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ON THE EVENTUAL LOCAL POSITIVITY FOR POLYHARMONIC HEAT EQUATIONS

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ABSTRACT. In this paper we show the eventual local positivity property for higher-order heat equations (including noninteger order). As a consequence, we give a positive answer for an open problem stated by Barbatis and Gazzola [Contemp. Math. 594 (2013)] for polyharmonic heat equations. Moreover, we obtain some polynomial decay properties of solutions.

1. INTRODUCTION

We are concerned with positivity issues for the higher-order linear parabolic equation

$$\begin{cases} u_t + (-\Delta)^\alpha u = 0, \text{ in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \text{ in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the positive parameter α can be a noninteger number. The number 2α is the order of the equation. Taking $\alpha = 1$ in (1.1), one obtains the classical heat equation

$$\begin{cases} u_t - \Delta u = 0, \text{ in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \text{ in } \mathbb{R}^n, \end{cases} \tag{1.2}$$

for which the positivity preserving property (for short, positivity) is well known because the heat fundamental solution

$$E(x,t) = t^{-n/2} f_{1,n}(t^{-1/2}|x|), \quad \text{where} \quad f_{1,n}(r) = \frac{1}{(4\pi)^{n/2}} \exp\left(-\frac{r^2}{4}\right),$$

is a positive function for $x \in \mathbb{R}^n$ and $t > 0$. In fact, the solution of (1.2), given by the convolution $u(x,t) = u_0 * E(\cdot,t)$, is positive provided that the initial condition u_0 is a nontrivial nonnegative continuous function. Positivity also holds for $\alpha \in (0,1)$, which corresponds to the fractional Laplacian case that has been widely studied in the literature (see, e.g., [6, 7]).

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On the other hand, the range $\alpha \in (1,\infty)$ corresponds to higher-order operators and the positivity type properties are less studied. In this case, comparison principles are unavailable and positivity often breaks down, leading to several difficulties in order to extend the results available for lower-order equations. We refer the reader to the book [17] for a nice survey about higher-order differential equations and their properties.

The biharmonic case $\alpha = 2$ was studied recently in [13, 15] and turned out to be notable. In contrast to the lower-order operator, positive initial conditions in (1.1) produce sign-changing solutions. In fact, it is always possible to find $\tau > 0$ such that for any $t > \tau$ there exists at least a point $x_t \in \mathbb{R}^n$ satisfying $u(x_t, t) < 0$.

Let the kernel $G_{\alpha,n}$ be defined by

$$G_{\alpha,n}(x,t) = (2\pi)^{-n/2} f_{\alpha,n}\left(|x|\, t^{-1/2\alpha}\right), \tag{1.3}$$

where

$$f_{\alpha,n}(r) = r^{-n} \int_0^\infty \exp\left(-\,(s/r)^{2\alpha}\right) s^{n/2}\, J_{(n-2)/2}(s)\, ds, \tag{1.4}$$

and let the J_m's stand for the Bessel functions of the first kind. Note that $f_{a,n}$ can be considered as a continuous functions on $[0,\infty)$ with $f_{\alpha,n}(0) > 0$ being the limit of the right-hand side of (1.4) as $r \to 0^+$. The solution of (1.1) can be expressed in the integral form

$$u(x,t) = (u_0 * G_{\alpha,n})(x,t) = \int_{\mathbb{R}^n} u_0(y)\, G_{\alpha,n}(y-x,t)\, dy, \tag{1.5}$$

for $u_0 \in C_c(\mathbb{R}^n)$ where C_c denotes the class of compactly supported continuous functions. For further details on the profile $f_{\alpha,n}$, see [2, 5]. It is worth noting that the oscillatory behavior of $f_{\alpha,n}$, which it inherits from the Bessel function in (1.4), is responsible for the failure of the positivity preserving property of (1.1).

As a general positivity result is unavailable, the question of the existence of weaker localized versions of that property arises. In this direction, in [15] there was shown eventual local positivity for the biharmonic equation with initial data $u_0 \in C_c(\mathbb{R}^n)$, as well as the fact that the negative part of the solution always exists. The term *eventual local positivity* means precisely: let u be the solution of (1.1) with initial data u_0. If $u_0 \geq 0$ is a nontrivial function and $K \subset \mathbb{R}^n$ is a given compact set, then for a certain $T > 0$ determined by K and u_0, we have that $u(x,t) > 0$ for all $x \in K$ and $t > T$. The results of [13] extended the eventual local positivity to initial data $u_0 = 1/(g(x) + |x|^\beta)$, where g is a nonnegative continuous function, $g(0) > 0$, and $g(x) = o(|x|^\beta)$ with $\beta \geq 0$ as $|x| \to \infty$. Eventual local positivity for the nonhomogeneous biharmonic equation $u_t + \Delta^2 u = h(x,t)$ was obtained in [3] under suitable conditions on the forcing term $h(x,t)$. Asymptotic behavior results as $t \to 0^+$ (and fixed $x \in \mathbb{R}^n$) for fundamental solutions of parabolic polyharmonic equations were proved in [11] by assuming that the symbol of the elliptical differential operator is a positive definite polynomial, homogeneous of degree $2m$, and convex in a suitable sense. For existence, blow-up, positivity, decay, and asymptotic behavior results on semilinear parabolic and elliptic problems involving higher order PDEs, see, e.g., [4, 8–10, 12, 14, 19] and their references.

