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Endomorphisms of power series fields and residue fields of Fargues-Fontaine curves

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ENDOMORPHISMS OF POWER SERIES FIELDS AND RESIDUE FIELDS OF FARGUES-FONTAINE CURVES

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ABSTRACT. Let k be a field. We show that every endomorphism of the completed algebraic closure of $k((t))$ which restricts to an automorphism on k is itself an automorphism. As a corollary, we resolve a question of Fargues and Fontaine by showing that for p a prime and $\mathbb{C}_p$ a completed algebraic closure of $\mathbb{Q}_p$, every closed point of the Fargues-Fontaine curve associated to $\mathbb{C}_p$ has residue field $\mathbb{C}_p$. The argument involves a bit of analysis of ramified covers of closed discs in Berkovich's nonarchimedean analytic geometry, which may be of independent interest.

1. INTRODUCTION

In this short note, we address the following question.

Question 1.1. *Let K be a nonarchimedean field, i.e., a field complete with respect to a nontrivial real valuation (which we always notate multiplicatively). Is every continuous homomorphism from K to itself which induces automorphisms of residue fields and value groups necessarily surjective (and hence an automorphism)?*

We will view Question 1.1 as a collection of distinct cases indexed by the choice of K. For example, it is easy to check (Proposition 3.1) that one has an affirmative answer if K is spherically complete, so in particular if K in discretely valued. On the other hand, one can construct examples of fields K for which Question 1.1 admits a negative answer (Example 3.2).

Our main result is an affirmative answer to Question 1.1 in the intermediate case where K is the completed algebraic closure of a power series field over a prime field. More precisely, we prove the following result.

Theorem 1.2. *Let K be a completed algebraic closure of $k((t))$ for some field k. Let $\tau : K \to K$ be a continuous homomorphism inducing an isomorphism on residue fields, such that $\tau(k) \subseteq k$. Then τ is an isomorphism.*

The condition on τ is vacuous if k is the algebraic closure of a prime field. Otherwise, we do not get a full affirmative answer to Question 1.1, and considerations similar to Example 3.2 show that the negative answer is sometimes possible.

This theorem was prompted by an application to a foundational question of p-adic Hodge theory, specifically in the *perfectoid correspondence* (commonly known as *tilting*) between

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nonarchimedean fields in mixed and equal characteristics (generalizing the *field of norms correspondence* of Fontaine and Wintenberger). A nonarchimedean field K of residue characteristic p is *perfectoid* if it is not discretely valued and the Frobenius automorphism on $\mathfrak{o}_K/(p)$ is surjective. Given such a field, let $K^\flat$ be the inverse limit of K under the p-power map; one then shows that $K^\flat$ naturally carries the structure of a perfectoid (and hence perfect) nonarchimedean field of equal characteristic p and that there is a canonical isomorphism between the absolute Galois groups of K and $K^\flat$ [10, 11, 12]. The functor $K \mapsto K^\flat$ is not fully faithful; for instance, one can construct many algebraic extensions of $\mathbb{Q}_p$ whose completions K map to the completed perfect closure of a power series field over $\mathbb{F}_p$ (e.g., the cyclotomic extension $\mathbb{Q}_p(\mu_{p^\infty})$ and the Kummer extension $\mathbb{Q}_p(p^{1/p^\infty})$). However, Fargues and Fontaine have asked [6] whether this can happen for a completed algebraic closure of $\mathbb{Q}_p$, and using Theorem 1.2 we are able to rule this out.

Theorem 1.3. *Let p be a prime number and let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$. If K is a perfectoid field of characteristic 0 such that $K^\flat \cong \mathbb{C}_p^\flat$ (i.e., $K^\flat$ is the completed algebraic closure of a power series field over $\mathbb{F}_p$), then K is itself isomorphic (though not canonically) to $\mathbb{C}_p$.*

This result admits the following geometric interpretation. For each perfectoid field K, Fargues and Fontaine define an associated scheme X_K which is a "complete curve" (i.e., a regular one-dimensional noetherian scheme equipped with a surjection of its Picard group onto $\mathbb{Z}$) in terms of which p-adic Hodge theory over K can be simply formulated. Theorem 1.3 implies that for $K = \mathbb{C}_p$, the residue fields of the closed points of X_K are all isomorphic to $\mathbb{C}_p$, though not canonically so. However, one does not expect the same result to hold with $\mathbb{C}_p$ replaced by a larger algebraically closed perfectoid field.

Using the Ax-Sen-Tate theorem on invariants of Galois actions on completed algebraic closures of henselian fields [1], we may formally promote Theorem 1.3 so as to identify those perfectoid fields which tilt to completed algebraic extensions of power series fields.

