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A NONLOCAL TRANSPORT EQUATION DESCRIBING ROOTS OF POLYNOMIALS UNDER DIFFERENTIATION

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Abstract. Let p_n be a polynomial of degree n having all its roots on the real line distributed according to a smooth function $u(0,x)$. One could wonder how the distribution of roots behaves under iterated differentation of the function, i.e., how the density of roots of $p_n^{(k)}$ evolves. We derive a nonlinear transport equation with nonlocal flux

$$u_t + \frac{1}{\pi}\left(\arctan\left(\frac{Hu}{u}\right)\right)_x = 0 \qquad \text{on } \operatorname{supp}\{u > 0\},$$

where H is the Hilbert transform. This equation has three very different compactly supported solutions: (1) the arcsine distribution $u(t,x) = (1-x^2)^{-1/2}\chi_{(-1,1)}$, (2) the family of semicircle distributions

$$u(t,x) = \frac{2}{\pi}\sqrt{(T-t)-x^2},$$

and (3) a family of solutions contained in the Marchenko–Pastur law.

1. Introduction

If p_n is a polynomial of degree n having n distinct roots on the real line, then Rolle's theorem implies that $p_n^{(k)}$ has all its $n-k$ roots on the real line as well. Moreover, there is an interlacing phenomenon. A result commonly attributed to Riesz [22, 40] implies that the minimum gap between consecutive roots of p_n' is bigger than that of p_n: zeroes even out and become more regular. It is classical (and follows from interlacing) that if p_n has its roots distributed according to some nice distribution function, then p_n' has its roots distributed according to the same function as $n \to \infty$. The detailed study of the distribution of roots of p_n' depending on p_n is an active field [5, 6, 13, 15, 23, 26, 27, 31, 32, 34–36, 39, 41–43]. By the same reasoning, $p_n^{(k)}$ is also distributed following the same distribution for every fixed k as $n \to \infty$. However, this is no longer true when k grows with n.

Problem. Let (p_n) be a polynomial with $\deg p_n = n$ and having only real roots whose distribution approximates a smooth probability distribution on $\mathbb{R}$ in a strong quantitative sense (say Kolmogorov–Smirnov or Wasserstein distance). What can be said about the distribution of roots of $p_n^{(0.001n)}$?

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If the roots of p_n are evenly spaced at scale $\sim n^{-1}$, then interlacing implies that roots of the derivative are shifted by at most $\sim n^{-1}$, which implies that the dynamical evolution starts happening when the number of derivatives is comparable to the number of roots.

The equation. In the process of investigating this question, we came across a mean-field approximation that leads to a linear transport equation with nonlocal flux that can describe the evolution of the distribution of roots under iterated differentiation. The main purpose of this paper is to derive (in §3) and introduce the nonlinear equation

$$\boxed{u_t + \frac{1}{\pi}\left(\arctan\left(\frac{Hu}{u}\right)\right)_x = 0},$$

where the equation is valid on the support $\operatorname{supp} u = \{x : u(x) > 0\}$ and

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy \qquad \text{is the Hilbert transform.}$$

The equation has the obvious symmetries under translation $u(x) \to u(x-\lambda)$ and reflection $u(x) \to u(-x)$. Moreover, since the Hilbert transform Hu commutes with dilation, there is an additional symmetry $u(t,x) \to \lambda u(t, x/\lambda)$. If $\operatorname{supp} u$ is an interval, then, assuming sufficient regularity (u vanishing on the boundary of the support), we formally have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} u(x)dx &= \int_{\mathbb{R}} u_t(x)dx \\ &= -\frac{1}{\pi}\int_{\operatorname{supp} u} \frac{d}{dx}\left(\arctan\left(\frac{Hu}{u}\right)\right)dx = -1.\end{aligned}$$

This is in line with how the equation was derived: there should be a constant loss of mass since $p_n^{(t\cdot n)}$ has $(1-t)n$ roots. In particular, the solution should vanish in finite time at $t = 1$.

Related equations. The equation is quite nonlinear but somewhat similar to a series of recently derived one-dimensional transport equations with nonlocal flux given by the Hilbert transform or the fractional Laplacian. These were introduced as models for the quasi-geostrophic equation and one-dimensional analogues of the three-dimensional Navier–Stokes and Euler equations: we refer to Balodis and Córdoba [1], Carrillo, Ferreira, and Precioso [7], Castro and Córdoba [8], Chae, Córdoba, Córdoba, and Fontelos [9], Constantin, Lax, and Majda [11], Córdoba, Córdoba, and Fontelos [12], Do, Hoang, Radosz, and Xu [16], Dong [17], Dong and Li [18], Lazar and Lemarié-Rieusset [29], Li and Rodrigo [30] and Silvestre and Vicol [37]. Note added in print: Granero-Belinchon [25] has since studied an analogue of our equation on the one-dimensional torus. We believe that it is conceivable that (a) techniques from that field could conceivably be useful in studying our transport equation (which is rather nonlinear) and (b) that, conversely, the transport equation may be of interest in other contexts as well.

