PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, Number 10, October 2019, Pages 4371–4378
https://doi.org/10.1090/proc/14619
Article electronically published on June 10, 2019

K-THEORY FOR GENERALIZED LAMPLIGHTER GROUPS

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(Communicated by Adrian Ioana)

ABSTRACT. We compute K-theory for the reduced group C*-algebras of generalized Lamplighter groups.

1. INTRODUCTION

The classical Lamplighter group is given by the semidirect product $(\bigoplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})) \rtimes \mathbb{Z}$, where the $\mathbb{Z}$-action on $\bigoplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ is induced by the canonical translation action of $\mathbb{Z}$ on itself. This construction can be generalized by replacing $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$ by other groups. The classical Lamplighter group and its generalizations are important examples in group theory which led to solutions of several open problems (see for instance [6, 7, 10]).

The goal of these notes is to derive a K-theory formula for group C*-algebras of generalized Lamplighter groups of the form $(\bigoplus_{\Gamma} \Sigma) \rtimes \Gamma$, where Σ is an arbitrary finite group and Γ is an arbitrary countable group. As in the classical setting, the Γ-action on $\bigoplus_{\Gamma} \Sigma$ is induced by the canonical left translation action of Γ on itself. Our computations are inspired by [9, 13], which treat the special case of free groups Γ ([9] deals with the case $\Gamma = \mathbb{Z}$). Our method, however, is completely different from the ones adopted in [9, 13].

Our main result reads as follows. Let Σ be a finite group and let Γ be a countable group. Let $\mathrm{con}\ \Sigma$ be the set of conjugacy classes in Σ, and let $\mathrm{con}^{\times}\ \Sigma := \mathrm{con}\ \Sigma \setminus \{\{1\}\}$ be the set of non-trivial conjugacy classes. Let $\mathcal{C}$ be the set of conjugacy classes of finite subgroups of Γ. For a finite subgroup C of Γ, let $F(C)$ be the set of non-empty finite subsets of the right coset space $C\backslash\Gamma$ which are not of the form $\pi^{-1}(Y)$ for a finite subgroup $D \subseteq \Gamma$ with $C \subsetneq D$ and $Y \subseteq D\backslash\Gamma$, where $\pi : C\backslash\Gamma \twoheadrightarrow D\backslash\Gamma$ is the canonical projection. The normalizer $N_C := \left\{\gamma \in \Gamma\colon \gamma C \gamma^{-1} = C\right\}$ acts on $F(C)$ by left multiplication, and we denote the set of orbits by $N_C\backslash F(C)$. Given $X \in F(C)$, we write $C \cdot X := \bigsqcup_{x \in X} C \cdot x$ and let $(\mathrm{con}^{\times}\ \Sigma)^{C \cdot X}$ be the set of functions $C \cdot X \to \mathrm{con}^{\times}\ \Sigma$. $\gamma \in C$ acts on $\varphi \in (\mathrm{con}^{\times}\ \Sigma)^{C \cdot X}$ via $(\gamma.\varphi)(x) = \varphi(\gamma^{-1}x)$, and we set $\mathrm{Stab}_C(\varphi) = \{\gamma \in C\colon \gamma.\varphi = \varphi\}$ for $\varphi \in (\mathrm{con}^{\times}\ \Sigma)^{C \cdot X}$.

Received by the editors April 9, 2018, and, in revised form, January 15, 2019.
2010 *Mathematics Subject Classification.* Primary 46L80.

Theorem 1.1. *If Γ satisfies the Baum-Connes conjecture with coefficients, then the K-theory of $C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)$ is given by*

$$\begin{aligned} &K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)) \\ &\cong K_*(C^*_\lambda(\Gamma)) \\ &\oplus \left(\bigoplus_{[C] \in \mathcal{C}} \ \bigoplus_{[X] \in N_C \backslash F(C)} \ \bigoplus_{[\varphi] \in C \backslash ((\mathrm{con}^\times \ \Sigma)^{C \cdot X})} K_*(C^*_\lambda(\mathrm{Stab}_C(\varphi))) \right). \end{aligned}$$

Here we take one representative C out of each class in $\mathcal{C}$, one representative X out of each class in $N_C \backslash F(C)$, and one representative φ out of each class in $C \backslash \left((\mathrm{con}^\times \ \Sigma)^{C \cdot X} \right)$.

