_{n=0}^{\infty} \frac{(\lambda \sqrt{t} \Lambda_t(b))^n}{\sqrt{n!}}.$$

Since  $(2n + 1)! \geq (2n)! \geq (n!)^2$  for integers  $n \geq 0$ , we have for  $z \geq 0$ ,

$$(2.8) \quad \begin{aligned} \Phi(z) &:= \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} \frac{z^{(2n)}}{\sqrt{(2n)!}} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{\sqrt{(2n+1)!}} \\ &\leq \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} + z \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} = (1+z)e^{z^2}. \end{aligned}$$

This combined with (2.7) yields that

$$(2.9) \quad \mathbb{E}_x \left[ e^{\lambda \int_0^t b(X_s) ds} \right] \leq \left( 1 + \lambda \Lambda_t(b) \sqrt{t} \right) e^{\lambda^2 \Lambda_t(b)^2 t}.$$

This completes the proof of the proposition as  $1 + a \leq 2e^{a^2}$  for every  $a \geq 0$ . □

*Proof of Theorem 2.1.* Fix  $x \in \mathbb{R}^d$ . Let  $\mu_\varepsilon^x$  denote the law of the Brownian motion  $\{x + W_{\varepsilon t}; t \in [0, 1]\}$  on the path space  $C([0, 1], \mathbb{R}^d)$ . Then it is well known that  $\{\mu_\varepsilon^x, \varepsilon > 0\}$  obeys a large deviation principle with a rate function  $I(\cdot)$  defined as in the statement of Theorem 2.1. Note that

$$X_{\varepsilon t} = x + W_{\varepsilon t} + A_{\varepsilon t}.$$

According to [DZ, Theorem 4.2.13], to prove that  $\nu_\varepsilon^x$  satisfies a large deviation principle with the same rate function as  $\mu_\varepsilon^x$  it is sufficient to show that the two families  $\{\nu_\varepsilon^x\}, \{\mu_\varepsilon^x\}$  are exponentially equivalent; namely, for any  $\delta > 0$ ,

$$(2.10) \quad \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_x \left( \sup_{0 \leq t \leq 1} |A_{\varepsilon t}| = \sup_{0 \leq t \leq 1} |X_{\varepsilon t} - (x + W_{\varepsilon t})| > \delta \right) = -\infty.$$

To this end, we first deduce an estimate similar to (2.3) for the Laplace transform of the process  $A$ ; namely, for any  $\lambda > 0$ ,

$$(2.11) \quad \mathbb{E}_x \left[ e^{\lambda \sup_{0 \leq s \leq t} |A_s|} \right] \leq 2 \exp \left( 2\lambda^2 t C_4^2 e^{2C_5} N_t^{C_6} \left( \sum_{i=1}^d |\mu_i| \right)^2 \right).$$

For  $1 \leq i \leq d$ , recall the following function defined in Section 1:

$$b_i^{(n)}(x) = \int \varphi_n(x - y) \mu_i(dy).$$

We claim that

$$(2.12) \quad \Lambda_t(|b_i^{(n)}|) \leq C_4 e^{C_5} N_t^{C_6} (|\mu_i|),$$

where  $N_t^{C_6}(|\mu_i|)$  was defined in (2.1). Note that

$$|b_i^{(n)}|(x) \leq \int \varphi_n(x - y) |\mu_i|(dy).$$

Using the upper Gaussian-type estimate (1.5) for  $q(s, x, y)$ , we have for  $t \leq 1$ ,

$$\begin{aligned} & \Lambda_t(|b_i^{(n)}|) \\ \leq & \sup_x \int_0^t \int_{\mathbb{R}^d} \frac{q(s, x, y)}{\sqrt{s}} \left( \int_{\mathbb{R}^d} \varphi_n(y - z) |\mu_i|(dz) \right) dy ds \\ \leq & \sup_x \int_0^t \int_{\mathbb{R}^d} |\mu_i|(dz) \frac{1}{\sqrt{s}} \left( \int_{\mathbb{R}^d} q(s, x, y' + z) \varphi_n(y') |dy' \right) ds \\ \leq & \sup_x \int_0^t \int_{\mathbb{R}^d} |\mu_i|(dz) \frac{1}{\sqrt{s}} \left( \int_{\mathbb{R}^d} C_4 e^{C_5 s} s^{-d/2} e^{-C_6 |x - y' - z|^2/s} \varphi_n(y') |dy' \right) ds \\ \leq & C_4 e^{C_5} \int_{\mathbb{R}^d} \varphi_n(y') |dy'| \sup_{x, y'} \int_0^t \left( \int_{\mathbb{R}^d} s^{-d+1/2} e^{-C_6 |x - y' - z|^2/s} |\mu_i|(dz) \right) ds \\ \leq & C_4 e^{C_5} N_t^{C_6}(|\mu_i|), \end{aligned}$$