In view of the above results, it is natural to wonder about some kind of positivity property for values of α higher than the biharmonic case. Motivated by that, in the survey [2] (see Problem 10, p. 18), the eventual local positivity problem for (1.1) with $u_0 \in C_c(\mathbb{R}^n)$ and $\alpha = m \geq 3$ integer was suggested as an open problem. Here we prove the eventual local positivity property for (1.1) with $\alpha \in (1, \infty)$ (including noninteger values of α), giving, in particular, a positive answer for that open problem (see Theorems 1.1 and 1.3 below). Moreover, we obtain some decay properties for solutions of (1.1) (see Section 4). Since the case $\alpha = 2$ is also covered here, we reobtain the eventual local positivity property in the biharmonic case; however, our arguments give a proof different from those of [13, 15].

For the reader's convenience, we summarize some notation that will be used throughout the paper:

(1.6)

2α	equation order
n	space dimension
$B_R(x_0)$	the open ball in $\mathbb{R}^n$ with radius R and center x_0
$\text{supp}(u_0)$	the support of u_0
γ_l	the lth positive zero of $f_{\alpha,n}$

Let $\text{supp}(u_0)$ be a nonempty compact $H \subset \mathbb{R}^n$. We recall that the minimal enclosing ball of H is the closed ball $\mathcal{B}[H]$ with minimal radius among all closed balls containing H. Notice that $\mathcal{B}[H]$ is well-defined since an intersection of two noncoinciding closed balls of equal radius is contained in a closed ball containing H of smaller radius. $\mathcal{B}[H]$ depends only on H. From now on, we denote the radius and center of $\mathcal{B}[H]$ by ρ and y_0, respectively (i.e., $\mathcal{B}[H] = \overline{B_\rho(y_0)}$).

Our first result extends eventual local positivity to real numbers $\alpha > 1$.

Theorem 1.1. *Assume that $\alpha > 1$. Let u_0 be a nontrivial nonnegative $C_c(\mathbb{R}^n)$-function, and let u be the solution of* (1.1). *Given $R > 0$ and $x_0 \in \mathbb{R}^n$, set*

$$\tau = \left(\frac{R + \rho + |x_0 - y_0|}{\gamma_1} \right)^{2\alpha}. \tag{1.7}$$

Then

$$u(x,t) > 0 \quad \text{for all } x \in B_R(x_0) \text{ and } t \geq \tau. \tag{1.8}$$

Remark 1.2. In the theorem above we can consider an arbitrary compact $K \subset \mathbb{R}^n$ instead of $B_R(x_0)$. In fact, it is sufficient to take R_K and x_K such that $K \subset B_{R_K}(x_K)$ and apply (1.8) with $\tau = \tau_K = [(R_K + \rho + |x_K - y_0|)/\gamma_1]^{2\alpha}$.

The next result shows that nontrivial nonnegative initial data give rise to sign-changing solutions to (1.1) when $\alpha > 1$ is an integer.

Theorem 1.3. *Assume that $\alpha = m > 1$ is an integer. Let $u_0 \geq 0$ be a nontrivial $C_c(\mathbb{R}^n)$-function, and let u be the solution of* (1.1). *Then, there exists*

$$\sigma > \left(\frac{3\rho}{\gamma_2} \right)^{2m} \tag{1.9}$$

such that the negative part u^- of u satisfies $u^-(\cdot, t) \not\equiv 0$ for all $t > \sigma$.

In view of (1.7), a natural question is: what is the threshold time value τ_* to obtain positivity on a given compact set? In the next remark, we discuss about the existence and estimates for τ_*.

Remark 1.4. From (1.7), we have that given $R > 0$, there exists a threshold time value τ_*, depending on the support of u_0, such that the solution $u(x,t)$ is positive in $B_R(x_0)$ for $t > \tau_*$. For simplicity, let $x_0 = 0$, $y_0 = 0$, and $\text{supp}(u_0) = \overline{B_\rho(0)}$. Combining Theorems 1.1 and 1.3, we are able to get lower and upper bounds for τ_*. Let γ_1, γ_2 be as in (1.6), and let $R > \frac{\gamma_2+\gamma_1}{\gamma_2-\gamma_1}\rho$. Let $\delta = \gamma_1/\gamma_2$ and x_δ be such that $r(x_\delta) = \delta R(x_\delta)$, where $r(x_\delta) = |x_\delta| - \rho$ and $R(x_\delta) = |x_\delta| + \rho$ (see Remark 3.2 and the definitions of $r(x)$ and $R(x)$ in (3.4)). Theorem 1.3 gives us that $u(x_\delta, t_\delta) < 0$ for some $t_\delta > \sigma$, and then $\tau_* > t_\delta$ because $x_\delta \in B_R(0)$. Thus, we can bound τ^* by

$$\left(\frac{3\rho}{\gamma_2}\right)^{2m} < \tau^* \leq \left(\frac{\rho + R}{\gamma_1}\right)^{2m}.$$

The paper is organized as follows. The next section is devoted to core properties of the profile $f_{\alpha,n}$. With these properties in hand, in Section 3 we prove Theorems 1.1 and 1.3. Finally, in Section 4 we show decay properties of solutions.