We refer the reader to [1, 5, 14] and the references therein for more information about the Baum-Connes conjecture. For instance, Theorem 1.1 applies to all groups with the Haagerup property [11] and all hyperbolic groups [12].

Note that Σ enters our formula only in the form of $\mathrm{con}^\times \ \Sigma$. What is more, if Γ is infinite, then for each $[C] \in \mathcal{C}$, we simply get a free abelian group of countably infinite rank, so that $K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma))$ does not depend on Σ at all. This becomes particularly evident in K_1, where Theorem 1.1 yields the following.

Corollary 1.2. *Let Σ be a finite group and let Γ be a countable group. If Γ satisfies the Baum-Connes conjecture with coefficients, then the canonical inclusion $\Gamma \hookrightarrow \Sigma \rtimes \Gamma$ induces an isomorphism*

$$K_1(C^*_\lambda(\Gamma)) \cong K_1(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)).$$

Moreover, for torsion-free Γ, our formula becomes particularly simple.

Corollary 1.3. *Let Σ and Γ be as in Theorem* 1.1*. Assume that Γ is torsion-free. Write $\mathrm{FIN}^\times$ for the set of non-empty finite subsets of Γ. Then, under the same assumptions as in Theorem* 1.1*, we have*

$$K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[X] \in \Gamma \backslash \mathrm{FIN}^\times} \ \bigoplus_{(\mathrm{con}^\times \ \Sigma)^X} K_*(\mathbb{C}) \right).$$

The proof of our main theorem proceeds in two steps. First, using the Going-Down principle from [2, 8] (see also [5, §3]), we show that $C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)$ has the same K-theory as the crossed product $C((\mathrm{con} \ \Sigma)^\Gamma) \rtimes_r \Gamma$ for the topological full shift $\Gamma \curvearrowright (\mathrm{con} \ \Sigma)^\Gamma$. Here we view $\mathrm{con} \ \Sigma$ as a finite alphabet. Secondly, we compute K-theory for $C((\mathrm{con} \ \Sigma)^\Gamma) \rtimes_r \Gamma$ using [3, 4]. As a byproduct, we obtain a general K-theory formula for crossed products of topological full shifts (see Proposition 2.4). Both steps require our assumption that Γ satisfies the Baum-Connes conjecture with coefficients.

We point out that it is not possible to apply the results in [3, 4] directly because [3, 4] only deal with crossed products attached to actions on commutative C*-algebras.

2. K-theory for certain crossed products and generalized Lamplighter groups

We first discuss the following abstract situation: Let $A = \bigoplus_{i=0}^n M_{k_i}$ be a finite dimensional C*-algebra, where M_k is the algebra of $k \times k$-matrices. We assume that $k_0 = 1$, i.e., $A = \mathbb{C} \oplus M_{k_1} \oplus \ldots \oplus M_{k_n}$. Let Γ be a countable group. We form the tensor product $\bigotimes_\Gamma A$ as follows: For every finite subset $F \subseteq \Gamma$, we form the ordinary tensor product $\bigotimes_F A$, and for $F_1 \subseteq F_2$, we have the canonical embedding $\bigotimes_{F_1} A \hookrightarrow \bigotimes_{F_2} A$, $x \mapsto x \otimes 1$ (here 1 denotes the unit of $\bigotimes_{F_2 \setminus F_1} A$, and we used the canonical isomorphism $\bigotimes_{F_2} A \cong (\bigotimes_{F_1} A) \otimes \left(\bigotimes_{F_2 \setminus F_1} A\right)$). Then set $\bigotimes_\Gamma A := \varinjlim_F \bigotimes_F A$. The left Γ-action on itself by translations induces an action $\Gamma \curvearrowright \bigotimes_\Gamma A$. Our goal is to compute the K-theory of $(\bigotimes_\Gamma A) \rtimes_r \Gamma$. The special case $A = C^*_\lambda(\Sigma)$ will lead to Theorem 1.1.