where the fact that  $\int_{\mathbb{R}^d} \varphi_n(y') |dy'| = 1$  was used in the last inequality. This proves the claim (2.12). We have by the definition of the process  $A$ , Fatou's lemma, Proposition 2.2, and (2.12) that for every  $\lambda > 0$ ,

$$\begin{aligned} & \mathbb{E}_x \left[ e^{\lambda \sup_{0 \leq s \leq t} |A_s|} \right] \\ \leq & \mathbb{E}_x \left[ e^{\lambda \sup_{0 \leq s \leq t} \sum_{i=1}^d |A_s^{(i)}|} \right] \\ \leq & \liminf_{n \rightarrow \infty} \mathbb{E}_x \left[ e^{\lambda \int_0^t \sum_{i=1}^d |b_i^{(n)}|(X_s) ds} \right] \\ \leq & 2 \exp \left( 2\lambda^2 t \Lambda_t \left( \sum_{i=1}^d |b_i^{(n)}| \right)^2 \right) \\ \leq & 2 \exp \left( 2\lambda^2 t C_4^2 e^{2C_5} N_t^{C_6} \left( \sum_{i=1}^d |\mu_i| \right)^2 \right). \end{aligned}$$

This establishes the claim (2.11). We are now ready to prove (2.10). For any  $\delta, \lambda > 0$ , we have by (2.11),

$$\begin{aligned} & \mathbb{P}_x \left( \sup_{0 \leq t \leq 1} |A_{\varepsilon t}| > \delta \right) = \mathbb{P}_x \left( \sup_{0 \leq s \leq \varepsilon} |A_s| > \delta \right) \\ \leq & e^{-\lambda \delta} \mathbb{E}_x \left[ e^{\lambda \sup_{0 \leq s \leq \varepsilon} |A_s|} \right] \\ (2.13) \quad & \leq 2e^{-\lambda \delta} \exp \left( 2\lambda^2 \varepsilon C_4^2 e^{2C_5} N_\varepsilon^{C_6} \left( \sum_{i=1}^d |\mu_i| \right)^2 \right). \end{aligned}$$

Taking

$$\lambda = \frac{\delta}{4\varepsilon C_4^2 e^{2C_5} N_\varepsilon^{C_6} \left( \sum_{i=1}^d |\mu_i| \right)^2}$$

yields

$$(2.14) \quad \mathbb{P}_x \left( \sup_{0 \leq t \leq 1} |A_{\varepsilon t}| > \delta \right) \leq 2 \exp \left( - \frac{\delta^2}{8\varepsilon C_4^2 e^{2C_5} N_\varepsilon^{C_6} \left( \sum_{i=1}^d |\mu_i| \right)^2} \right).$$

Hence

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_x \left( \sup_{0 \leq t \leq 1} |A_{\varepsilon t}| > \delta \right) \\ & \leq \limsup_{\varepsilon \rightarrow 0} \left( \varepsilon \log 2 - \frac{\delta^2}{8C_4^2 e^{2C_5} N_\varepsilon^{C_6} \left( \sum_{i=1}^d |\mu_i| \right)^2} \right) \\ & = -\infty, \end{aligned}$$

where the last inequality is due to the fact that  $\lim_{\varepsilon \rightarrow 0} N_\varepsilon^{C_6} \left( \sum_{i=1}^d |\mu_i| \right) = 0$ . This completes the proof of Theorem 2.1.  $\square$

### 3. SMALL TIME ASYMPTOTICS

Recall that  $q(t, x, y)$  denotes the transition density of the Markov process  $\{X_t, t \geq 0\}$ . In this section we aim to establish the following Varadhan asymptotics.