2. Three explicit solutions

We derive and describe three explicit compactly supported solutions (see Figure 1):

(1) the stationary arcsine solution (not on all of $\mathbb{R}$ but only on $(-1,1)$)

$$u(t,x) = \frac{c}{\sqrt{1-x^2}}\chi_{(-1,1)} \qquad \text{where} \quad c \in \mathbb{R},$$

(2) the semicircle solution

$$u(t,x) = \frac{2}{\pi}\sqrt{(T-t)-x^2} \cdot \chi_{|x| \le \sqrt{T-t}} \qquad \text{for } 0 \le t \le T,$$

(3) the Marchenko–Pastur solution: introducing, for $c \ge 0$,

$$v(c,x) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi x}\chi_{(x_-,x_+)} \quad \text{where} \quad x_\pm = (\sqrt{c+1} \pm 1)^2,$$

that solution is given by

$$u_c(t,x) = v\left(\frac{c+t}{1-t}, \frac{x}{1-t}\right).$$

2.1. **The arcsine solution.**

We first describe the stationary solution when considering the equation only in the interval $(-1,1)$; in contrast to the other two solutions, the solution has singularities at the boundary of its support. If a function $f : \mathbb{R} \to \mathbb{R}$ is compactly supported on $(-1,1)$ and has its Hilbert transform Hf vanish on its support, then it is given by the arcsine distribution

$$u(t,x) = \frac{c}{\sqrt{1-x^2}}\chi_{(-1,1)} \qquad \text{where} \quad c \in \mathbb{R}.$$

This is true in a rather strong sense: Coifman and the author [10] recently established that if $f(x)(1-x^2)^{1/4} \in L^2(-1,1)$ and $f(x)\sqrt{1-x^2}$ has mean value 0 on $(-1,1)$ (this enforces a form of orthogonality to the arcsine distribution), then

$$\int_{-1}^{1} (Hf)(x)^2 \sqrt{1-x^2}dx = \int_{-1}^{1} f(x)^2\sqrt{1-x^2}dx.$$

This is mirrored in the classical fact that orthogonal polynomials on $(-1,1)$ with respect to a fairly large class of weights have their distribution of roots converge to the arcsine distribution (see Erdős and Turan [20], Erdős and Freud [21], Ullman [42] and Van Assche [43]). Since the solution u is time-independent, it is tempting to linearize around it. The linearization is given by

$$w_t + \left(\sqrt{1-x^2}Hw\right)_x = 0.$$

If the initial datum $w(0,\cdot)$ is compactly supported on $(-1,1)$ and $w(0,x)\sqrt{1-x^2}$ has mean value 0, then the linearized equation has an explicit solution formula (derived in §4.2)

$$w(t,x) = \sum_{k=1}^{\infty}\left(\frac{2}{\pi}\int_{-1}^{1} w(0,x)T_k(x)dx\right)e^{kt}T_k(x),$$

where T_k denotes the Chebyshev polynomials of the first kind. We believe this to be interesting in its own right. This simple solution formula shows exponential growth of all nonzero solutions. Moreover, there is a stronger result.

Proposition. *If a solution exists for all $t \geq 0$ and $\|w(t,x)\sqrt{1-x^2}\|_{L^\infty_x} \leq c \cdot e^{dt}$ for some constant $c > 0$, then*

$$w(0,x)\sqrt{1-x^2} \qquad \text{is a polynomial of degree at most } d.$$

This follows immediately from the explicit solution formula which also implies that existence up to some time $t_0 > 0$ requires exponential decay of the inner products with Chebyshev polynomials, the function has to be almost polynomial. We emphasize that this is a *very* strong form of linear instability. It would be interesting to understand whether the linearizations around the other two solutions have comparable instability properties or whether they are stable (with the obvious dynamical implications for roots of polynomials); the arcsine distribution has a vanishing Hilbert transform which leads to a very simple linearization; for the other two explicit solutions of the transport equation the Hilbert transform does not vanish and understanding the linearizations seems more challenging.

2.2. **The semicircle distribution.** The construction of the semicircle solution is motivated by the behavior of the Hermite polynomials H_n. It is known that

(1) the roots of the Hermite polynomial H_n are approximately (in the sense of weak convergence after rescaling) given by the measure
$$\mu = \frac{1}{\pi}\sqrt{2n - x^2}dx;$$
(2) the derivatives of Hermite polynomials are again Hermite polynomials
$$\frac{d^m}{dx^m}H_n(x) = \frac{2^n n!}{(n-m)!}H_{n-m}(x).$$

This suggests that if our transport equation models the flow of roots, then the semicircle solution should turn into a self-similar one parameter family of solutions. A computation (carried out in §4.3) shows that, for every $T > 0$,

$$u(t,x) = \frac{2}{\pi}\sqrt{(T-t) - x^2} \cdot \chi_{|x| \leq \sqrt{T-t}} \qquad \text{for } t \leq T$$

is indeed a solution for $0 \leq t \leq T$.

