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CR REGULAR EMBEDDINGS OF $S^{4n-1}$ IN $\mathbb{C}^{2n+1}$

NAOHIKO KASUYA

Abstract. Ahern and Rudin have given an explicit construction of a totally real embedding of $S^3$ in $\mathbb{C}^3$. As a generalization of their example, we give an explicit example of a CR regular embedding of $S^{4n-1}$ in $\mathbb{C}^{2n+1}$. Consequently, we show that the odd dimensional sphere $S^{2m-1}$ with $m > 1$ admits a CR regular embedding in $\mathbb{C}^{m+1}$ if and only if $m$ is even.

1. Introduction

Suppose $F: M^n \to \mathbb{C}^q$ is a smooth embedding of an $n$-manifold in $\mathbb{C}^q$. Then, for any point $x \in M^n$ and the standard complex structure $J$ on $\mathbb{C}^q$, the following inequality holds:

$$\dim_{\mathbb{C}}(dF_x(T_xM^n) \cap JdF_x(T_xM^n)) \geq n - q.$$ 

If the equality holds for each point $x \in M^n$, the embedding $F$ is called a CR regular embedding, and when $n = q$, we say that $F$ is a totally real embedding and $F(M^n)$ is a totally real submanifold.

Totally real submanifolds have been investigated by many geometers and topologists. Especially, the problem of determining which manifolds admit a totally real embedding in $\mathbb{C}^n$ has been widely studied from the viewpoint of the $h$-principle (Gromov [6, 7, 8], Lees [13], Forstnerič [5], Audin [2]). On the other hand, Ahern and Rudin [1] have constructed an explicit example of a totally real embedding of $S^3$ in $\mathbb{C}^3$. In the following, let $z = (z_1, z_2, \ldots, z_m)$ be the coordinates on $\mathbb{C}^m$ and we regard $S^{2m-1}$ as the unit sphere in $\mathbb{C}^m$.

Theorem 1.1 (Ahern-Rudin [1]). Let $P(z_1, z_2) = z_2 z_1 z_3^3 + i z_1 z_2^2 z_2$. Then, the embedding $F: S^3 \to \mathbb{C}^3$ defined by $F(z_1, z_2) = (z_1, z_2, P(z_1, z_2))$ is a totally real embedding.

CR regular embeddings also have been studied by many authors from various viewpoints (Cartan [3], Tanaka [16, 17], Wells [20, 21], Lai [12], Jacobowitz-Landweber [9], Slapar [14, 15], Torres [18, 19], Elgindi [4]). In [10, Section 5] and [11], the author and Takase have worked on the problem of determining when the $n$-sphere $S^n$ admits a CR regular embedding in $\mathbb{C}^q$ and have given some necessary conditions on $(n, q)$. In particular, we have proved that the $(4n + 1)$-dimensional sphere $S^{4n+1}$ does not admit a CR regular embedding in $\mathbb{C}^{2n+2}$ ([11, Theorem 5.2 (c)]). In this paper, we settle the remaining codimension three case by generalizing Theorem 1.1. The following is our main theorem.

Theorem 1.2. Let

$$Q(z_1, z_2, \ldots, z_{2n-1}, z_{2n}) = \sum_{k=1}^{n} P(z_{2k-1}, z_{2k}),$$

where $P(x, y) = y^3 - 2x^2y$. Then, the embedding $F: S^{4n-1} \to \mathbb{C}^{2n+1}$ defined by

$$F(z_1, z_2, \ldots, z_{2n-1}, z_{2n}) = \left( z_1, z_2, \ldots, z_{2n-1}, z_{2n}, Q(z_1, z_2, \ldots, z_{2n-1}, z_{2n}) \right)$$

is a CR regular embedding.

Corollary 1.3. Let $m$ be an integer greater than $1$. The odd dimensional sphere $S^{2m-1}$ admits a CR regular embedding in $\mathbb{C}^{m+1}$ if and only if $m$ is even.

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2. Proof of Main Theorem

For a smooth complex-valued function $f$ on $\mathbb{C}^m$, we use the following notations:

$$\frac{\partial f}{\partial z_j} = \sum_{j=1}^m \frac{\partial f}{\partial z_j} \, dz_j, \quad \hat{f} = \sum_{j=1}^m \frac{\partial f}{\partial \bar{z}_j},$$

$$\frac{\partial f}{\partial ar{z}} = (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_m}), \quad \frac{\partial f}{\partial z} = (\frac{\partial f}{\partial ar{z}_1}, \ldots, \frac{\partial f}{\partial ar{z}_m}).$$

Lemma 2.1. Let $u$ and $v$ be the real part and the imaginary part of a smooth function $f: \mathbb{C}^m \to \mathbb{C}$, respectively. Then, $\partial u \wedge \partial v = \frac{i}{2} \partial f \wedge \partial \bar{f}$.

Proof. Since $f = u + iv$, we have $\partial f = \partial u + i\partial v$ and $\partial \bar{f} = \partial u - i\partial v$. Hence,

$$\partial f \wedge \partial \bar{f} = (\partial u + i\partial v) \wedge (\partial u - i\partial v) = -2i\partial u \wedge \partial v.$$

$\square$

In [9], Jacobowitz and Landweber have given a necessary and sufficient condition for an embedding to be a CR regular embedding.