**Theorem 3.1.**

$$(3.1) \quad \lim_{t \rightarrow 0} t \log q(t, x, y) = -\frac{|x - y|^2}{2}$$

uniformly in  $x, y \in \mathbb{R}^d$  such that  $|x - y|$  is bounded.

We will establish the upper bound and the lower bound separately. For  $a > 0$ , let

$$G_a(s, x, y) = s^{-d/2} \exp \left( -\frac{a|x - y|^2}{2s} \right).$$

Recall the following result from [Z1, Lemma 3.1].

**Lemma 3.2.** *Suppose  $0 < a_1 < a_2$ . There exist positive constants  $C_0$  and  $\alpha$  depending only on  $a_1, a_2$  such that*

$$\int_0^t \int_{\mathbb{R}^d} G_{a_1}(t - s, x, z) |b(z)| \frac{G_{a_2}(s, z, y)}{s^{1/2}} dz ds \leq C_0 N_t^\alpha (|b|) G_{a_1}(t, x, y).$$

**Proposition 3.3.** *Assume the drift measure  $\mu = (\mu_1, \dots, \mu_d)$  belongs to  $\mathbf{K}_{d,1}$ . Then*

$$\limsup_{t \rightarrow 0} t \log q(t, x, y) \leq -\frac{|x - y|^2}{2}$$

uniformly in  $x, y \in \mathbb{R}^d$  such that  $|x - y|$  is bounded.

*Proof.* We will prove the proposition by showing that for any  $\delta > 0$ , there exist constants  $T_\delta > 0, C_{2,\delta}$  such that for  $t \in (0, T_\delta]$ ,

$$(3.2) \quad q(t, x, y) \leq C_{2,\delta} t^{-d/2} \exp \left( -(1 - \delta) \frac{|x - y|^2}{2t} \right).$$

Using the approximation arguments as in [KS], it suffices to show that (3.2) holds for drift measure  $\mu$  given by  $\mu(dy) = b(y)dy$  and that the constant  $T_\delta$  is determined only by the quantity  $N_t^\alpha(\mu)$ . From now on we suppose  $\mu(dy) = b(y)dy$  for some function  $b = (b_1, \dots, b_d) \in \mathbf{K}_{d,1}$ . Note that for  $0 < \delta < 1$ , we have

$$\begin{aligned} |\nabla_z G_1(s, z, y)| &= \frac{|z - y|}{s} \exp \left( -\frac{\delta}{2} \frac{|z - y|^2}{2s} \right) G_{(1-\frac{\delta}{2})}(s, z, y) \\ (3.3) \quad &\leq m_\delta s^{-1/2} G_{(1-\frac{\delta}{2})}(s, z, y), \end{aligned}$$

where  $m_\delta := \sup_{r>0} r e^{-\delta r^2/2} = 1/\sqrt{e\delta}$ . Hence for every  $0 < \delta < 1$ , by Lemma 3.2 (with  $a_1 = 1 - \delta$  and  $a_2 = 1 - (\delta/2)$ ), there are positive constants  $C_\delta$  and  $c_\delta$  depending only on  $\delta$  so that

$$\begin{aligned}
 J(t, x, y) &:= \int_0^t \int_{\mathbb{R}^d} G_{(1-\delta)}(t-s, x, z) |b(z)| |\nabla_z G_1(s, z, y)| dz ds \\
 (3.4) \qquad &\leq C_\delta N_t^{c_\delta} (|b|) G_{(1-\delta)}(t, x, y).
 \end{aligned}$$

Define recursively  $I_k(t, x, y)$  as follows:

$$\begin{aligned}
 I_0(t, x, y) &= p(t, x, y) = (2\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{2t}\right), \\
 I_{k+1}(t, x, y) &= \int_0^t \int_{\mathbb{R}^d} I_k(t-s, x, z) b(z) \cdot \nabla_z p(s, z, y) dz ds \quad \text{for } k \geq 0.
 \end{aligned}$$