2. Some properties of the profile $f_{\alpha,n}$

This section is devoted to some properties of $f_{\alpha,n}$ and related functions. In view of (1.3), such properties can be naturally extended for the kernel $G_{\alpha,n}$ and are useful to study solutions of (1.1). Positivity properties involving weighted integrals of the profile of polyharmonic heat kernels, such as Lorch–Szegö-type monotonicity results, were obtained in [16] by techniques different from ours.

Applying the change of variables $s \mapsto s/r$ in (1.4), we obtain

$$f_{\alpha,n}(r) = r^{1-n/2}\int_0^\infty \exp(-s^{2\alpha}) s^{n/2} J_{(n-2)/2}(rs)\, ds. \tag{2.1}$$

In order to prove Theorem 1.1, we establish positivity of $f_{\alpha,n}$ in a neighborhood of 0. To do so, we use a recursive procedure with the base steps $n = 2$ and $n = 3$.

Consider the family of functions

$$\varphi_{\alpha,\beta,\gamma}(r) = \int_0^\infty \exp(-s^{2\alpha}) J_{\frac{\beta-2}{2}}(rs) s^{\frac{\gamma-1}{2}}\, ds, \tag{2.2}$$

where $\alpha > 0$, $\beta \geqslant 1$, and $\gamma \geqslant 1$. It follows from (2.1) that

$$f_{\alpha,n}(r) = r^{1-n/2}\varphi_{\alpha,n,n+1}(r). \tag{2.3}$$

Then, for appropriate values of α, β, γ, the functions $\varphi_{\alpha,\beta,\gamma}$ have the same sign as our kernel profile $f_{\alpha,n}$. The next lemma collects some facts about that family.

Lemma 2.1. *Let $\alpha > 0$, $\beta \geqslant 1$, and $\gamma \geqslant 1$. Then,*

$$(r\,\varphi_{\alpha,\beta,\gamma}(r))' = r\,\varphi_{\alpha,\beta-2,\gamma+2}(r). \tag{2.4}$$

Moreover, for $k \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(\alpha, k) > 0$ such that

$$\varphi_{\alpha,2,1+2k}(r) > 0, \quad r \in (0, \varepsilon), \tag{2.5}$$

and

$$\varphi_{\alpha,1,2+2k}(r) > 0, \quad r \in (0, \varepsilon). \tag{2.6}$$

Assuming further $\beta \geq 2$, we have that

$$\lim_{r\to 0^+} r\varphi_{\alpha,\beta,\gamma}(r) = 0. \tag{2.7}$$

Proof. Taking the derivative in (2.2) and recalling that (see, e.g., [1])

$$J_\nu(r)' = J_{\nu-1}(r) - r^{-1}J_\nu(r),$$

we get

$$(\varphi_{\alpha,\beta,\gamma}(r))' = \int_0^\infty \exp(-s^{2\alpha}) J_{\frac{\beta-4}{2}}(rs)s^{\frac{\gamma+1}{2}}\,ds - r^{-1}\int_0^\infty \exp(-s^{2\alpha})J_{\frac{\beta-2}{2}}(rs)s^{\frac{\gamma-1}{2}}\,ds$$

$$= \varphi_{\alpha,\beta-2,\gamma+2}(r) - r^{-1}\varphi_{\alpha,\beta,\gamma}(r). \tag{2.8}$$

Multiplying by r and collecting like terms, it follows that

$$(r\,\varphi_{\alpha,\beta,\gamma}(r))' = r\,\varphi_{\alpha,\beta,\gamma}(r)' + \varphi_{\alpha,\beta,\gamma}(r) = r\,\varphi_{\alpha,\beta-2,\gamma+2}(r),$$

which gives (2.4).

Next we turn to (2.5) and (2.6). For $\beta = 2$, we have that

$$\varphi_{\alpha,2,1+2k}(r) = \int_0^\infty \exp(-s^{2\alpha})\,s^k\,J_0(rs)\,ds.$$

For $\beta = 1$, the Bessel function takes the simpler form

$$J_{-1/2}(s) = \sqrt{\frac{2}{\pi s}}\cos(s),$$

and then

$$\varphi_{\alpha,1,2+2k}(r) = (2/\pi)^{1/2}r^{-1/2}\int_0^\infty \exp(-s^{2\alpha})\,s^k\,\cos(rs)\,ds.$$

Thus, for $\beta = 1, 2$, we can write

$$\varphi_{\alpha,\beta,3-\beta+2k}(r) = C_\beta(r)\int_0^\infty \exp(-s^{2\alpha})\,s^k\,q(rs)\,ds,$$

where $C_1(r) = (2/\pi)^{1/2}r^{-1/2}$ and $C_2(r) = 1$; $q(0) > 0$, $q(s) > 0$, and $q'(s) < 0$ in $(0, z_1)$ for some suitable $z_1 > 0$; and $|q(s)| \leqslant 1$ for all s.