Proposition 2.2 (Jacobowitz-Landweber [9]). An embedding $F: M^{2n+1} \to \mathbb{C}^{n+q}$ is a CR regular embedding if and only if the submanifold $F(M^{2n+1})$ is given by simultaneous real equations

$$\rho_j(z_1, z_2, \ldots, z_{n+q}) = 0 \quad (j = 1, \ldots, k)$$

satisfying $\partial \rho_1 \wedge \cdots \wedge \partial \rho_k \neq 0$ at each point of $F(M^{2n+1})$.

Applying this proposition to the case where the submanifold is the graph of a function, we obtain the following.

Proposition 2.3. Let $f_j$ ($j = 1, \ldots, q$) be smooth complex-valued functions on $\mathbb{C}^m$ with $1 \leq q \leq m-1$. The embedding $F: S^{2m-1} \to \mathbb{C}^{m+q}$ defined by

$$F(z) = (z, f_1(z), \ldots, f_q(z))$$

is a CR regular embedding if and only if the $(q+1)$ complex vectors $z$, $\frac{\partial f_j}{\partial \bar{z}}(z)$ ($j = 1, \ldots, q$) are linearly independent over $\mathbb{C}$ for each $z \in S^{2m-1}$.

Proof. The submanifold $F(S^{2m-1})$ is described as

$$\{ (z, z_{m+1}, \ldots, z_{m+q}) \in \mathbb{C}^{m+q} \mid \|z\|^2 = 1, z_{m+1} = f_1(z), \ldots, z_{m+q} = f_q(z) \}.$$

We define smooth real functions $\rho_1(z), \ldots, \rho_{2q+1}(z)$ by

$$\rho_1 = -1 + \sum_{k=1}^m z_k \bar{z}_k, \quad \rho_j(z) = f_j(z) + i\rho_{2j+1} \quad (j = 1, \ldots, q),$$

for which we have

$$F(S^{2m-1}) = \rho_1^{-1}(0) \cap \rho_2^{-1}(0) \cap \cdots \cap \rho_{2q+1}^{-1}(0).$$

By Lemma 2.1,

$$\partial \rho_{2j} \wedge \partial \rho_{2j+1} = \frac{i}{2} \partial (z_{m+1} - f_j(z)) \wedge (\partial z_{m+1} - \partial f_j) = \frac{i}{2} (\partial f_j) \wedge (\partial z_{m+1} - \partial f_j).$$

Therefore, $\partial \rho_1 \wedge \partial \rho_2 \wedge \cdots \wedge \partial \rho_{2q+1} \neq 0$ holds if and only if

$$\left( \bar{z}_1 dz_1 + \cdots + \bar{z}_m dz_m \right) \wedge \left( \frac{\partial f_1}{\partial \bar{z}_1} dz_1 + \cdots + \frac{\partial f_1}{\partial \bar{z}_m} dz_m \right) \wedge \cdots \wedge \left( \frac{\partial f_q}{\partial \bar{z}_1} dz_1 + \cdots + \frac{\partial f_q}{\partial \bar{z}_m} dz_m \right) \neq 0.$$
holds. This condition is equivalent to the complex vectors
\[ z = (z_1, \ldots, z_m), \frac{\partial f_1}{\partial z_j} = (\frac{\partial f_1}{\partial z_1}, \ldots, \frac{\partial f_1}{\partial z_m}), \ldots, \frac{\partial f_n}{\partial z_j} = (\frac{\partial f_n}{\partial z_1}, \ldots, \frac{\partial f_n}{\partial z_m}) \]
being linearly independent over \( \mathbb{C} \).

Now, we are ready to prove our main theorem. First we reprove Ahern-Rudin’s result from the viewpoint of Proposition 2.3, and then, prove Theorem 1.2.

**Proof of Theorem 1.1.** When \( (z_1, z_2) \neq (0, 0) \), the two vectors \( (z_1, z_2) \) and \( \left( \frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2} \right) \) are linearly independent over \( \mathbb{C} \). Indeed, the function
\[ z_2 \frac{\partial P}{\partial z_1} - z_1 \frac{\partial P}{\partial z_2} = |z_1|^2(|z_2|^2 - 2|z_1|^2) - i|z_1|^2(|z_1|^2 - 2|z_2|^2) \]
vanishes only at the origin \( (0, 0) \in \mathbb{C}^2 \). Therefore, by Proposition 2.3, the embedding \( F \) is a totally real embedding.

**Proof of Theorem 1.2.** Suppose \( z = (z_1, z_2, \ldots, z_{2n-1}, z_{2n}) = (0, 0, \ldots, 0) \). Then there exists \( j \) such that \( (z_{j-1}, z_j) \neq (0, 0) \). For such a \( j \), the two vectors \( (z_{j-1}, z_j) \) and \( \left( \frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2} \right) \) are linearly independent over \( \mathbb{C} \) by the proof of Theorem 1.1. Hence, the two vectors \( z = (z_1, z_2, \ldots, z_{2n-1}, z_{2n}) \) and
\[ \frac{\partial Q}{\partial z}(z) = \left( \frac{\partial P}{\partial z}(z_1, z_2), \ldots, \frac{\partial P}{\partial z}(z_{2n-1}, z_{2n}) \right) \]
are linearly independent over \( \mathbb{C} \). Then, by Proposition 2.3, the embedding \( F \) is CR regular.

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**References**


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