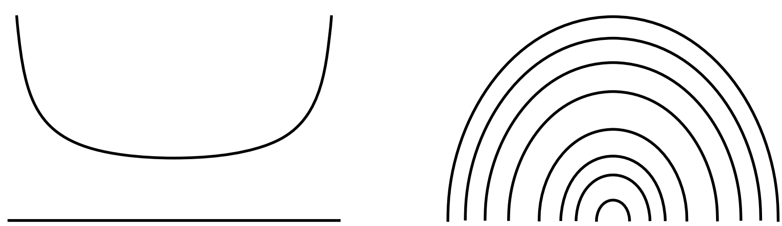

FIGURE 1. The arcsine solution (left) and the semicircle solution for $T = 1$ (right) both shown for $t \in \{0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99\}$.

2.3. **The Marchenko–Pastur solutions.** Our construction of the Marchenko–Pastur solution is motivated by the behavior of Laguerre polynomials. Laguerre polynomials L_n do not form an Appell sequence, i.e., they are not closed under differentiation, however, the larger family of associated Laguerre polynomials $L_n^{(\alpha)}$ satisfies

$$\frac{d^k}{dx^k} L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x).$$

Moreover, the asymptotic distribution of roots is given by a Marchenko–Pastur distribution (indexed by a parameter α): more precisely, it is classical [28] that for n large, the roots of $L_n^{(c\cdot n)}$ rescaled by a factor of n converge in distribution to the Marchenko–Pastur distribution

$$v(c,x) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi x} \chi_{(x_-,x_+)} dx, \quad \text{where} \quad x_\pm = (\sqrt{c+1} \pm 1)^2.$$

Combining these two facts, we see that, asymptotically and for $0 < t < 1$,

$$\frac{d^{t\cdot n}}{dx^{t\cdot n}} L_n^{c\cdot n} \sim \text{const} \cdot L_{(1-t)n}^{(c+t)\cdot n},$$

and this suggests that our nonlocal transport equation should have a solution of the form

$$u_c(t,x) = v\left(\frac{c+t}{1-t}, \frac{x}{1-t}\right).$$

This is indeed the case. For large values of c, the profile approximates that of the semicircle distribution (see Figure 2). Presumably this will have implications for the stability analysis around a semicircle distribution with Marchenko–Pastur solutions.

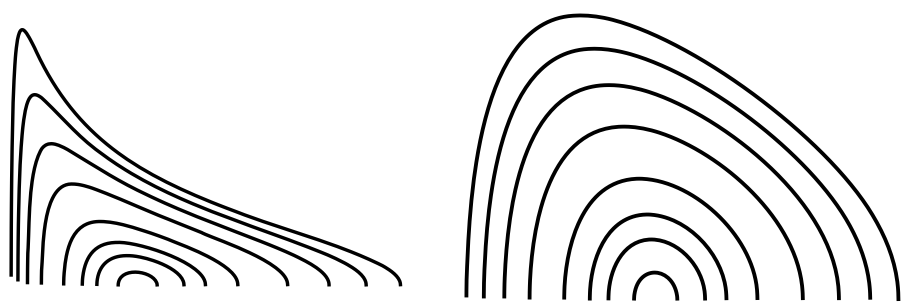

FIGURE 2. Marchenko–Pastur solutions $u_c(t,x)$: $c = 1$ (left) and $c = 15$ (right) shown for $t \in \{0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99\}$.

2.4. **Outlook.** We believe that this motivates a rather large number of problems; it is natural to ask about the properties of the transport equation itself: for which initial conditions is it well-posed? Is there a possibility of shock formation or finite-time blow-up? These questions might conceivably have direct analogues for roots of polynomials under differentiation; presumably Riesz' theorem [22, 40] implies some basic form of regularity. Another natural question is whether there is a rigorous derivation of the equation from polynomial dynamics in the small scale limit (this is likely to require a proper understanding of the microstructure of roots). Are there other explicit solutions of the equation that that can be derived? What can be said about the stability properties of the semicircle solution and the Marchenko–Pastur solution and does it correspond to polynomial dynamics? Finally, it seems natural to ask whether there is an analogous equation (or possibly systems of equations) for polynomials with roots in the complex plane.