Now we split the integral at z_1/r to obtain

$$\begin{aligned}\varphi_{\alpha,\beta,3-\beta+2k}(r) &= C_\beta(r)\int_0^{z_1/r} \exp(-s^{2\alpha})\,s^k\,q(rs)\,ds \\ &\quad + C_\beta(r)\int_{z_1/r}^\infty \exp(-s^{2\alpha})\,s^k\,q(rs)\,ds \\ &:= I_1 + I_2.\end{aligned} \tag{2.9}$$

First, we estimate I_2. For that, let r_0 be such that

$$C_\beta(r)\exp(-s^{2\alpha})s^k \leqslant C\exp(-s^{2\alpha})s^{k+\frac{1}{2}} \leqslant C\exp(-s^\alpha),$$

for all $s > z_1/r$ and $0 < r \leqslant r_0$. It follows that

$$|I_2| \leqslant \int_{z_1/r}^\infty C\exp(-s^\alpha)ds \to 0, \quad z_1/r \to \infty.$$

Now notice that $q(rs) \to q(0) > 0$ as $r \to 0$ for each $s > 0$. Also, we have the inequality

$$q(rs) < q(ts), \text{ where } r > t > 0,$$

for all $s \in (0, z_1/r)$. From the Monotone Convergence Theorem, it follows that

$$\int_0^{z_1/r} \exp(-s^{2\alpha})\,s^k\,q(rs)\,ds \to q(0)\int_0^\infty \exp(-s^{2\alpha})\,s^k\,ds > 0, \tag{2.10}$$

where the last integral in (2.10) is independent of r. From (2.9), we have the lower bound
(2.11)

$$\varphi_{\alpha,\beta,3-\beta+2k}(r) \geqslant C_\beta(r)\int_0^{z_1/r} \exp(-s^{2\alpha})\, s^k\, q(rs)\, ds - o(r) > 0, \text{ for } 0<r<r_0.$$

Properties (2.5) and (2.6) follow from (2.10) and (2.11).

Finally, we turn to (2.7). Recall that $|J_d(s)| \leqslant C(1+s)^d$, for $s \geq 0$, when $d \geq 0$. Since $\gamma \geq 1$, $\beta \geq 2$, and $\exp(-s^{2\alpha})\,(1+s)^{\frac{\beta+\gamma-3}{2}} \in L^1([0,\infty))$, for $0<r<1$ we can estimate

$$\begin{aligned} |r\varphi_{\alpha,\beta,\gamma}(r)| &= \left| r\int_0^\infty \exp(-s^{2\alpha}) s^{\frac{\gamma-1}{2}}\, J_{\frac{\beta-2}{2}}(rs) ds\right| \\ &\leqslant Cr\int_0^\infty \exp(-s^{2\alpha}) s^{\frac{\gamma-1}{2}}\,(1+rs)^{\frac{\beta-2}{2}} ds \\ &\leqslant Cr\int_0^\infty \exp(-s^{2\alpha})\,(1+s)^{\frac{\beta+\gamma-3}{2}} ds \to 0, \text{ as } r\to 0^+, \end{aligned}$$

and we are done. □

Next we are ready to prove that $f_{\alpha,n}(s)$ is positive near $s=0$.

Lemma 2.2. *Let $\alpha \geq 1$ and $n \geqslant 2$. Then $f_{\alpha,n}(s) > 0$ in $(0,\varepsilon)$ for sufficiently small ε.*

Proof. First consider $n = 2j$. We proceed by induction on j to prove that for all $k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\varphi_{\alpha,2j,1+2k}(s) > 0$ in $(0,\varepsilon)$. For $j = 1$, this is exactly (2.5). Assume now that for all $k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\varphi_{\alpha,2j-2,2k+1}(s) > 0$ in $(0,\varepsilon)$. In particular, by replacing k by $k+1$, it also holds true for $\varphi_{\alpha,2j-2,2k+3}(s)$ (possibly changing ε). In view of (2.4), $r\,\varphi_{\alpha,2j,1+2k}(r)$ is increasing in $(0,\varepsilon)$. Thus, it follows from (2.7) that $r\,\varphi_{\alpha,2j,1+2k}(r) > 0$, and so $\varphi_{\alpha,2j,1+2k}(r) > 0$ in $(0,\varepsilon)$.

As for the case $n = 2j+1$, we proceed similarly by employing (2.6) and (2.7) to show that $\varphi_{\alpha,2j+1,2+2k}(s) > 0$ in $(0,\varepsilon)$ for some $\varepsilon > 0$ depending on α, k, and j. Putting together both cases, notice that, in particular, we obtain the positivity of $\varphi_{\alpha,n,n+1}(s)$ in $(0,\varepsilon)$ for some $\varepsilon > 0$ and $n \geq 2$. The proof is concluded by using (2.3). □

For the reader convenience, in Figure 2 we display the functions $f_{m,2}$, corresponding to the m-harmonic kernel in two dimensions for integer m from 2 through 10, illustrating the oscillatory behavior, as well as the decay. In Figures 2 and 2 the reader can find the graphs of $f_{\alpha,n}$ for $n = 2,3,4,5$ in the cases $\alpha = 3/2$ and $\alpha = 5/2$, respectively.