Recall the following identity (see e.g. [KS, Z1]):

$$(3.5) \qquad q(t, x, y) = \sum_{k=0}^\infty I_k(t, x, y).$$

We next show that for any  $\delta \in (0, 1)$ , it holds that

$$(3.6) \qquad |I_k(t, x, y)| \leq (C_\delta N_t^{c_\delta} (|b|))^k t^{-d/2} \exp\left(- (1-\delta) \frac{|x-y|^2}{2t}\right).$$

In view of (3.4), (3.6) clearly holds for  $k = 0, 1$ . Suppose (3.6) holds for  $k \geq 1$ . By induction and (3.4), we have

$$\begin{aligned}
 I_{k+1}(t, x, y) &\leq (C_\delta N_t^{\delta/2} (|b|))^k \int_0^t \int_{\mathbb{R}^d} G_{1-\delta}(t-s, x, z) |b(z)| |\nabla_z G_1(s, z, y)| dz ds \\
 &\leq (C_\delta N_t^{c_\delta} (|b|))^k C_\delta N_t^{c_\delta} (|b|) G_{1-\delta}(t, x, y) \\
 &= (C_\delta N_t^{c_\delta} (|b|))^{k+1} t^{-d/2} \exp\left(- (1-\delta) \frac{|x-y|^2}{2t}\right).
 \end{aligned}$$

Since  $\mu = b(x)dx \in \mathbf{K}_{d,1}$ , there exists a constant  $T_\delta > 0$  such that  $C_\delta N_t^{\delta/2} (|b|) \leq \frac{1}{2}$  for  $t \leq T_\delta$ . This together with (3.5) and (3.6) gives that

$$q(t, x, y) \leq 2t^{-d/2} \exp\left(- (1-\delta) \frac{|x-y|^2}{2t}\right).$$

From the proof above, we see that the constant  $T_\delta$  depends only on the rate at which  $N_t^{\delta/2} (|b|)$  tends to zero. Consequently,

$$t \log q(t, x, y) \leq - (1-\delta) \frac{|x-y|^2}{2} - (d/2)t \log(2t) \quad \text{for } t \in (0, T_\delta].$$

It follows that  $\limsup_{t \rightarrow 0} t \log q(t, x, y) \leq -\frac{|x-y|^2}{2}$  uniformly on compact subsets of  $\mathbb{R}^d \times \mathbb{R}^d$ . By an approximation procedure as in [KS], we assert that the proposition holds also for  $\mu \in \mathbf{K}_{d,1}$ . □

**Proposition 3.4.** *Suppose  $\mu = (\mu_1, \dots, \mu_d) \in \mathbf{K}_{d,1}$ . Then*

$$(3.7) \qquad \liminf_{t \rightarrow 0} t \log q(t, x, y) \geq -\frac{|x-y|^2}{2}$$

*uniformly in  $x, y \in \mathbb{R}^d$  such that  $|x-y|$  is bounded.*

*Proof.* For  $\varepsilon > 0$ , set  $B(y, \varepsilon) = \{z; |z - y| < \varepsilon\}$ . Let  $r > 0$ . We first would like to give an estimate for the probability  $\mathbb{P}_x(X_r \in B(y, \varepsilon))$ . Let  $\delta \in (0, \varepsilon)$ . We have

$$\begin{aligned}
 & \mathbb{P}_x(W_r + x \in B(y, \varepsilon - \delta)) = \mathbb{P}_x(X_r - A_r \in B(y, \varepsilon - \delta)) \\
 & \leq \mathbb{P}_x(X_r - A_r \in B(y, \varepsilon - \delta), |A_r| < \delta) + \mathbb{P}_x(|A_r| \geq \delta) \\
 (3.8) \quad & \leq \mathbb{P}_x(X_r \in B(y, \varepsilon)) + \mathbb{P}_x(|A_r| \geq \delta).
 \end{aligned}$$

In view of (2.14),

$$(3.9) \quad \mathbb{P}_x(|A_r| \geq \delta) \leq 2 \exp\left(-\frac{\delta^2}{8rC_4^2 e^{2C_5} N_r^{C_6} (\sum_{i=1}^d |\mu_i|)^2}\right).$$