Let e_i be a minimal projection in $M_{k_i} \subseteq A$. In particular, $e_0 = 1 \in \mathbb{C} \subseteq A$. For $F \subseteq \Gamma$ finite, let $\varphi \in \{1, \ldots, n\}^F$, i.e., φ is a function $\varphi: F \to \{1, \ldots, n\}$. Define $e_\varphi := \bigotimes_{f \in F} e_{\varphi(f)} \in \bigotimes_F A \subseteq \bigotimes_\Gamma A$. If $F = \emptyset$, then for $\varphi: \emptyset \to \{1, \ldots, n\}$, we set $e_\varphi := 1$ (where 1 denotes the unit of $\bigotimes_\Gamma A$). The set

$$\left\{e_\varphi \colon \varphi \in \{1, \ldots, n\}^F, \; F \subseteq \Gamma \text{ finite}\right\} \tag{1}$$

is a Γ-invariant family of commuting non-zero projections, which is closed under multiplication up to zero (i.e., the product of two projections in the family is either zero or again a projection in the family). We do not need it now, but the family is also linearly independent (see Lemma 2.3 and the proof of (2)). Let D be the C*-subalgebra of $\bigotimes_\Gamma A$ generated by the projections in (1). Let $\iota: D \hookrightarrow \bigotimes_\Gamma A$ be the canonical embedding. Note that ι is Γ-equivariant.

Proposition 2.1. *If Γ satisfies the Baum-Connes conjecture with coefficients, then $\iota \rtimes_r \Gamma$ induces an isomorphism $K_*(D \rtimes_r \Gamma) \cong K_*((\bigotimes_\Gamma A) \rtimes_r \Gamma)$.*

Proof. By the Going-Down principle (see [5, §3]), it suffices to show that for every finite subgroup $H \subseteq \Gamma$, $\iota \rtimes_r H$ induces an isomorphism $K_*(D \rtimes_r H) \cong K_*((\bigotimes_\Gamma A) \rtimes_r H)$.

Let us first treat the case of the trivial subgroup, $H = \{1\}$. For a fixed finite subset $F \subseteq \Gamma$, let

$$D_F = C^*(\left\{e_\varphi \colon \varphi \in \{1, \ldots, n\}^{F'} \text{ for } F' \subseteq F\right\}).$$

Then $D = \varinjlim_F D_F$. We also have $\bigotimes_\Gamma A = \varinjlim_F \bigotimes_F A$. As K-theory is continuous, i.e., preserves direct limits, it suffices to show that $\iota_F := \iota|_{D_F}: D_F \to \bigotimes_F A$ induces an isomorphism in K_*. Let $[\iota_F] \in KK(D_F, \bigotimes_F A)$ be the KK-element determined by ι_F. Consider the projection $e = \sum_{i=0}^n e_i$ in A. e is a full projection in A, and we have $eAe = \bigoplus_{i=0}^n \mathbb{C} e_i$. The $\bigotimes_F A$–$\bigotimes_F eAe$-imprimitivity bimodule $\bigotimes_F Ae$ gives rise to a KK-element $\mathbf{j}_F \in KK(\bigotimes_F A, \bigotimes_F eAe)$. $\mathbf{j}_F$ is invertible, and its inverse is the KK-element induced by the inclusion $\bigotimes_F eAe \hookrightarrow \bigotimes_F A$. Hence it suffices to show that the Kasparov product $[\iota_F] \cdot \mathbf{j}_F \in KK(D_F, \bigotimes_F eAe)$ induces an isomorphism $K_*(D_F) \to K_*(\bigotimes_F eAe)$.

First, consider the special case of a single element subset, $F = \{f\}$ for some $f \in \Gamma$. Let us write $D_f := D_{\{f\}}$, $\iota_f := \iota_{\{f\}}$ and $\mathbf{j}_f := \mathbf{j}_{\{f\}}$. Since $D_f = \mathbb{C} \cdot 1 + \mathbb{C} e_1 + \ldots + \mathbb{C} e_n$ (where 1 denotes the unit of $\bigotimes_\Gamma A$) and $eAe = \mathbb{C} e_0 \oplus \mathbb{C} e_1 \oplus \ldots \oplus \mathbb{C} e_n$, we can describe the map $K_*(D_F) \to K_*(\bigotimes_F eAe)$ induced by $[\iota_f] \cdot \mathbf{j}_f$ by the commutative diagram

