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EQUIVARIANT KAZHDAN-LUSZTIG POLYNOMIALS OF THAGOMIZER MATROIDS

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Abstract. The equivariant Kazhdan-Lusztig polynomial of a matroid was introduced by Gedeon, Proudfoot, and Young. Gedeon conjectured an explicit formula for the equivariant Kazhdan-Lusztig polynomials of thagomizer matroids with an action of symmetric groups. In this paper, we discover a new formula for these polynomials which is related to the equivariant Kazhdan-Lusztig polynomials of uniform matroids. Based on our new formula, we confirm Gedeon's conjecture by the Pieri rule.

1. Introduction

Given a matroid M, Elias, Proudfoot, and Wakefield [1] introduced the Kazhdan-Lusztig polynomial $P_M(t)$. If M is equipped with an action of a finite group W, Gedeon, Proudfoot, and Young [3] defined the W-equivariant Kazhdan-Lusztig polynomial $P_M^W(t)$, whose coefficients are graded virtual representations of W and from which $P_M(t)$ can be recovered by sending virtual representations to their dimensions. The equivariant Kazhdan-Lusztig polynomials have been computed for uniform matroids [3] and q-niform matroids [8] and conjectured for thagomizer matroids [2].

The thagomizer matroid M_n is isomorphic to the graphic matroid of the complete tripartite graph $K_{1,1,n}$ or the graph obtained by adding an edge between the two distinguished vertices of bipartite graph $K_{2,n}$. Gedeon [2] computed the polynomial $P_{M_n}(t)$ and presented a conjecture for the equivariant polynomial $P_{M_n}^{S_n}(t)$, where S_n is the symmetric group of order n. Let Υ_n be the set of partitions of n of the form $(a, n-a-2i-\eta, 2^i, \eta)$, where $\eta \in \{0,1\}$, $i \geq 0$, and $1 < a < n$. For any partition λ of n, we let V_λ denote the irreducible representation of S_n indexed by λ. We also set

$$\kappa(\lambda) = \begin{cases} \lambda_1 - 1, & \lambda = (n-1,1), \\ \lambda_1 - \lambda_2 + 1, & \text{otherwise}, \end{cases}$$

and

$$\omega(\lambda) = \begin{cases} 0, & \lambda_{\ell(\lambda)} = 1, \\ 1, & \text{otherwise}. \end{cases}$$

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Gedeon [2] conjectured an explicit formula for $P_{M_n}^{S_n}(t)$.

Conjecture 1.1. *For any positive integer n,*

$$P_{M_n}^{S_n}(t) = \sum_{\lambda \in \Upsilon_n} \kappa(\lambda) V_\lambda t^{\ell(\lambda)-1}(t+1)^{\omega(\lambda)} + V_{(n)}((n-1)t+1).$$

In this paper, we shall confirm Conjecture 1.1. To this end, we find a new formula for $P_{M_n}^{S_n}(t)$ which is related to the equivariant Kazhdan-Lusztig polynomials of uniform matroids. Let $U_{1,n}$ be the uniform matroid of rank n on $n+1$ elements which is isomorphic to the graphic matroid of the cycle graph with $n+1$ vertices. One of the main results of this paper is as follows.

Theorem 1.2. *For any positive integer n, we have*

$$P_{M_n}^{S_n}(t) = V_{(n)} + t \sum_{k=2}^{n} \mathrm{Ind}_{S_{n-k}\times S_k}^{S_n} \left(V_{(n-k)} \otimes P_{U_{1,k-1}}^{S_k}(t) \right), \tag{1.1}$$

where $P_{U_{1,k-1}}^{S_k}(t) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor -1} V_{k-2i,2^i} t^i$.

Note that for any parition λ of k there holds that

$$\begin{aligned} \dim \mathrm{Ind}_{S_{n-k}\times S_k}^{S_n} \left(V_{(n-k)} \otimes V_\lambda \right) &= |S_n : S_{n-k} \times S_k| \times \dim V_{(n-k)} \times \dim V_\lambda \\ &= \frac{n!}{(n-k)!k!} \dim V_\lambda = \binom{n}{k} \dim V_\lambda, \end{aligned} \tag{1.2}$$

where $|S_n : S_{n-k} \times S_k|$ is the index of $S_{n-k} \times S_k$ in S_n in the sense of isomorphism. Hence, the following formula for the non-equivalent Kazhdan-Lusztig polynomials which inspires this paper can be derived from Theorem 1.2.