3. Derivation of the equation

Our derivation is based on two ingredients: (1) the Gauss interpretation of roots of derivatives as electrostatic equilibria (see [23, 31, 38]) and (2) Euler's cotangent identity. Regarding (1), we note that for any polynomial p_n having roots in $\{x_1, \dots, x_n\} \subset \mathbb{R}$,

$$\frac{p_n'(x)}{p_n(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}.$$

This identity is also valid for polynomials in the complex plane with complex roots (thus suggesting that perhaps part of the derivation can be carried out in the complex plane; what is missing is an analogue of the cotangent identity and the additional difficulty that density no longer uniquely defines a lattice). The electrostatic interpretation also allows for an immediate proof of the Gauss–Lucas theorem [23, 31, 38, 39]: the roots of p_n' are contained in the convex hull of the roots of p_n. This, in terms of our transport equation, implies that compactly supported initial conditions give rise to compactly supported solutions (and that there is an inclusion relation for the support which is shrinking over time). Our second ingredient is the equation

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right) \qquad \text{for } x \in \mathbb{R} \setminus \mathbb{Z}$$

dating back to Euler's *Introductio in Analysis Infinitorum* (there is a particularly simple proof due to Herglotz [3, 19]). We now assume that the roots of a polynomial p_n of very large degree n are distributed according to a smooth density $u_0(x)$ and try to understand the microscopic movement of roots when passing from p_n to p_n' at the local scale n^{-1}. Let us fix a root $p_n(y) = 0$. Recalling

$$\frac{p_n'(x)}{p_n(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i},$$

we split the right-hand side of that equation around y into a far-field and a near-field. The far-field is approximately given by

$$\sum_{|x_i - y| \text{ large}}^{n} \frac{1}{x - x_i} \sim n \int_{\mathbb{R}} \frac{u_0(y)}{x - y} dy = n\pi (H u_0)(y),$$

where H is the Hilbert transform. Here, $|x_i - y|$ being "large" is to be understood as $n^{-1} \ll |x_i - y| \ll 1$. It remains to understand the near-field. Since the distribution u_0 is smooth, the local density does not vary on short scales and we may approximate the near-field created by the local roots with a lattice structure; since the local density is given by u_0, the spacing of the roots is given by $u_0(y)^{-1} n^{-1}$

and

$$\sum_{|x_i - y| \text{ small}}^{n} \frac{1}{x - x_i} \sim \frac{1}{x - y_0} + \sum_{k \in \mathbb{N}} \left(\frac{1}{x - k u_0(y)^{-1} n^{-1}} + \frac{1}{x + k u_0(y)^{-1} n^{-1}} \right)$$
$$= u_0(y) n\pi \cot\left(n\pi u_0(y)(x - y)\right).$$

The approximation is justified by the extremely fast convergence of the cotangent identity (assuming, of course, the underlying density to indeed be smooth). Roots of p_n' are created in places where the near-field and the far-field add up to 0; this leads to the equation

$$u_0(y) \cot\left(n\pi u_0(y)(x - y)\right) = (Hu_0)(y).$$

This equation can be solved leading to

$$x - y = -\frac{1}{n\pi} \arctan\left(\frac{(Hu_0)(y)}{u_0(y)}\right),$$

which informs us about the microscopix flux at scale $\sim n^{-1}$. This microscopic flow then gives rise to the transport equation

$$u_t + \frac{1}{\pi}\left(\arctan\left(\frac{Hu}{u}\right)\right)_x = 0.$$

Missing ingredients. There are two missing ingredients in making the derivation rigorous: (1) a rigorous understanding of the dynamics at the boundary of the support and (2) a rigorous understanding of what is happening in the bulk (this distinction is admittedly somewhat tautological).

(1) It is clear that our derivation, which assumes a flat background density of roots, must fail at the boundary where roots may have a different asymptotic behavior. We emphasize that ignoring these issues (which only affect a very small proportion of roots) still seems to result in a reasonable equation that is solved by at least three classical distributions. It might be that the contribution that the boundary has on the global dynamics is somewhat negligible, but this remains to be rigorously proven.

(2) The derivation in the bulk is accurate as long as $u(\cdot, t)$ is essentially constant on length scales slightly larger than n^{-1}. This requires the equation to somehow undergo a smoothing effect: the spacing between the roots becomes more regular and the change in scale of spacing evens out. This would perhaps not be all that surprising: we refer to the paper of Farmer and Rhoades [22] which discusses the possible existence of such a phenomenon and connects it to a series of classical results (i.e., an argument of M. Riesz [40], which shows that the smallest gap between roots increases when going from p_n to p_n'). These phenomena do not seem to be currently understood.