Theorem 1.4. *Any perfectoid field K for which $K^\flat$ is the completion of an algebraic extension of $\mathbb{F}_p((t))$ is itself the completion of an algebraic extension of $\mathbb{Q}_p$.*

In particular, while (as noted above) the completed perfect closure of $\mathbb{F}_p((t))$ appears as the tilt of many different completed algebraic extensions of $\mathbb{Q}_p$, it cannot additionally occur as the tilt of a perfectoid field which does not contain a dense subfield algebraic over $\mathbb{Q}_p$.

2. Covers of Berkovich discs

As a technical input into the proof of Theorem 1.2, we study the geometry of ramified covers of a closed unit disc over a nonarchimedean field in the sense of Berkovich [4]. The resulting statements may be of independent interest.

Hypothesis 2.1. Throughout §2, let K be an algebraically closed nonarchimedean field. The algebraically closed hypothesis could be relaxed at the expense of some complexity, but would provide no benefit for the applications in this paper.

Definition 2.2. For A a commutative nonarchimedean Banach ring, let $\mathcal{M}(A)$ denote the *Gel'fand spectrum* of A in the sense of Berkovich [4, Chapter 1], i.e., the set of bounded multiplicative seminorms on A equipped with the evaluation topology (the subspace topology for the inclusion $\mathcal{M}(A) \hookrightarrow \mathbb{R}^A$). For each $\alpha \in \mathcal{M}(A)$, let $\mathcal{H}(\alpha)$ be the *residue field* of α, obtained by completing $\mathrm{Frac}(A/\ker(\alpha))$ for the induced multiplicative norm.

Definition 2.3. Let $K\{t\}$ be the Tate algebra over K, i.e., the completion of $K[t]$ for the Gauss norm. For each $z \in \mathfrak{o}_K$ and each $\rho \in [0,1]$, the ρ-Gauss seminorm on $K[t-z]$ extends to a seminorm $\alpha_{z,\rho} \in \mathcal{M}(K\{t\})$. Following Berkovich [4, (1.4.4)], [3, §1.2], [8, Proposition 2.2.7], [9, Theorem 2.26], it is customary to classify points of $\mathcal{M}(K\{t\})$ into the following four types.

1. Points of the form $\alpha_{z,0}$ for some $z \in \mathfrak{o}_K$. For any such point, the map $K \to K\{t\} \to \mathcal{H}(\alpha)$ is an isomorphism.
2. Points of the form $\alpha_{z,\rho}$ for some $z \in \mathfrak{o}_K$ and some $\rho \in (0,1] \cap |K^\times|$. This includes the point $\alpha_{0,1}$, commonly called the *Gauss point.*
3. Points of the form $\alpha_{z,\rho}$ for some $z \in \mathfrak{o}_K$ and some $\rho \in (0,1) - |K^\times|$.
4. All other points.

There is a canonical retraction of $\mathcal{M}(K\{t\})$ onto the Gauss point [9, Theorem 2.5]. We will need an extension of the construction to integral extensions of $K\{t\}$.

Definition 2.4. Let S_0 be the integral closure of $K\{t\}$ in a finite separable extension of its fraction field. For i a nonnegative integer, define the map $d^i : K\{t\} \to K\{t\}$ by the formula

$$d^i\left(\sum_{j=0}^{\infty} a_j t^j\right) = \sum_{j=0}^{\infty} \binom{j+i}{i} a_{j+i} t^j.$$

In case K is of characteristic zero, we have $d^i = \frac{1}{i!}\frac{d^i}{dt^i}$, so the unique extension of $d^1 = \frac{d}{dt}$ to S_0 defines an extension of each d^i to S_0.

In case K is of characteristic $p > 0$, this argument only provides an extension of d^i for $i < p$; to go further, we argue as follows. Let φ denote the absolute Frobenius on either $K\{t\}$ or S_0; we then have a well-defined *Cartier operator*

$$C = \varphi^{-1} \circ d^{p-1}$$

on both $K\{t\}$ and S_0. We may use the Cartier operator to see that the natural map

$$S_0^p \otimes_{K\{t^p\}} K\{t\} \to S_0$$

is bijective, or equivalently that every $s \in S$ has a unique representation in the form $\sum_{j=0}^{p-1} \varphi(s_j) t^j$: namely, we can and must take

$$s_{p-1} = C(s), \qquad s_{p-2} = C(t(s - \varphi(s_{p-1})t^{p-1})), \qquad \dots.$$