3. Proof of Theorems 1.1 and 1.3

Proof of Theorem 1.1. According to Lemma 2.2, we have that $f_{\alpha,n}$ is positive in a neighborhood of 0. It follows that it remains positive until its first zero γ_1, i.e.,

$$f_{\alpha,n}(r) > 0 \text{ for all } r \in (0,\gamma_1). \tag{3.1}$$

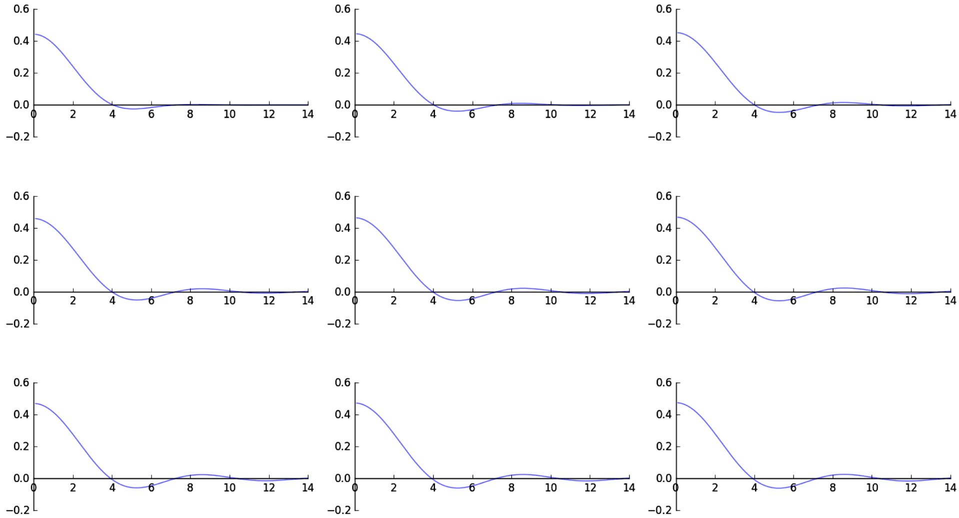

FIGURE 1. The family of functions $f_{m,2}$ for m from 2 to 10

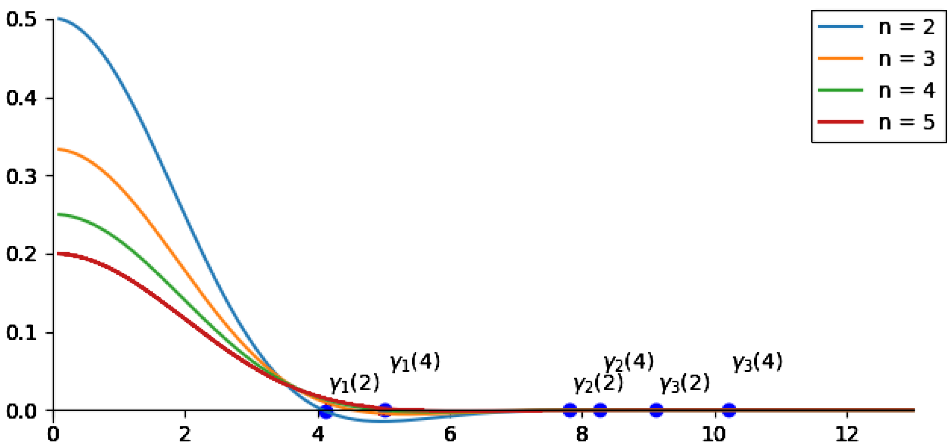

FIGURE 2. The function $f_{\alpha,n}$ and its zeros for $\alpha = 3/2$

Let $x_0 \in \mathbb{R}^n$ and $R > 0$, and take $x \in B_R(x_0)$. Making the change of variables $z = t^{-1/2\alpha}(y - x)$ in (1.5) and using (1.3), we can write the solution u as

$$u(x,t) = (2\pi)^{-n/2} t^{-n/2\alpha} \int_{\mathbb{R}^n} u_0(x + t^{1/2\alpha} z)\, f_{\alpha,n}(|z|)\, dz. \tag{3.2}$$

From (3.2) and (3.1), in order to obtain $u(x,t) > 0$, it is sufficient to have

$$u_0(x + t^{1/2\alpha} z) = 0, \text{ when } |z| > \gamma_1.$$

Assume that $|z| > \gamma_1$. We know that $u_0(x + t^{1/2\alpha} z) = 0$ if $x + t^{1/2\alpha} z \notin \overline{B_\rho(y_0)}$. Also, we have that

$$|x + t^{1/2\alpha} z - y_0| \geqslant t^{1/2\alpha}|z| - |x - y_0| > t^{1/2\alpha}\gamma_1 - |x - x_0| - |x_0 - y_0|.$$