4. Verification of the solutions

4.1. **The arcsine solution.**

We recall an argument given by Coifman and the author in [10]: for this we introduce the Chebyshev polynomials T_k (that will also play a role in the subsequent sections) via

$$T_0(x) = 1, T_1(x) = x \quad \text{and} \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),$$

as well as Chebyshev polynomials of the second kind U_k given by

$$U_0(x) = 1, U_1(x) = 2x \quad \text{and} \quad U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x).$$

These sequences of polynomials are orthogonal, and for $n, m \geq 1$,

$$\frac{2}{\pi}\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \delta_{nm} \quad \text{and} \quad \frac{2}{\pi}\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = \delta_{nm}.$$

The crucial identity is

$$\frac{1}{\pi}\int_{-1}^{1} \frac{a_k T_k(y)}{(x-y)\sqrt{1-y^2}}dy = a_k U_{k-1}(x).$$

In particular, considering the function $g(x) = f(x)\sqrt{1-x^2}$ and expanding it into Chebyshev polynomials, we see that the Hilbert transform acts as a shift operator. That shift operator annihilates exactly constants. The shift operator is also responsible for the fact that if $f(x)(1-x^2)^{1/4} \in L^2(-1,1)$ and $f(x)\sqrt{1-x^2}$ has mean value 0 on $(-1,1)$, then

$$\int_{-1}^{1} (Hf)(x)^2\sqrt{1-x^2}dx = \int_{-1}^{1} f(x)^2\sqrt{1-x^2}dx.$$

This shows that if Hu vanishes on $(-1,1)$ for some u compactly supported on $(-1,1)$, then u is necessarily the arcsine distribution. This also shows that this is the only time-independent solution of our transport equation when restricted to an open interval.

4.2. **Linearization around the arcsine.** We now linearize the transport equation around the arcsine solution $u(t,x) = (1-x^2)^{-1/2}\chi_{(-1,1)}$. This linearization is given by

$$w_t + \left(\sqrt{1-x^2}Hw\right)_x = 0 \qquad \text{on } (-1,1).$$

We introduce a new function v by weighing w,

$$v(t,x) = w(t,x)\sqrt{1-x^2},$$

and obtaining

$$\frac{v_t}{\sqrt{1-x^2}} = -\left(\sqrt{1-x^2}H\left(\frac{v}{\sqrt{1-x^2}}\right)\right)_x.$$

We expand v into Chebyshev polynomials

$$v(t,x) = \sum_{k=0}^{\infty} a_k(t)T_k(x)$$

and use the identity

$$\frac{1}{\pi}\int_{-1}^{1} \frac{a_k T_k(y)}{(x-y)\sqrt{1-y^2}}dy = a_k U_{k-1}(x)$$

to conclude that

$$H\frac{v}{\sqrt{1-x^2}} = \sum_{k=1}^{\infty} a_k(t)U_{k-1}(x).$$

This shows that

$$\frac{1}{\sqrt{1-x^2}}\frac{\partial}{\partial t}\sum_{k=0}^{\infty} a_k(t)T_k(x) = -\frac{\partial}{\partial x}\sum_{k=1}^{\infty} a_k(t)\sqrt{1-x^2}U_{k-1}(x).$$

We now compute, using $T_k' = kU_{k-1}$ and the differential equation for Chebyshev polynomials of the first kind

$$(1-x^2)y'' - xy' + n^2 y = 0,$$

that the partial derivative in x simplifies to

$$\begin{aligned}\frac{\partial}{\partial x}\sqrt{1-x^2}U_{k-1}(x) &= -\frac{x}{\sqrt{1-x^2}}U_{k-1}(x) + \sqrt{1-x^2}U_{k-1}'(x)\\ &= \frac{1}{\sqrt{1-x^2}}\left((1-x^2)U_{k-1}'(x) - xU_{k-1}\right)\\ &= \frac{1}{\sqrt{1-x^2}}\frac{1}{k}\left((1-x^2)T_k''(x) - xT_k'(x)\right)\\ &= \frac{1}{\sqrt{1-x^2}}\frac{1}{k}\left(-k^2T_k(x)\right) = -\frac{kT_k(x)}{\sqrt{1-x^2}}\end{aligned}$$

to conclude

$$\frac{\partial}{\partial t}\sum_{k=0}^{\infty} a_k(t)T_k(x) = \sum_{k=1}^{\infty} a_k(t)kT_k(x),$$

and thus

$$v(t,x) = a_0 + \sum_{k=1}^{\infty} a_k(0)e^{kt}T_k(x),$$

where a_0 is a constant. This immediately implies the proposition. Moreover, we can compute the initial condition by using orthogonality,

$$a_k(0) = \frac{2}{\pi}\int_{-1}^{1} v(0,x)T_k(x)\frac{dx}{\sqrt{1-x^2}} = \frac{2}{\pi}\int_{-1}^{1} w(0,x)T_k(x)dx,$$

and this implies the solution formula written in terms of $w(0,\cdot)$.