By induction on j, we see that the natural maps

$$S_0^{p^j} \otimes_{K\{t^{p^j}\}} K\{t\} \to S_0$$

are all isomorphisms; we may thus extend d^{p^j} to S_0 by first extending it to $S_0^{p^j}$ via the following valid formula on $K\{t^{p^j}\}$:

$$d^{p^j} = \varphi^j \circ d^1 \circ \varphi^{-j}.$$

For general i, we extend d^i to S_0 via the formula

$$d^i = \prod_j \frac{(d^{p^j})^{a_j}}{a_j!} \qquad (i = \sum a_j p^j, a_j \in \{0, \dots, p-1\}).$$

For K of any characteristic, the operators d^i commute pairwise and satisfy the Leibniz rule:

$$d^i(xy) = \sum_{j=0}^{i} d^j(x) d^{i-j}(y) \qquad (x, y \in S_0).$$

We may then argue as in [9, Lemma 2.3, Theorem 2.5] to see that the formula

$$H(\beta,\rho)(f) = \max_i \left\{\rho^i \beta\left(d^i(f)\right)\right\} \qquad (\beta \in \mathcal{M}(S_0), f \in S_0, \rho \in [0,1])$$

defines a continuous map $H : \mathcal{M}(S_0) \times [0,1] \to \mathcal{M}(S_0)$. Note that if $S_0 = K\{t\}$, then $H(\alpha_{z,\rho}, \sigma) = \alpha_{z,\max\{\rho,\sigma\}}$ [9, Lemma 2.4].

Now let S be the completed integral closure of $K\{t\}$ in a completed separable closure of its fraction field (noting that a completed separable closure is also algebraically closed). By interpolating the maps defined above, we again obtain a continuous map $H : \mathcal{M}(S) \times [0,1] \to \mathcal{M}(S)$.

The following is an instance of the "nonarchimedean Hadamard three circles theorem"; compare [7, Proposition 8.2.3(c)].

Lemma 2.5. *With notation as in Definition 2.4, for $f \in S$, $\beta \in \mathcal{M}(S)$, $\rho, \sigma \in (0,1]$, and $s \in [0,1]$, we have*

$$H(\beta,\rho)(f)^s H(\beta,\sigma)(f)^{1-s} \geq H(\beta, \rho^s \sigma^{1-s})(f).$$

Proof. By continuity, it suffices to check the claim with S replaced by some S_0. We may reinterpret the claim as the statement that the map $r \mapsto \log H(\beta, e^{-r})(f)$ is convex for $r > 0$. Note this map is continuous and piecewise linear, with changes of slopes only at values where $H(\beta, e^{-r})$ lifts a point of $\mathcal{M}(K\{t\})$ of type 2; it thus suffices to compare directional derivatives at such points.

Let k be the residue field of K (which is algebraically closed because K is). Viewing the residue field $\mathcal{H}(H(\beta, e^{-r}))$ of $H(\beta, e^{-r})$ as a nonarchimedean field, it then has its own residue field, which may be identified with the function field of some curve C over k (see for example [2, Definition 4.16]). The points of C correspond to valuations on the function field, which in turn correspond to the local connected components of $\mathcal{M}(S_0) - \{H(\beta, e^{-r})\}$ (and to *type 5 points* of the associated adic space in the sense of Huber; see for example [5, §3.4]). In particular, there are two distinguished points $0, \infty$ of C corresponding to the local connected components containing $H(\beta, e^{-r-\epsilon}), H(\beta, e^{-r+\epsilon})$, respectively, for $\epsilon > 0$.

Since K is algebraically closed and $H(\beta, e^{-r})$ is of type 2, we can choose a nonzero element $\lambda \in K$ such that $H(\beta, e^{-r})(\lambda f) = 1$. The image of λf in the residue field of $\mathcal{H}(H(\beta, e^{-r}))$ then corresponds to a rational function g on C with poles only at ∞; the left and right directional derivatives of $\log H(\beta, e^{-r})(f)$ are unchanged by replacing f with λf, and thus must equal $\operatorname{ord}_\infty(g)$ and $-\operatorname{ord}_0(g)$, respectively. Since

$$-\operatorname{ord}_0(g) - \operatorname{ord}_\infty(g) = \sum_{P \in C - \{0,\infty\}} \operatorname{ord}_P(g) \geq 0,$$

we conclude that $r \mapsto \log H(\beta, e^{-r})(f)$ is convex, as desired. $\square$

One standard property of the Berkovich classification is that if $\alpha \in \mathcal{M}(K\{t\})$ is of type other than 1, then the map $K\{t\} \to \mathcal{H}(\alpha)$ is injective; that is, α defines a true norm on $K\{t\}$ rather than a seminorm. Using Lemma 2.5, we may extend this fact to the ring S.