Corollary 1.3. *For any positive integer n, we have*

$$P_{M_n}(t) = 1 + t \sum_{k=2}^{n} \binom{n}{k} P_{U_{1,k-1}}(t). \tag{1.3}$$

This paper is organized as follows. Section 2 is dedicated to the proof of Theorem 1.2. The main tools used in our proof of Theorem 1.2 are the Frobenius characteristic map and the generating functions of symmeric functions. In Section 3, based on Theorem 1.2, we confirm Conjecture 1.1 by the Pieri rule.

2. Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. We first review the definition of the Frobenius characteristic map and then show in Theorem 2.1 that Theorem 1.2 can be translated into a symmetric function equality. Once Theorem 2.1 is proved, the proof of Theorem 1.2 is done since they are equivalent under the Frobenius characteristic map.

Following Gedeon, Proudfoot, and Young [8], let $\mathrm{VRep}(S_n)$ be the $\mathbb{Z}$-module of isomorphism classes of virtual representations of S_n and set $\mathrm{grVRep}(W) := \mathrm{VRep}(W) \otimes_{\mathbb{Z}} \mathbb{Z}[t]$. Consider the Frobenius characteristic map

$$\mathrm{ch} : \mathrm{grVRep}(S_n) \longrightarrow \Lambda_n \otimes_{\mathbb{Z}} \mathbb{Z}[t],$$

where Λ_n is the $\mathbb{Z}$-module of symmetric functions of degree n in the variables $\mathbf{x} = (x_1, x_2, \ldots)$; see [7, Section I.7]. We refer the reader to [7, 10] for undefined terminology from the theory of symmetric functions. Given two graded virtual representations $V_1 \in \mathrm{grVRep}(S_{n_1})$ and $V_2 \in \mathrm{grVRep}(S_{n_2})$, we have

$$\mathrm{ch}\left(\mathrm{Ind}_{S_{n_1}\times S_{n_2}}^{S_{n_1+n_2}} V_1 \otimes V_2\right) = \mathrm{ch}(V_1)\,\mathrm{ch}(V_2).$$

The image of the irreducible representation V_λ under ch is the Schur function s_λ, and, in particular, the image of the trivial representation $V_{(n)}$ is the complete symmetric function $h_n(\mathbf{x})$. Define $Q_n(\mathbf{x};t)$ as

$$Q_n(\mathbf{x};t) = \begin{cases} 0, & n = 0 \text{ or } 1, \\ \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor -1} s_{n-2i,2^i}(\mathbf{x})t^i, & n \geq 2. \end{cases} \tag{2.1}$$

When $n \geq 2$, $Q_n(\mathbf{x};t)$ is the image under the Frobenius map of $P^{S_n}_{U_{1,n-1}}(t)$; see [9]. Let $P_n(\mathbf{x};t)$ be the image under the Frobenius map of $P^{S_n}_{M_n}(t)$. Since the Frobenius characteristic map is an isomorphism between $\mathrm{grVRep}(S_n)$ and $\Lambda_n \otimes_{\mathbb{Z}} \mathbb{Z}[t]$, the following theorem is equivalent to Theorem 1.2.

Theorem 2.1. *For any positive integer n, we have*

$$P_n(\mathbf{x};t) = h_n(\mathbf{x}) + t\sum_{k=2}^{n} h_{n-k}(\mathbf{x})\, Q_k(\mathbf{x};t). \tag{2.2}$$

The rest of this section is dedicated to the proof of Theorem 2.1. It is known from [2] that the polynomial $P_n(\mathbf{x};t)$ is uniquely determined by the following three conditions:

(i) $P_0(\mathbf{x};t)$=1,
(ii) the degree of $P_n(\mathbf{x};t)$ is less than $(n+1)/2$ for any positive integer n, and
(iii) for any positive integer n the polynomial $P_n(\mathbf{x};t)$ satisfies that

$$\begin{aligned} t^{n+1}P_n(\mathbf{x};t^{-1}) =&(t-1)\sum_{\ell=0}^{n} h_\ell[(t-2)X]h_{n-\ell}(\mathbf{x}) \\ &+ \sum_{i+j+m=n} P_i(\mathbf{x};t)h_j[(t-1)X]h_m[(t-1)X], \end{aligned}$$

where the square bracket denotes the plethystic substitution [5, 6], and it is a convention that $X = x_1 + x_2 + \cdots$.