4.3. **The semicircle solution.** As discussed above, the asymptotics of roots of Hermite polynomials combined with the fact that Hermite polynomials form an Appell sequence (closure under differentiation) suggests that

$$u(t,x) = \frac{2}{\pi}\sqrt{(T-t-x^2)}\cdot\chi_{|x|\le\sqrt{T-t}} \qquad \text{for } 0\le t\le T$$

should be a solution of the equation. Clearly,

$$\frac{\partial}{\partial t}u = -\frac{1}{\pi\sqrt{T-t-x^2}}.$$

It remains to compute the Hilbert transform Hu. The Hilbert transform commutes with positive dilations and is linear. We thus scale the function by a factor of $\sqrt{T-t}$ to reduce it to the computation of the Hilbert transform of $(1-x^2)_+^{1/2}$ supported on $(-1,1)$. This reduces to a simple identity for Chebyshev polynomials of the second kind U_k,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-y^2}U_{n-1}(y)}{x-y}dy = T_n(x),$$

since $U_0(x)=1$ and $T_1(x)=x$, and thus, for x in the support of u,

$$Hu(t,x) = \frac{2x}{\pi}\chi_{(-\sqrt{T-t},\sqrt{T-t})},$$

where χ is the characteristic function. A simple computation shows that

$$\frac{1}{\pi}\left(\arctan\left(\frac{Hu}{u}\right)\right)_x = \frac{1}{\pi}\left(\arctan\frac{x}{\sqrt{(T-t-x^2)}}\right)_x = \frac{1}{\pi\sqrt{T-t-x^2}},$$

and this shows that the semicircle solution solves the transport equation.

4.4. **The Marchenko–Pastur solution.** Laguerre polynomials $L_n^{(\alpha)}$ are given by the recursion formula

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}}{n!}\left(\frac{d}{dx}-1\right)^n x^{n+\alpha}.$$

Their behavior under differentiation is fairly easy to describe:

$$\frac{d^k}{dx^k}L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x).$$

The behavior of the roots of $L_n^{(\alpha)}$ for $\alpha \geq 0$ is essentially classical [4, 14, 24, 28, 33]. The result that will inspire the construction of our solution uses that if α_n is a sequence such that $\alpha_n/n \to c \in (-1,\infty)$, then the empirical distribution of the roots of $L_n^{(\alpha_n)}$ rescaled by a factor of n converges weakly to the Marchenko–Pastur distribution

$$v(c,x) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi x}\chi_{(x_-,x_+)}dx, \quad \text{where} \quad x_\pm = (\sqrt{c+1}\pm 1)^2.$$

Heuristically, we see that if

$$\text{roots of } L_n^{(c\cdot n)} \sim v(c,x), \quad \text{then} \quad \text{roots of } L_{n(1-\varepsilon)}^{((c+\varepsilon)\cdot n)} \sim v\left(\frac{c+\varepsilon}{1-\varepsilon}, \frac{x}{1-\varepsilon}\right).$$

This suggests the existence of a solution of the form

$$u(t,x) = v\left(\frac{c+t}{1-t}, \frac{x}{1-t}\right).$$

We now verify the existence of the solution. The Hilbert transform of the Marchenko–Pastur distribution is known (see, e.g., [2, §5.5.2]) and given by

$$Hv(c,x) = \frac{x-c}{2\pi x} \qquad \text{on } (x_-,x_+).$$

A somewhat lengthy computation then shows that

$$\frac{1}{\pi}\left(\arctan\left(\frac{Hv(\frac{c+t}{1-t},\frac{1}{t-t})}{v(\frac{c+t}{1-t},\frac{1}{t-t})}\right)\right)_x = \frac{c+t+x}{2\pi x\sqrt{2(2+c-t)x-(c+t)^2-x^2}},$$

while

$$\frac{\partial}{\partial t}v\left(\frac{c+t}{1-t},\frac{x}{1-t}\right) = -\frac{c+t+x}{2\pi x\sqrt{2(2+c-t)x-(c+t)^2-x^2}},$$

as desired.

References

[1] Pedro Balodis and Antonio Córdoba, *An inequality for Riesz transforms implying blow-up for some nonlinear and nonlocal transport equations*, Adv. Math. **214** (2007), no. 1, 1–39, DOI 10.1016/j.aim.2006.07.021. MR2348021

[2] Gordon Blower, *Random matrices: high dimensional phenomena*, London Mathematical Society Lecture Note Series, vol. 367, Cambridge University Press, Cambridge, 2009. MR2566878

[3] Salomon Bochner, *Book Review: Gesammelte Schriften*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 6, 1020–1022, DOI 10.1090/S0273-0979-1979-14724-4. MR1567203