On the other hand,

$$\begin{aligned}
 & \mathbb{P}_x(W_r + x \in B(y, \varepsilon - \delta)) = \int_{B(0, \varepsilon - \delta)} (2\pi r)^{-\frac{d}{2}} e^{-\frac{|z - (y-x)|^2}{2r}} dz \\
 (3.10) \quad & \geq (2\pi r)^{-d/2} \omega_d (\varepsilon - \delta)^d \exp\left(-\frac{(|x - y| + \varepsilon - \delta)^2}{2r}\right),
 \end{aligned}$$

where  $\omega_d$  is the volume of the unit ball in  $\mathbf{R}^d$ . Thus for  $0 < \delta < \varepsilon$ ,

$$\begin{aligned}
 & \mathbb{P}_x(X_r \in B(y, \varepsilon)) \geq (2\pi r)^{-d/2} \omega_d (\varepsilon - \delta)^d \exp\left(-\frac{(|x - y| + \varepsilon - \delta)^2}{2r}\right) \\
 (3.11) \quad & - 2 \exp\left(-\frac{\delta^2}{8rC_4^2 e^{2C_5} N_r^{C_6} (\sum_{i=1}^d |\mu_i|)^2}\right).
 \end{aligned}$$

Note that for every  $0 < \eta < 1$  and  $\varepsilon > 0$ ,

$$\begin{aligned}
 q(t, x, y) &= \int_{\mathbf{R}^d} q((1 - \eta)t, x, z) q(\eta t, z, y) dz \\
 &\geq \int_{B(y, \varepsilon)} q((1 - \eta)t, x, z) q(\eta t, z, y) dz \\
 &\geq \inf_{z \in B(y, \varepsilon)} q(\eta t, z, y) \int_{B(y, \varepsilon)} q((1 - \eta)t, x, z) dz \\
 (3.12) \quad &= \inf_{z \in B(y, \varepsilon)} q(\eta t, z, y) \mathbb{P}_x(X_{(1-\eta)t} \in B(y, \varepsilon)).
 \end{aligned}$$

Let  $M > 0$ . We have by (3.12), (1.5), and (3.11) with  $r = (1 - \eta)t$  that for any  $x, y \in \mathbf{R}^d$  with  $|x - y| \leq M$ ,

$$\begin{aligned}
 & t \log q(t, x, y) \\
 & \geq t \log \inf_{z \in B(y, \varepsilon)} q(\eta t, z, y) + t \log \mathbb{P}_x(X_{(1-\eta)t} \in B(y, \varepsilon)) \\
 & \geq t \log\left(C_1 e^{-C_2 t} (\eta t)^{-d/2}\right) - \frac{C_3 \varepsilon^2}{2\eta} + t \log\left((2\pi(1 - \eta)t)^{-d/2} \omega_d (\varepsilon - \delta)^d\right) \\
 & \quad - \frac{(|x - y| + \varepsilon - \delta)^2}{2(1 - \eta)} + t \log\left(1 - 2(2\pi(1 - \eta)t)^{d/2} \omega_d^{-1} (\varepsilon - \delta)^{-d}\right) \\
 & \quad \times \exp\left(\frac{(M + \varepsilon - \delta)^2}{2(1 - \eta)t} - \frac{\delta^2}{8(1 - \eta)t C_4^2 e^{2C_5} N_{(1-\eta)t}^{C_6} (\sum_{i=1}^d |\mu_i|)^2}\right).
 \end{aligned}$$

Since  $N_{(1-\eta)t}^{C_6} \left( \sum_{i=1}^d |\mu_i| \right)^2 \rightarrow 0$  as  $t \rightarrow 0$ , it follows that

$$(3.13) \quad \liminf_{t \rightarrow 0} t \log q(t, x, y) \geq -\frac{C_3 \varepsilon^2}{2\eta} - \frac{(|x - y| + \varepsilon - \delta)^2}{2(1 - \eta)}$$

uniformly in  $x, y$  with  $|x - y| \leq M$ . Now letting first  $\varepsilon \rightarrow 0$  and then  $\eta \rightarrow 0$ , we have

$$\liminf_{t \rightarrow 0} t \log q(t, x, y) \geq -\frac{|x - y|^2}{2}$$

uniformly in  $x, y \in \mathbb{R}^d$  with  $|x - y| \leq M$ . □

Combining Propositions 3.3 and 3.4 establishes Theorem 3.1.

#### ACKNOWLEDGMENTS

The authors thank James Norris and Renming Song for useful discussions.

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