The third condition can also be expressed in terms of its generating function. Let

$$\phi(t,u) = \sum_{n=0}^{\infty} P_n(\mathbf{x};t)u^{n+1}.$$

It is known by Gedeon [2, Proposition 4.7] that the condition (iii) is equivalent to saying that the function $\phi(t,u)$ satisfies

$$\phi(t^{-1},tu) = (t-1)uH(u)v(t,u) + \frac{H(tu)^2}{H(u)^2}\phi(t,u), \tag{2.3}$$

where

$$v(t,u) = \sum_{n=0}^{\infty} h_n[(t-2)X]u^n \quad ([4, \text{ p. } 8]) \quad \text{and} \quad H(u) = \sum_{n=0}^{\infty} h_n(\mathbf{x})u^n.$$

We note that the equation (2.3) can be simplified as follows.

Lemma 2.2. *The function $\phi(t,u)$ satisfies*

$$\phi(t^{-1},tu)=(t-1)u\frac{H(tu)}{H(u)}+\frac{H(tu)^2}{H(u)^2}\phi(t,u). \tag{2.4}$$

Proof. It suffices to show that

$$v(t,u)=\frac{H(tu)}{H(u)^2}.$$

By the formula [5, Theorem 1.27]

$$h_n[E+F]=\sum_{k=0}^{n}h_k[E]h_{n-k}[F], \tag{2.5}$$

where $E=E(t_1,t_2,\ldots)$ and $F=F(w_1,w_2,\ldots)$ are two formal series of rational terms in their indeterminates, we have

$$h_n[2X]=\sum_{k=0}^{n}h_k[X]h_{n-k}[X] \quad \text{and} \quad h_n[tX]=\sum_{k=0}^{n}h_k[(t-2)X]h_{n-k}[2X].$$

Note that $h_n[X]=h_n(\mathbf{x})$. Hence, it follows that

$$\sum_{n=0}^{\infty}h_n[2X]u^n=\left(\sum_{n=0}^{\infty}h_n[X]u^n\right)^2=H(u)^2,$$

and thus

$$\sum_{n=0}^{\infty}h_n[tX]u^n=\left(\sum_{n=0}^{\infty}h_n[(t-2)X]u^n\right)\left(\sum_{n=0}^{\infty}h_n[2X]u^n\right)=v(t,u)H(u)^2.$$

By the definition of plethysm, we have $H(tu)=\sum_{n=0}^{\infty}h_n[tX]u^n$. Thus $H(tu)=v(t,u)H(u)^2$ as desired. This completes the proof. □

In order to prove Theorem 2.1, we shall prove that for every positive integer n the polynomial on the right hand side of (2.2) also satisfies the three conditions (i), (ii), and (iii). For convenience, we define $R_n(\mathbf{x};t)$ as

$$R_n(\mathbf{x};t)=\begin{cases}1, & n=0,\\ h_n(\mathbf{x})+t\sum_{k=2}^{n}h_{n-k}(\mathbf{x})\,Q_k(\mathbf{x};t), & n\geq 1.\end{cases} \tag{2.6}$$

By (2.6), we know that $R_0(\mathbf{x};t)=1$ and the degree of $R_n(\mathbf{x};t)$ is $\lfloor \frac{n}{2}\rfloor$. Hence $R_n(\mathbf{x};t)$ satisfies the first two conditions. For the condition (iii), let us consider the generating function of $R_n(\mathbf{x};t)$. Denote

$$\rho(t,u)=\sum_{n=0}^{\infty}R_n(\mathbf{x};t)u^{n+1}.$$

We have the following result.