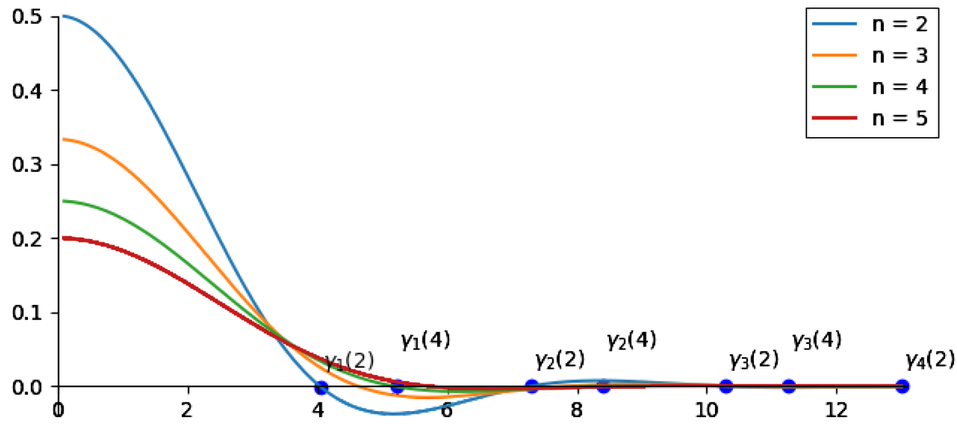

FIGURE 3. The function $f_{\alpha,n}$ and its zeros for $\alpha = 5/2$

Thus, we obtain $|x + t^{1/2\alpha}z - y_0| > \rho$, provided that

$$|x - x_0| \leqslant t^{1/2\alpha}\gamma_1 - |x_0 - y_0| - \rho. \tag{3.3}$$

We finish the proof by noting that (3.3) holds true for all

$$t \geq \left(\frac{R + \rho + |x_0 - y_0|}{\gamma_1}\right)^{2\alpha} = \tau.$$

□

For $u_0 \in C_c(\mathbb{R}^n)$, set

$$R(x) = \sup_{y \in \text{supp}(u_0)} |y - x| \quad \text{and} \quad r(x) = \inf_{y \in \text{supp}(u_0)} |y - x|. \tag{3.4}$$

Recalling the definition of the minimal enclosing ball $\overline{B_\rho(y_0)}$ of $\text{supp}(u_0)$, it follows that

$$|x - y_0| - \rho \leqslant r(x) \leqslant R(x) \leqslant |x - y_0| + \rho.$$

Notice that if $u_0 \not\equiv 0$, then $r(x) < R(x)$ for all $x \in \mathbb{R}^n$. Finally, for $\kappa > 0$ we have the estimate

$$\sup_{z \in B_\kappa(x)} R(z) \leqslant \kappa + \rho + |x - y_0|.$$

For Theorem 1.3 we need the following lemma.

Lemma 3.1. *Let $f : [0, \infty) \to \mathbb{R}$ be continuous, and let $0 < a < b$ be such that $f(s) < 0$ for $s \in (a, b)$. If $u_0 \in C_c(\mathbb{R}^n)$, $0 \not\equiv u_0 \geq 0$, and $x \in \mathbb{R}^n \setminus \text{supp}(u_0)$ satisfies*

$$\frac{a}{b} \leq \frac{r(x)}{R(x)}, \tag{3.5}$$

then

$$\int_{\mathbb{R}^n} u_0(x + t^{1/\alpha} z) f(|z|)\, dz < 0, \quad \textit{for } t \in [(R(x)/b)^{2\alpha}, (r(x)/a)^{2\alpha}].$$

Proof. Consider the set

$$A_{x,t} = \left\{ z \in \mathbb{R}^n ; u_0(x + t^{1/2\alpha} z) \neq 0 \right\}.$$

By continuity of $u_0 \not\equiv 0$, it follows that $A_{x,t}$ is a nonempty open set and

$$I_{x,t} = \int_{\mathbb{R}^n} u_0(x + t^{1/2\alpha}\, z) f(|z|)\, dz = \int_{A_{x,t}} u_0(x + t^{1/2\alpha}\, z) f(|z|)\, dz. \tag{3.6}$$

Moreover, for any $z \in A_{x,t}$ we have that $x + t^{1/2\alpha} z \in \operatorname{supp}(u_0)$, and then

$$r(x) \leqslant |x - (x + t^{1/2\alpha}\, z)| = |z| t^{1/2\alpha} \leqslant R(x).$$

Thus, in order to obtain $I_{x,t} < 0$, it suffices to have

$$a \leqslant r(x)\, t^{-1/2\alpha} \quad \text{and} \quad R(x)\, t^{-1/2\alpha} \leqslant b,$$

which is equivalent to

$$\frac{R(x)}{b} \leqslant t^{1/2\alpha} \leq \frac{r(x)}{a}.$$

Due to (3.5), we can choose t satisfying both inequalities. □

Remark 3.2. Note that $0 < \frac{a}{b} < 1$ and the quotient $r(x)/R(x)$ has limit 1 as $|x| \to \infty$, whereas it vanishes for $x \in \operatorname{supp}(u_0)$. As $r(x)$ and $R(x)$ are continuous functions and $R(x) > 0$ (for $u_0 \not\equiv 0$), for any $\frac{a}{b} < \delta < 1$ the level set δ is nonempty. In other words, there exists $x_\delta \in \mathbb{R}^n$ such that

$$\frac{r(x_\delta)}{R(x_\delta)} = \delta \in \left(\frac{a}{b}, 1\right).$$

For such x_δ, we have that

$$u(x_\delta, t) < 0 \text{ whenever } \left(\frac{R(x_\delta)}{b}\right)^{2\alpha} \leq\ t\ \leq \left(\frac{\delta R(x_\delta)}{a}\right)^{2\alpha}. \tag{3.7}$$