$$\begin{array}{ccc} K_*(D_f) & \longrightarrow & K_*(eAe) \\ \| & & \| \\ \mathbb{Z}[1] \oplus \bigoplus_{i=1}^n \mathbb{Z}[e_i] & \xrightarrow{M_f} & \bigoplus_{i=0}^n \mathbb{Z}[e_i] \end{array}$$

where the upper horizontal map is the map we want to describe, and M_f is the $(n+1) \times (n+1)$-matrix

$$M_f = \begin{pmatrix} 1 & 0 & \dots & 0 \\ k_1 & 1 & & 0 \\ \vdots & & \ddots & \\ k_n & 0 & & 1 \end{pmatrix}.$$

Obviously, M_f is invertible. Note that everything is independent of f.

Now consider the case of a general finite subset $F \subseteq \Gamma$. Since $D_F = \bigotimes_{f \in F} D_f$, we have $K_*(D_F) \cong \bigotimes_{f \in F} K_*(D_f)$, and we also have $K_*(\bigotimes_F eAe) \cong \bigotimes_{f \in F} K_*(eAe)$. The homomorphism $K_*(D_F) \to K_*(\bigotimes_F eAe)$ induced by $[\iota_F] \cdot \mathbf{j}_F$ respects this tensor product decomposition, in the sense that we have a commutative diagram

$$\begin{array}{ccc} \bigotimes_{f \in F} K_*(D_f) \overset{\sim}{=\!=} K_*(D_F) & \to & K_*(\bigotimes_F eAe) \overset{\sim}{=\!=} \bigotimes_{f \in F} K_*(eAe) \\ \| & & \| \\ \bigotimes_{f \in F} (\mathbb{Z}[1] \oplus \bigoplus_{i=1}^n \mathbb{Z}[e_i]) & \xrightarrow{M_F = \otimes_{f \in F} M_f} & \bigotimes_{f \in F} (\bigoplus_{i=0}^n \mathbb{Z}[e_i]) . \end{array}$$

Again, we see that M_F is invertible because all the M_f, $f \in F$, are.

Now let us deal with the case of an arbitrary finite subgroup $H \subseteq \Gamma$. If we choose an increasing sequence of H-invariant finite subsets $F \subseteq \Gamma$ whose union is Γ, we obtain H-equivariant inductive limit decompositions $D = \varinjlim_F D_F$ and $\bigotimes_\Gamma A = \varinjlim_F \bigotimes_F A$. Hence, again by continuity of K-theory, it suffices to show that, for every F, $\iota_F \rtimes_r H : D_F \rtimes_r H \to (\bigotimes_F A) \rtimes_r H$ induces an isomorphism in K_*. Let $\mathbf{j}_F \in KK(\bigotimes_F A, \bigotimes_F eAe)$ be as before. Since the full projection $\bigotimes_F e \in \bigotimes_F A$ giving rise to $\mathbf{j}_F$ is H-invariant, $\mathbf{j}_F$ is a KK^H-equivalence (see [5, Remark 3.3.16]). Thus, to show that $\iota_F \rtimes_r H : D_F \rtimes_r H \to (\bigotimes_F A) \rtimes_r H$ induces an isomorphism in K_*, it suffices to show that $[\iota_F] \cdot \mathbf{j}_F \in KK^H(D_F, \bigotimes_F eAe)$ induces an isomorphism $K_*(D_F \rtimes_r H) \to K_*((\bigotimes_F eAe) \rtimes_r H)$, for which in turn it is enough to prove that $[\iota_F] \cdot \mathbf{j}_F$ is a KK^H-equivalence.