Lemma 2.3. *The function $\rho(t,u)$ satisfies*

$$\rho(t^{-1},tu)=(t-1)u\frac{H(tu)}{H(u)}+\frac{H(tu)^2}{H(u)^2}\rho(t,u).$$

Proof. Let $\psi(t,u)=\sum_{n=2}^{\infty} Q_n(\mathbf{x};t)u^{n-1}$. Since $Q_0(\mathbf{x};t)=Q_1(\mathbf{x};t)=0$, it follows from (2.6) that

$$\rho(t,u) = uH(u)+tu^2H(u)\psi(t,u)$$

$$= uH(u)\left(1+tu\,\psi(t,u)\right). \tag{2.7}$$

Hence $\rho(t^{-1},tu)$ turns out to be

$$\rho(t^{-1},tu)=tuH(tu)\left(1+u\,\psi(t^{-1},tu)\right). \tag{2.8}$$

On the other hand, taking the coefficient of x in [3, equation (4)], the function $\psi(t,u)$ satisfies

$$\left(\frac{1}{u}+\psi(t^{-1},tu)\right)H(u)-\left(\frac{1}{u}+h_1(\mathbf{x})\right)=\left(\frac{1}{tu}+\psi(t,u)\right)H(tu)-\left(\frac{1}{tu}+h_1(\mathbf{x})\right).$$

Hence, we have that the function $\psi(t,u)$ satisfies the equation

$$\psi(t^{-1},tu)=-\frac{1}{u}+\frac{t-1}{tuH(u)}+\frac{H(tu)}{H(u)}\left(\psi(t,u)+\frac{1}{tu}\right),$$

and thus it follows from (2.7) that

$$\psi(t^{-1},tu)=-\frac{1}{u}+\frac{t-1}{tuH(u)}+\frac{H(tu)}{tu^2H(u)^2}\rho(t,u). \tag{2.9}$$

Substituting (2.9) into the right hand side of (2.8), we have that the function $\rho(t,u)$ satisfies that

$$\rho(t^{-1},tu)=tuH(tu)+tu^2H(tu)\left(-\frac{1}{u}+\frac{t-1}{tuH(u)}+\frac{H(tu)}{tu^2H(u)^2}\rho(t,u)\right)$$

$$=(t-1)u\frac{H(tu)}{H(u)}+\frac{H(tu)^2}{H(u)^2}\rho(t,u).$$

This completes the proof. □

We are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. As shown previously, the polynomial $R_n(\mathbf{x};t)$ satisfies the first two conditions (i) and (ii). By Lemma 2.2, the condition (iii) is equivalent to the generating function $\phi(t,u)$ of $P_n(\mathbf{x};t)$ satisfying (2.4). Compared with Lemma 2.3, the generating function $\psi(t,u)$ of $R_n(\mathbf{x};t)$ satisfies the same function. Thus, we obtain that the condition (iii) is true for $R_n(\mathbf{x};t)$ as well. Since these three conditions uniquely determine a polynomial sequence, we get that $P_n(\mathbf{x};t)=R_n(\mathbf{x};t)$ for every positive integer n. This completes the proof of Theorem 2.1. □

3. Proof of Conjecture 1.1

In this section, we shall prove the following theorem, which is equivalent to Conjecture 1.1 in the sense of Frobenius map. Our proof is based on the Pieri rule.

Theorem 3.1. *For any positive integer n, we have*

$$P_n(\mathbf{x};t)=\sum_{\lambda\in\Upsilon_n}\kappa(\lambda)s_\lambda(\mathbf{x})t^{\ell(\lambda)-1}(t+1)^{\omega(\lambda)}+h_n(\mathbf{x})((n-1)t+1). \tag{3.1}$$

Proof. By Theorem 2.1 we need to prove that $R_n(\mathbf{x};t)$ is equal to the right side of (3.1), namely,

$$\sum_{\lambda\in\Upsilon_n}\kappa(\lambda)s_\lambda(\mathbf{x})t^{\ell(\lambda)-1}(t+1)^{\omega(\lambda)}+h_n(\mathbf{x})((n-1)t+1)$$
$$= h_n(\mathbf{x})+t\sum_{k=2}^{n}h_{n-k}(\mathbf{x})\,Q_k(\mathbf{x};t).$$

Recall that $Q_n(\mathbf{x};t)=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1}s_{n-2i,2^i}(\mathbf{x})t^i$ for $n\geq 2$. It suffices to prove that

$$\sum_{\lambda\in\Upsilon_n}\kappa(\lambda)s_\lambda(\mathbf{x})t^{\ell(\lambda)-1}(t+1)^{\omega(\lambda)}+h_n(\mathbf{x})(n-1)t = \sum_{k=2}^{n}h_{n-k}(\mathbf{x})\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor-1}s_{k-2i,2^i}(\mathbf{x})t^{i+1}. \tag{3.2}$$

For convenience, we denote by $A_n(\mathbf{x};t)$ and $B_n(\mathbf{x};t)$ the left side and the right side of (3.2), respectively.