Proof of Theorem 1.3. From [18], we know that $G_{\alpha,n}$ changes sign infinitely many times for $\alpha = m \in \mathbb{N}$ with $m \geq 2$. We can set $\gamma_1 = \sup\{s > 0;\ f_{m,n} \geqslant 0 \text{ in } [0, s]\}$ and $\gamma_2 = \sup\{s > \gamma_1;\ f_{m,n} \leqslant 0 \text{ in } [\gamma_1, s]\}$. It follows that $0 < \gamma_1 < \gamma_2$. Taking f as the kernel profile $f_{m,n}$, and $a = \gamma_1$ and $b = \gamma_2$ in Lemma 3.1, we get

$$u(x,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u_0(x + t^{1/2m}\, y) f_{m,n}(|y|)\, dy < 0,$$

provided that $x \in \mathbb{R}^n \setminus \operatorname{supp}(u_0)$ and $t \in J_x$, where $J_x = [(R(x)/\gamma_2)^{2m}, (r(x)/\gamma_1)^{2m}]$ and $\gamma_1/\gamma_2 \leqslant r(x)/R(x)$. Notice that $\rho < R(x) < 3\rho$ for $x \in \partial B_\rho(y_0)$ and recall that $R(x) \to \infty$ as $|x| \to \infty$. For each $t > \sigma$, take $x_t \in \mathbb{R}^n \setminus \operatorname{supp}(u_0)$ such that $|x_t|$ is sufficiently large and

$$\sigma = \left(\frac{3\rho}{\gamma_2}\right)^{2m} < \left(\frac{R(x_t)}{\gamma_2}\right)^{2m} = t \leq \left(\frac{r(x_t)}{\gamma_1}\right)^{2m}.$$

It follows that $t \in J_{x_t}$ and then $u(x_t, t) < 0$ and $u^-(\cdot, t) \not\equiv 0$. □

4. Polynomial decay of solutions

In this last section, we show that solutions of (1.1) satisfy $u(x,t) = \mathcal{O}(|x|^{-(n+2\alpha)})$, as $|x| \to \infty$, for each fixed $t > 0$. For that, first recall that the kernel profile $G_{\alpha,n}$ can be implicitly defined via Fourier transform as

$$e^{-t|\eta|^{2\alpha}} = \int_{\mathbb{R}^n} e^{-i(x,\eta)} G_{\alpha,n}(t,x) dx \tag{4.1}$$

and the scaling property

$$G_{\alpha,n}(x,t) = t^{-\frac{n}{2\alpha}} G_{\alpha,n}(t^{-\frac{1}{2\alpha}} x, 1). \tag{4.2}$$

Due to (4.2), we can deduce decay properties for $G_{\alpha,n}(x,t)$ by studying $G_{\alpha,n}(x,1)$. Since the latter is radially symmetric, expressing the inverse Fourier transform of a radial function as a Hankel integral, we get

$$G_{\alpha,n}(x,1) = (2\pi)^{-\frac{n}{2}} |x|^{1-\frac{n}{2}} \int_0^\infty e^{-s^{2\alpha}} s^{\frac{n}{2}} J_{\frac{n-2}{2}}(s|x|) ds = (2\pi)^{-\frac{n}{2}} f_{\alpha,n}(|x|).$$

By scaling, we arrive at

$$G_{\alpha,n}(x,t) = (2\pi)^{-\frac{n}{2}} |x|^{1-\frac{n}{2}} t^{1-\frac{n}{2}-\frac{n}{2\alpha}} \int_0^\infty e^{-s^{2\alpha}} s^{\frac{n}{2}} J_{\frac{n-2}{2}}(s|x|t^{-\frac{1}{2\alpha}}) ds.$$

The limiting value of $G_{\alpha,n}(x,1)$ is expressed as (see [5, Theorem 2.1, p. 263])

$$\lim_{|x|\to\infty} |x|^{n+2\alpha} G_{\alpha,n}(x,1) = \alpha 2^{2\alpha} \pi^{-\frac{n}{2}-1} \sin(\alpha\pi) \Gamma(\alpha + n/2) \Gamma(\alpha) \tag{4.3}$$

$$= C_\alpha \sin(\alpha\pi); \quad C_\alpha > 0. \tag{4.4}$$

We will explore (4.3) in the remainder of this section. For large $|x|$, we have $G_{\alpha,n}(x,1) \approx C_\alpha \sin(\alpha\pi)|x|^{-2\alpha-n}$; therefore, for $x = t^{-\frac{1}{2\alpha}} y$ with $|y|t^{-\frac{1}{2\alpha}}$ large, we obtain

$$\begin{aligned} G_{\alpha,n}(y,t) &\approx t^{-\frac{n}{2\alpha}} C_\alpha \sin(\alpha\pi) |y|^{-2\alpha-n} (t^{-\frac{1}{2\alpha}})^{-2\alpha-n} \\ &= t^{-\frac{n}{2\alpha}} t^{1+\frac{n}{2\alpha}} |y|^{-2\alpha-n} C_\alpha \sin(\alpha\pi) \\ &= C_\alpha \sin(\alpha\pi) t |y|^{-2\alpha-n}. \end{aligned}$$