Now both D_F and $\bigotimes_F eAe$ are finite dimensional commutative C*-algebras with an H-action, so that we are exactly in the setting of [4, Appendix]. It is straightforward to check that $[\iota_F] \cdot \mathbf{j}_F = x^H_{M_F}$, where $x^H_{M_F}$ is the element in $KK^H(D_F, \bigotimes_F eAe)$ corresponding to the matrix M_F, as constructed in [4, Appendix]. By [4, Lemma A.2], $x^H_{M_F}$ is a KK^H-equivalence because M_F is an invertible matrix. The inverse of $x^H_{M_F}$ is given by $x^H_{M_F^{-1}}$. □

Remark 2.2. Note that our assumption on A that $\mathbb{C}$ appears as a direct summand is really necessary. For instance, if $A = M_2$, then $\bigotimes_\Gamma A$ would be the UHF algebra M_{2^∞} (as soon as Γ is infinite). But we have $K_0(M_{2^\infty}) \cong \mathbb{Z}[\frac{1}{2}]$, while our method would always yield a free abelian group for K_0. Hence our method fails.

Let us now compare with the topological full shift $\Gamma \curvearrowright \{0,\dots,n\}^\Gamma$. For a finite subset $F \subseteq \Gamma$, let π_F be the canonical projection $\{0,\dots,n\}^\Gamma \twoheadrightarrow \{0,\dots,n\}^F$. Given $\varphi \in \{0,\dots,n\}^F$, we have the cylinder set $\pi_F^{-1}(\varphi)$ and its characteristic function $1_{\pi_F^{-1}(\varphi)} \in C(\{0,\dots,n\}^\Gamma)$. The following is now easy to see.

Lemma 2.3. *The Γ-equivariant isomorphism $D \cong C(\{0,\dots,n\}^\Gamma)$, $e_\varphi \mapsto 1_{\pi_F^{-1}(\varphi)}$ induces an isomorphism $D \rtimes_r \Gamma \cong C(\{0,\dots,n\}^\Gamma) \rtimes_r \Gamma$.*

We now compute K-theory for $C(\{0,\dots,n\}^\Gamma) \rtimes_r \Gamma$.

Proposition 2.4. *If Γ satisfies the Baum-Connes conjecture with coefficients, then*

$$\begin{aligned} &K_*(C(\{0,\dots,n\}^\Gamma) \rtimes_r \Gamma) \\ &\quad \cong K_*(C^*_\lambda(\Gamma)) \\ &\quad \oplus \left(\bigoplus_{[C]\in\mathcal{C}} \; \bigoplus_{[X]\in N_C\backslash F(C)} \; \bigoplus_{[\varphi]\in C\backslash(\{1,\dots,n\}^{C\cdot X})} K_*(C^*_\lambda(\mathrm{Stab}_C(\varphi))) \right). \end{aligned}$$

Here we use the same notation as in Theorem 1.1.

Proof. First of all, the same arguments as for [4, Examples 2.13 & 3.1] show that the family

$$\left\{ \pi_F^{-1}(\varphi) \colon \varphi \in \{1,\dots,n\}^F, \, F \subseteq \Gamma \text{ finite} \right\}$$

is a Γ-invariant regular basis for the compact open sets in $\{0,\dots,n\}^\Gamma$. Here Γ acts via $\gamma.\pi_F^{-1}(\varphi) = \pi_{\gamma\cdot F}^{-1}(\gamma.\varphi)$, where $\gamma.\varphi \in \{1,\dots,n\}^{\gamma\cdot F}$ is given by $(\gamma.\varphi)(x) = \varphi(\gamma^{-1}x)$. Therefore, using the bijection

$$\begin{aligned} &\bigsqcup_{[F]\in\Gamma\backslash \mathrm{FIN}} \mathrm{Stab}_\Gamma(F)\backslash \left(\{1,\dots,n\}^F \right) \\ &\qquad \cong \Gamma \backslash \left\{ \pi_F^{-1}(\varphi) \colon \varphi \in \{1,\dots,n\}^F, \, F \subseteq \Gamma \text{ finite} \right\}, \; [\varphi] \mapsto [\varphi], \end{aligned}$$

and the observation that for $\gamma \in \Gamma$ and $\varphi \in \{1,\dots,n\}^F$ we have $\gamma.\pi_F^{-1}(\varphi) = \pi_F^{-1}(\varphi)$ if and only if $\gamma \cdot F = F$ and $\gamma.\varphi = \varphi$, we may apply [4, Corollary 3.14], and obtain

$$K_*\left(C\left(\{0,\dots,n\}^\Gamma \right) \rtimes_r \Gamma \right) \cong \bigoplus_{[F]\in\Gamma\backslash \mathrm{FIN}} \; \bigoplus_{[\varphi]\in\mathrm{Stab}_\Gamma(F)\backslash(\{1,\dots,n\}^F)} K_*(C^*_\lambda(\mathrm{Stab}_\Gamma(\varphi))). \tag{2}$$