[4] Christof Bosbach and Wolfgang Gawronski, *Strong asymptotics for Laguerre polynomials with varying weights*, Proceedings of the VIIIth Symposium on Orthogonal Polynomials and Their Applications (Seville, 1997), J. Comput. Appl. Math. **99** (1998), no. 1-2, 77–89, DOI 10.1016/S0377-0427(98)00147-2. MR1662685

[5] N. G. de Bruijn, *On the zeros of a polynomial and of its derivative*, Nederl. Akad. Wetensch., Proc. **49** (1946), 1037–1044 = Indagationes Math. 8, 635–642 (1946). MR0019157

[6] N. G. de Bruijn and T. A. Springer, *On the zeros of a polynomial and of its derivative. II*, Nederl. Akad. Wetensch., Proc. **50** (1947), 264–270=Indagationes Math. 9, 458–464 (1947). MR0021148

[7] José A. Carrillo, Lucas C. F. Ferreira, and Juliana C. Precioso, *A mass-transportation approach to a one dimensional fluid mechanics model with nonlocal velocity*, Adv. Math. **231** (2012), no. 1, 306–327, DOI 10.1016/j.aim.2012.03.036. MR2935390

[8] A. Castro and D. Córdoba, *Global existence, singularities and ill-posedness for a nonlocal flux*, Adv. Math. **219** (2008), no. 6, 1916–1936, DOI 10.1016/j.aim.2008.07.015. MR2456270

[9] Dongho Chae, Antonio Córdoba, Diego Córdoba, and Marco A. Fontelos, *Finite time singularities in a 1D model of the quasi-geostrophic equation*, Adv. Math. **194** (2005), no. 1, 203–223, DOI 10.1016/j.aim.2004.06.004. MR2141858

[10] R. Coifman and S. Steinerberger, *A Remark on the Arcsine Distribution and the Hilbert transform*, arXiv:1810.10128.

[11] P. Constantin, P. D. Lax, and A. Majda, *A simple one-dimensional model for the three-dimensional vorticity equation*, Comm. Pure Appl. Math. **38** (1985), no. 6, 715–724, DOI 10.1002/cpa.3160380605. MR812343

[12] Antonio Córdoba, Diego Córdoba, and Marco A. Fontelos, *Formation of singularities for a transport equation with nonlocal velocity*, Ann. of Math. (2) **162** (2005), no. 3, 1377–1389, DOI 10.4007/annals.2005.162.1377. MR2179734

[13] Branko Ćurgus and Vania Mascioni, *A contraction of the Lucas polygon*, Proc. Amer. Math. Soc. **132** (2004), no. 10, 2973–2981, DOI 10.1090/S0002-9939-04-07231-4. MR2063118

[14] H. Dette and W. J. Studden, *Some new asymptotic properties for the zeros of Jacobi, Laguerre, and Hermite polynomials*, Constr. Approx. **11** (1995), no. 2, 227–238, DOI 10.1007/BF01203416. MR1342385

[15] Dimitar K. Dimitrov, *A refinement of the Gauss-Lucas theorem*, Proc. Amer. Math. Soc. **126** (1998), no. 7, 2065–2070, DOI 10.1090/S0002-9939-98-04381-0. MR1452801

[16] Tam Do, Vu Hoang, Maria Radosz, and Xiaoqian Xu, *One-dimensional model equations for hyperbolic fluid flow*, Nonlinear Anal. **140** (2016), 1–11, DOI 10.1016/j.na.2016.03.002. MR3492724

[17] Hongjie Dong, *Well-posedness for a transport equation with nonlocal velocity*, J. Funct. Anal. **255** (2008), no. 11, 3070–3097, DOI 10.1016/j.jfa.2008.08.005. MR2464570

[18] Hongjie Dong and Dong Li, *On a one-dimensional α-patch model with nonlocal drift and fractional dissipation*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 2041–2061, DOI 10.1090/S0002-9947-2013-06075-8. MR3152722

[19] Jürgen Elstrodt, *Partialbruchentwicklung des Kotangens, Herglotz-Trick und die Weierstraßsche stetige, nirgends differenzierbare Funktion* (German, with German summary), Math. Semesterber. **45** (1998), no. 2, 207–220, DOI 10.1007/s005910050046. MR1684563

[20] Paul Erdős and Paul Turán, *On interpolation. III. Interpolatory theory of polynomials*, Ann. of Math. (2) **41** (1940), 510–553, DOI 10.2307/1968733. MR0001999

[21] P. Erdős and G. Freud, *On orthogonal polynomials with regularly distributed zeros*, Proc. London Math. Soc. (3) **29** (1974), 521–537, DOI 10.1112/plms/s3-29.3.521. MR0420119

[22] David W. Farmer and Robert C. Rhoades, *Differentiation evens out zero spacings*, Trans. Amer. Math. Soc. **357** (2005), no. 9, 3789–3811, DOI 10.1090/S0002-9947-05-03721-9. MR2146650

[23] C. F. Gauss, *Werke*, Band 3, Göttingen 1866, S. 120:112.