Case 1. Let $j \in \mathbb{N}$, let $\alpha \in (2j, 2j+1)$, and note that $\sin(\alpha\pi) > 0$. Thus, there exists $R_0 = R_0(\alpha) > 0$ such that

$$\frac{1}{2} C_\alpha \sin(\alpha\pi) |x|^{-2\alpha-n} t \leqslant G_{\alpha,n}(x,t) \leq \frac{3}{2} C_\alpha \sin(\alpha\pi) |x|^{-2\alpha-n} t,$$

whenever $|x| \geqslant R_0\, t^{\frac{1}{2\alpha}}$. Without loss of generality, assume that the center of the minimal enclosing ball $\overline{B_\rho(y_0)}$ of $\mathrm{supp}(u_0)$ is zero. We have that $\mathrm{diam}(\mathrm{supp}(u_0)) \leq 2\rho$. Since $G_{\alpha,n}(\cdot,t)$ is positive in $\mathbb{R}^n \setminus B_{R_0\, t^{\frac{1}{2\alpha}}}(0)$, it follows that

$$u(x,t) = \int_{x-\mathrm{supp}(u_0)} u_0(x-y) G_{\alpha,n}(y,t)\, dy > 0,$$

when $x - \mathrm{supp}(u_0) \subset \mathbb{R}^n \setminus B_{R_0\, t^{\frac{1}{2\alpha}}}(0)$; in particular, for $|x| \geqslant \rho + R_0\, t^{\frac{1}{2\alpha}}$. In this case, we can estimate

$$\begin{aligned} u(x,t) &\geqslant \int_{x-\mathrm{supp}(u_0)} u_0(x-y) \frac{1}{2} C_\alpha \sin(\alpha\pi)\, t\, |y|^{-2\alpha-n}\, dy \\ &= \frac{1}{2} C_\alpha \sin(\alpha\pi)\, t \int_{x-\mathrm{supp}(u_0)} u_0(x-y) |y|^{-2\alpha-n} dy \\ &\geq \frac{1}{2} C_\alpha \sin(\alpha\pi)\, t (|x|+\rho)^{-2\alpha-n} \int_{x-\mathrm{supp}(u_0)} u_0(x-y) dy \\ &\geqslant \frac{1}{2} C_\alpha \sin(\alpha\pi)\, t\, (|x|+\rho)^{-2\alpha-n} \|u_0\|_{L^1} > 0, \end{aligned} \tag{4.5}$$

for $|x| \geqslant \rho + R_0\, t^{\frac{1}{2\alpha}}$. Also, we have the upper estimate

$$\begin{aligned} u(x,t) \leqslant & \int_{x-\mathrm{supp}(u_0)} u_0(x-y)\frac{3}{2}C_\alpha \sin(\alpha\pi)\, t\, |y|^{-2\alpha-n}\, dy \\ =& \frac{3}{2}C_\alpha \sin(\alpha\pi)\, t \int_{x-\mathrm{supp}(u_0)} u_0(x-y)|y|^{-2\alpha-n} dy \\ \leqslant& \frac{3}{2}C_\alpha \sin(\alpha\pi)\, t\, (|x|-\rho)^{-2\alpha-n}\|u_0\|_{L^1}. \end{aligned}$$

Case 2. For $\alpha \in (2j+1, 2j+2)$, we have $\sin(\alpha\pi) < 0$, and $G_{\alpha,n}(\cdot,t)$ is negative in the complement of the ball $B_{R_0\, t^{\frac{1}{2\alpha}}}(0)$ where $R_0 = R_0(\alpha) > 0$. So, proceeding analogously to Case 1, we can obtain

$$-\frac{3}{2}C_\alpha|\sin(\alpha\pi)|\|u_0\|_{L^1} t \leqslant (|x|-\rho)^{2\alpha+n} u(x,t)$$

and

$$(|x|+\rho)^{2\alpha+n} u(x,t) \leq -\frac{1}{2}C_\alpha|\sin(\alpha\pi)|\|u_0\|_{L^1} t.$$

Case 3. For $\alpha = m$ a positive integer, it follows that

$$\lim_{|x|\to\infty} |x|^{n+\alpha} G_{\alpha,n}(x,1) = 0,$$

and then, given $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$-\varepsilon|x|^{-n-2\alpha} \leqslant G_{\alpha,n}(x,1) \leq \varepsilon|x|^{-n-2\alpha}, \text{ for } |x| > R_\varepsilon.$$

Using (4.2), we arrive at

$$-\varepsilon t|x|^{-n-2\alpha} \leqslant G_{\alpha,n}(x,t) \leq \varepsilon t|x|^{-n-2\alpha}, \text{ for } |x| > R_\varepsilon t^{\frac{1}{2\alpha}}.$$

Thus, for $|x|$ sufficiently large w.r.t. t (e.g., $|x| > R_\varepsilon t^{\frac{1}{2\alpha}} + \rho$), we obtain that

$$\begin{aligned} |u(x,t)| \leqslant & \int_{x-\mathrm{supp}(u_0)} u_0(x-y) G_{\alpha,n}(y,t) dy \\ \leqslant& \varepsilon t \int_{x-\mathrm{supp}(u_0)} u_0(x-y)|y|^{-n-2\alpha} dy \\ \leqslant& \varepsilon t(|x|-\rho)^{-n-2\alpha}\|u_0\|_{L^1}, \end{aligned}$$

which gives the desired decay property in this last case.

Remark 4.1. Notice that the estimate (4.5) allows us to obtain positivity of the solution for x sufficiently large with respect to the time variable. More precisely, let $u_0 \in C_c(\mathbb{R}^n)$ and $\alpha \in (2j, 2j+1)$ for some $j \in \mathbb{N}$. There exists $R = R(\alpha)$ such that the solution u satisfies

$$u(x,t) > 0 \;\text{ for }\; |x| > \rho + R\, t^{\frac{1}{2\alpha}}.$$

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