Here FIN is the set of all finite subsets of Γ, and $\Gamma\backslash \mathrm{FIN}$ is the set of orbits of the left translation action $\Gamma \curvearrowright \mathrm{FIN}$. Moreover, $\mathrm{Stab}_\Gamma(F)$ and $\mathrm{Stab}_\Gamma(\varphi)$ denote the stabilizer groups $\mathrm{Stab}_\Gamma(F) = \{\gamma \in \Gamma \colon \gamma \cdot F = F\}$ and $\mathrm{Stab}_\Gamma(\varphi) = \{\gamma \in \Gamma \colon \gamma.\varphi = \varphi\}$, and $C^*_\lambda(\mathrm{Stab}_\Gamma(\varphi))$ is the reduced group C*-algebra of $\mathrm{Stab}_\Gamma(\varphi)$.

Now we analyze $\Gamma\backslash \mathrm{FIN}$ and $\mathrm{Stab}_\Gamma(F)$ for $[F] \in \Gamma\backslash \mathrm{FIN}$. For $F = \emptyset$, we have $\mathrm{Stab}_\Gamma(\varphi) = \Gamma$. This yields $K_*(C^*_\lambda(\Gamma))$ as one direct summand on the right-hand side of (2). We set $\mathrm{FIN}^\times := \mathrm{FIN} \setminus \{\emptyset\}$. Let us describe $\Gamma\backslash \mathrm{FIN}^\times$. Let $\mathcal{C}$, $F(C)$ and N_C be as in Theorem 1.1. Then we claim that

$$\bigsqcup_{[C]\in\mathcal{C}} N_C\backslash F(C) \to \Gamma\backslash \mathrm{FIN}^\times, \; [X] \mapsto [C \cdot X] \tag{3}$$

is a bijection, and that for every $[C] \in \mathcal{C}$, $[X] \in N_C \backslash F(C)$, we have

$$\mathrm{Stab}_\Gamma(C \cdot X) = C. \tag{4}$$

First note that the map (3) is well-defined. Moreover, this map is surjective because every $F \in \mathrm{FIN}^\times$ with $\mathrm{Stab}_\Gamma(F) = C$ is of the form $F = C \cdot X$ for some finite, non-empty subset $X \subseteq C\backslash\Gamma$. Now, X must lie in $F(C)$. Suppose not, i.e., $X = \pi^{-1}(Y)$ for a finite subgroup $D \subseteq \Gamma$ with $C \subsetneq D$ and $Y \subseteq D\backslash\Gamma$, where $\pi : C\backslash\Gamma \twoheadrightarrow D\backslash\Gamma$ is the canonical projection. Then $F = C \cdot X = D \cdot Y$, so that $D \subseteq \mathrm{Stab}_\Gamma(F)$, in contradiction to $\mathrm{Stab}_\Gamma(F) = C$. Not only does this show surjectivity, but it also proves (4). To see injectivity of (3), assume that $X \in F(C)$ and $X' \in F(C')$ satisfy $[C \cdot X] = [C' \cdot X']$, say $C' \cdot X' = \gamma \cdot C \cdot X$. It follows that $C' = \mathrm{Stab}_\Gamma(C' \cdot X') = \gamma \, \mathrm{Stab}_\Gamma(C \cdot X)\gamma^{-1} = \gamma C \gamma^{-1}$. Hence $[C] = [C']$, and since we are taking one representative out of each class, we must actually have $C = C'$. But then γ must lie in N_C, and we must have $C \cdot X' = \gamma \cdot C \cdot X = C \cdot \gamma \cdot X$, so that $X' = \gamma \cdot X$, i.e., $[X'] = [X]$ in $N_C \backslash F(C)$. This shows injectivity.