[24] Wolfgang Gawronski, *Strong asymptotics and the asymptotic zero distributions of Laguerre polynomials $L_n^{(an+\alpha)}$ and Hermite polynomials $H_n^{(an+\alpha)}$*, Analysis **13** (1993), no. 1-2, 29–67, DOI 10.1524/anly.1993.13.12.29. MR1245742

[25] R. Granero-Belinchon, *On a nonlocal differential equation describing roots of polynomials under differentiation*, arXiv:1812.00082.

[26] Boris Hanin, *Pairing of zeros and critical points for random polynomials* (English, with English and French summaries), Ann. Inst. Henri Poincaré Probab. Stat. **53** (2017), no. 3, 1498–1511, DOI 10.1214/16-AIHP767. MR3689975

[27] Zakhar Kabluchko, *Critical points of random polynomials with independent identically distributed roots*, Proc. Amer. Math. Soc. **143** (2015), no. 2, 695–702, DOI 10.1090/S0002-9939-2014-12258-1. MR3283656

[28] Miklós Kornyik and György Michaletzky, *On the moments of roots of Laguerre-polynomials and the Marchenko-Pastur law*, Ann. Univ. Sci. Budapest. Sect. Comput. **46** (2017), 137–151. MR3722670

[29] Omar Lazar and Pierre-Gilles Lemarié-Rieusset, *Infinite energy solutions for a 1D transport equation with nonlocal velocity*, Dyn. Partial Differ. Equ. **13** (2016), no. 2, 107–131, DOI 10.4310/DPDE.2016.v13.n2.a2. MR3520809

[30] Dong Li and José L. Rodrigo, *On a one-dimensional nonlocal flux with fractional dissipation*, SIAM J. Math. Anal. **43** (2011), no. 1, 507–526, DOI 10.1137/100794924. MR2783214

[31] F. Lucas, *Sur une application de la Mécanique rationnelle à la théorie des équations*, in: Comptes Rendus de l'Académie des Sciences (89), Paris 1979, S. 224–226.

[32] S. M. Malamud, *Inverse spectral problem for normal matrices and the Gauss-Lucas theorem*, Trans. Amer. Math. Soc. **357** (2005), no. 10, 4043–4064, DOI 10.1090/S0002-9947-04-03649-9. MR2159699

[33] Andrei Martínez-Finkelshtein, Pedro Martínez-González, and Ramón Orive, *On asymptotic zero distribution of Laguerre and generalized Bessel polynomials with varying parameters*, Proceedings of the Fifth International Symposium on Orthogonal Polynomials, Special Functions and their Applications (Patras, 1999), J. Comput. Appl. Math. **133** (2001), no. 1-2, 477–487, DOI 10.1016/S0377-0427(00)00654-3. MR1858305

[34] Sean O'Rourke and Noah Williams, *Pairing between zeros and critical points of random polynomials with independent roots*, Trans. Amer. Math. Soc. **371** (2019), no. 4, 2343–2381, DOI 10.1090/tran/7496. MR3896083

[35] Robin Pemantle and Igor Rivin, *The distribution of zeros of the derivative of a random polynomial*, Advances in combinatorics, Springer, Heidelberg, 2013, pp. 259–273. MR3363974

[36] M. Ravichandran, *Principal submatrices, restricted invertibility and a quantitative Gauss-Lucas theorem*, arXiv:1609.04187.

[37] Luis Silvestre and Vlad Vicol, *On a transport equation with nonlocal drift*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6159–6188, DOI 10.1090/tran6651. MR3461030

[38] Stefan Steinerberger, *Electrostatic interpretation of zeros of orthogonal polynomials*, Proc. Amer. Math. Soc. **146** (2018), no. 12, 5323–5331, DOI 10.1090/proc/14226. MR3866871

[39] S. Steinerberger, *A Stability Version of the Gauss-Lucas Theorem and Applications*, to appear in J. Austral. Math. Soc.

[40] A. Stoyanoff, *Sur un Theorem de M. Marcel Riesz*, Nouv. Annal. de Mathematique **1** (1926), 97–99.

[41] Vilmos Totik, *The Gauss-Lucas theorem in an asymptotic sense*, Bull. Lond. Math. Soc. **48** (2016), no. 5, 848–854, DOI 10.1112/blms/bdw047. MR3556367

[42] J. L. Ullman, *On the regular behaviour of orthogonal polynomials*, Proc. London Math. Soc. (3) **24** (1972), 119–148, DOI 10.1112/plms/s3-24.1.119. MR0291718

[43] Walter Van Assche, *Asymptotics for orthogonal polynomials*, Lecture Notes in Mathematics, vol. 1265, Springer-Verlag, Berlin, 1987. MR903848

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