We first show that $B_n(\mathbf{x};t)$ is of the form

$$\sum_{\lambda\in\Upsilon_n}s_\lambda(\mathbf{x})a_\lambda(t)+h_n(\mathbf{x})a_n(t),$$

where $a_\lambda(t)$ and $a_n(t)$ are polynomials of t with non-negative integer coefficients. In fact, by the Pieri rule we have that for $2\leq k\leq n$,

$$h_{n-k}(\mathbf{x})h_k(\mathbf{x})=\sum_{p=\max(0,n-2k)}^{n-k}s_{k+p,n-k-p}(\mathbf{x}),$$

and for $2\leq k\leq n$ and $1\leq i\leq\lfloor\frac{k}{2}\rfloor-1$,

$$h_{n-k}(\mathbf{x})s_{k-2i,2^i}(\mathbf{x})=\sum_{p=\max(0,n-2k+2i+2)}^{n-k}s_{k+p-2i,n-k-p+2,2^{i-1}}(\mathbf{x})$$
$$+\sum_{p=\max(0,n-2k+2i+1)}^{n-k-1}s_{k+p-2i,n-k-p+1,2^{i-1},1}(\mathbf{x})$$
$$+\sum_{p=\max(0,n-2k+2i)}^{n-k-2}s_{k+p-2i,n-k-p,2^i}(\mathbf{x}).$$

Set

$$B_n^{\langle 1\rangle}(\mathbf{x};t) := \sum_{k=2}^{n} \sum_{p=\max(0,n-2k)}^{n-k} s_{k+p,n-k-p}(\mathbf{x})\, t,$$

$$B_n^{\langle 2\rangle}(\mathbf{x};t) := \sum_{k=2}^{n} \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor -1} \sum_{p=\max(0,n-2k+2i+2)}^{n-k} s_{k+p-2i,n-k-p+2,2^{i-1}}(\mathbf{x})\, t^{i+1},$$

$$B_n^{\langle 3\rangle}(\mathbf{x};t) := \sum_{k=2}^{n-1} \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor -1} \sum_{p=\max(0,n-2k+2i+1)}^{n-k-1} s_{k+p-2i,n-k-p+1,2^{i-1},1}(\mathbf{x})\, t^{i+1},$$

$$B_n^{\langle 4\rangle}(\mathbf{x};t) := \sum_{k=2}^{n-2} \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor -1} \sum_{p=\max(0,n-2k+2i)}^{n-k-2} s_{k+p-2i,n-k-p,2^{i}}(\mathbf{x})\, t^{i+1}.$$

Hence,

$$B_n(\mathbf{x};t) = B_n^{\langle 1\rangle}(\mathbf{x};t) + B_n^{\langle 2\rangle}(\mathbf{x};t) + B_n^{\langle 3\rangle}(\mathbf{x};t) + B_n^{\langle 4\rangle}(\mathbf{x};t).$$

We proceed to prove that $a_\lambda(t)$ and $a_n(t)$ agree with the corresponding polynomials of t appearing in $A_n(\mathbf{x};t)$. Since $h_n(\mathbf{x})$ can be obtained only from $B_n^{\langle 1\rangle}(\mathbf{x};t)$, where k ranges from 2 to n, we obtain that $a_n(t) = (n-1)t$. We shall prove $a_\lambda(t) = \kappa(\lambda)t^{\ell(\lambda)-1}(t+1)^{\omega(\lambda)}$ for $\lambda \in \Upsilon_n$. To this end, we divide the proof into the following three cases according to the definitions of $\kappa(\lambda)$ and $\omega(\lambda)$:

Case 1. $\lambda = (n-1,1)$. In this case, $s_{n-1,1}(\mathbf{x})$ can be obtained only from $B_n^{\langle 1\rangle}(\mathbf{x};t)$, where k ranges from 2 to $n-1$. Thus we have

$$a_{n-1,1}(t) = (n-2)t = \kappa(\lambda)t^{\ell(\lambda)-1}(t+1)^{\omega(\lambda)}.$$

Case 2. $\lambda_{\ell(\lambda)} = 1$ and $\lambda \neq (n-1,1)$. In this case, λ must be of the form $(\lambda_1, \lambda_2, 2^{i-1}, 1)$, where $i = \ell(\lambda) - 2 \geq 1$. Hence, we get that $s_\lambda(\mathbf{x})$ can be obtained only from $B_n^{\langle 3\rangle}(\mathbf{x};t)$. We next compute the coefficient of $s_{\lambda_1,\lambda_2,2^{i-1},1}(\mathbf{x})$ in $B_n^{\langle 3\rangle}(\mathbf{x};t)$. From the betweenness condition of the Pieri rule, we know that

$$\lambda_2 \leq k - 2i \leq \lambda_1,$$

and thus

$$2 < \lambda_2 + 2i \leq k \leq \lambda_1 + 2i = n - \lambda_2 + 1 \leq n.$$

When i and k are fixed, p is uniquely determined, since $p = \lambda_1 + 2i - k$. Since $\lambda_1 \geq \lambda_2 \geq 2$ and $\lambda_1 + \lambda_2 = n - 2i + 1$, we have

$$\left\lceil \frac{n+1}{2} \right\rceil - i \leq \lambda_1 \leq n - 2i - 1,$$

and thus

$$\left\lceil \frac{n+1}{2} \right\rceil - k + i \leq p = \lambda_1 + 2i - k \leq n - k - 1.$$

Hence when λ is fixed, k is bounded by the inequality

$$\lambda_2 + 2i \leq k \leq \lambda_1 + 2i,$$

and any possible integer k in this interval makes an occurrence of $s_\lambda(\mathbf{x})$. Since $\omega(\lambda) = 0$ and $i = \ell(\lambda) - 2$, we have that

$$a_\lambda(t) = (\lambda_1 - \lambda_2 + 1)t^{i+1} = \kappa(\lambda)t^{\ell(\lambda)-1}(t+1)^{\omega(\lambda)}.$$

Case 3. $\lambda_{\ell(\lambda)} \neq 1$. In this case, we have that λ is of the form $(\lambda_1, \lambda_2, 2^j)$, where $j = \ell(\lambda) - 2 \geq 0$.

When $j = 0$, $s_\lambda(\mathbf{x})$ can be obtained only from $B_n^{\langle 2\rangle}(\mathbf{x}; t)$ and $B_n^{\langle 1\rangle}(\mathbf{x}; t)$. When $j \geq 1$, $s_\lambda(\mathbf{x})$ can be obtained only from $B_n^{\langle 2\rangle}(\mathbf{x}; t)$ and $B_n^{\langle 4\rangle}(\mathbf{x}; t)$. Note that when $s_\lambda(\mathbf{x})$ is obtained from $B_n^{\langle 2\rangle}(\mathbf{x}; t)$, i should be $j + 1$, and thus t^{i+1} will be t^{j+2}. Along similar lines with Case 2, we have that

$$\begin{aligned}
a_\lambda(t) &= (\lambda_1 - \lambda_2 + 1)\, t^{j+2} + (\lambda_1 - \lambda_2 + 1)\, t^{j+1} \\
&= (\lambda_1 - \lambda_2 + 1)\, t^{\ell(\lambda)} + (\lambda_1 - \lambda_2 + 1)\, t^{\ell(\lambda)-1} \\
&= (\lambda_1 - \lambda_2 + 1)\, t^{\ell(\lambda)-1}(t + 1) \\
&= \kappa(\lambda)\, t^{\ell(\lambda)-1}(t + 1)^{\omega(\lambda)}.
\end{aligned}$$

Therefore, we have shown that for each partition λ of n, the coefficients of s_λ in $A_n(\mathbf{x}; t)$ and $B_n(\mathbf{x}; t)$ are equal. Thus $A_n(\mathbf{x}; t) = B_n(\mathbf{x}; t)$, which completes the proof. □

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