We now complete the proof by plugging the bijections (3), (4) into (2) and observing that for $X \in F(C)$ and $\varphi \in \{1, \dots, n\}^{C \cdot X}$, we have

$$\mathrm{Stab}_\Gamma(\varphi) \subseteq \mathrm{Stab}_\Gamma(C \cdot X) = C.$$

□

Combining Proposition 2.1, Lemma 2.3, and Proposition 2.4, and using the concrete construction in [4, §3] for our following assertion on K_1, we obtain the following.

Corollary 2.5. *In the situation of Proposition* 2.1, *if* Γ *satisfies the Baum-Connes conjecture with coefficients, then we have*

$$\begin{aligned} &K_*((\bigotimes_\Gamma A) \rtimes_r \Gamma) \\ &\cong K_*(C^*_\lambda(\Gamma)) \\ &\oplus \left(\bigoplus_{[C] \in \mathcal{C}} \; \bigoplus_{[X] \in N_C \backslash F(C)} \; \bigoplus_{[\varphi] \in C \backslash (\{1,\dots,n\}^{C \cdot X})} K_*(C^*_\lambda(\mathrm{Stab}_C(\varphi))) \right). \end{aligned}$$

In K_1, *the canonical map* $C^*_\lambda(\Gamma) \hookrightarrow (\bigotimes_\Gamma A) \rtimes_r \Gamma$ *induces an isomorphism*

$$K_1(C^*_\lambda(\Gamma)) \cong K_1((\bigotimes_\Gamma A) \rtimes_r \Gamma).$$

If Γ *is in addition torsion-free, then we obtain*

$$K_*((\bigotimes_\Gamma A) \rtimes_r \Gamma) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[X] \in \Gamma \backslash \mathrm{FIN}^\times} \; \bigoplus_{\{1,\dots,n\}^X} K_*(\mathbb{C}) \right).$$

Now let us apply our K-theory formula to generalized Lamplighter groups. Consider the case $A = C^*_\lambda(\Sigma)$ for a finite group Σ. Our assumption on A that $\mathbb{C}$ appears as a direct summand is satisfied because the trivial representation gives rise to a direct summand $\mathbb{C}$ in $C^*_\lambda(\Sigma)$. The remaining direct summands of A are in one-to-one correspondence with $\mathrm{con}^\times\, \Sigma$. Hence we obtain

Corollary 2.6. *Let Σ be a finite group. If Γ satisfies the Baum-Connes conjecture with coefficients, then we have*

$$\begin{aligned} &K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)) \\ &\quad \cong K_*(C^*_\lambda(\Gamma)) \\ &\quad \oplus \left(\bigoplus_{[C]\in\mathcal{C}} \ \bigoplus_{[X]\in N_C\backslash F(C)} \ \bigoplus_{[\varphi]\in C\backslash((\mathrm{con}^\times\ \Sigma)^{C\cdot X})} K_*(C^*_\lambda(\mathrm{Stab}_C(\varphi))) \right). \end{aligned}$$

In K_1, the canonical inclusion $\Gamma \hookrightarrow \Sigma \rtimes \Gamma$ induces an isomorphism

$$K_1(C^*_\lambda(\Gamma)) \cong K_1(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)).$$

If Γ is in addition torsion-free, then we obtain

$$K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[X]\in\Gamma\backslash\,\mathrm{FIN}^\times} \ \bigoplus_{(\mathrm{con}^\times\ \Sigma)^X} K_*(\mathbb{C}) \right).$$

This completes the proofs of Theorem 1.1 and Corollaries 1.2 and 1.3.

Remark 2.7. As in [4, Corollary 3.14], we get KK-equivalences in Proposition 2.4 and Corollaries 2.5 and 2.6 if Γ satisfies the strong Baum-Connes conjecture.

Remark 2.8. Moreover, as in [4, Corollary 3.14], we could allow coefficients in Proposition 2.4 and Corollaries 2.5 and 2.6. However, in Corollary 2.6, we would only get a K-theory formula for crossed products $B \rtimes_r ((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)$ where the $(\bigoplus_\Gamma \Sigma)$-action on the C*-algebra B is trivial.

Acknowledgments

The author is indebted to Alain Valette for the invitation to Neuchatel, and to Sanaz Pooya and Alain Valette for interesting discussions which led to these notes. Moreover, the author thanks the referee for helpful comments and careful proofreading.

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