Lemma 2.6. *With notation as in Definition 2.4, if $\beta \in \mathcal{M}(S)$ restricts to $\alpha \in \mathcal{M}(K\{t\})$ of type other than 1, then the map $S \to \mathcal{H}(\beta)$ is injective.*

Proof. Since α is not of type 1, it follows (see for instance [9, Definition 2.10 and Theorem 2.26]) that there exists $\rho \in (0, 1]$ such that $H(\alpha, \rho) = \alpha$; we must then also have $H(\beta, \rho) = \beta$. If $f \in \ker(S \to \mathcal{H}(\beta))$, we may then apply Lemma 2.5 to deduce that $H(\beta, \sigma)(f) = 0$ for all $\sigma \in [\rho, 1)$, and hence also for $\sigma = 1$ by continuity. This forces $f = 0$. $\square$

3. Proofs and examples

We now settle the questions raised in the introduction.

Proposition 3.1. *Question 1.1 admits an affirmative answer if K is spherically complete.*

Proof. Let $\tau : K \to K$ be a homomorphism as in Question 1.1. Suppose by way of contradiction that there exists $x \in K$ with $x \notin \tau(K)$. Since K is spherically complete, the set of possible valuations of $x - \tau(y)$ for $y \in K$ has a least element. If y realizes this valuation, then by the matching of value groups, we can find $y' \in K$ such that $\tau(y')$ and $x - \tau(y)$ have the same valuation; by the matching of residue fields, we can further choose y' such that $(x - \tau(y))/\tau(y')$ maps to 1 in k. But then $x - \tau(y + y')$ has smaller valuation than $x - \tau(y)$, a contradiction. $\square$

Example 3.2. Let K_0 be a nonarchimedean field with nondiscrete valuation v, and choose a sequence $x_1, x_2, \ldots \in K_0^\times$ whose valuations are positive and form a convergent series. (For a more concrete example, take K_0 to be a completed algebraic closure of $\mathbb{C}((t))$ and take $x_n = t^{2^{-n}}$.) Let K be the completion of $K_0(t_1, t_2, \ldots)$ for the Gauss valuation (i.e., the valuation of a nonzero polynomial is the maximum valuation of its coefficients); then K admits a unique valuation-preserving endomorphism τ fixing K_0 and taking t_n to $t_n - x_n t_{n+1}$ for each n. We will show that the image of τ does not contain t_1, and hence τ is not an isomorphism.

Suppose to the contrary that there exists $y \in K$ with $\tau(y) = t_1$. By hypothesis, there exists some $\lambda \in K_0$ such that $v(\lambda) < v(x_1 \cdots x_n)$ for all n. We may then choose $y' \in K_0(t_1, \ldots, t_n)$ for some positive integer n in such a way that $v(y - y') < v(\lambda)$. Put $y'' = t_1 + x_1 t_2 + \cdots + x_1 \cdots x_n t_{n+1}$; then $\tau(y'') = t_1 - x_1 \cdots x_{n+1} t_{n+2}$, so $v(y'' - y) = v(\tau(y'' - y)) = v(x_1 \cdots x_{n+1}) > v(\lambda)$. Hence $v(y'' - y') = v(x_1 \cdots x_{n+1})$, but $y'' - y'$ equals $x_1 \cdots x_n t_{n+1}$ plus an element of $K_0(t_1, \ldots, t_n)$ and so cannot have valuation less than $v(x_1 \cdots x_n)$. This yields the desired contradiction.

Proof of Theorem 1.2. The hypotheses are preserved by replacing k with its algebraic closure in K, so we may assume at once that k is algebraically closed. Since τ induces an isomorphism of residue fields, it restricts to an automorphism of k. We may thus assume from the outset that τ fixes k.

Equip $R = k[\![t,u]\!]$ with the (t,u)-adic valuation (normalized arbitrarily), which extends multiplicatively to $F = \mathrm{Frac}(R)$. Let $\mathbb{C}_F$ be a completed algebraic closure of F and let S be the completed integral closure of R in $\mathbb{C}_F$; we may identify $\mathfrak{o}_K$ with a subring of S and thus identify $\tau(t)$ with an element of S. Let S' be the completed integral closure of $k((u))[t]$ in $\mathbb{C}_F$; then $S \subset S'$. Let L be the completed algebraic closure of $k((u))$ within S'; then S' is the completed integral closure of $L\{t/u\}$ within a completed algebraic closure of the fraction field of the latter. Choose a homomorphism $S' \to K$ mapping $\mathfrak{o}_K$ to itself and taking u to $\tau(t)$; this map defines a point $\alpha \in \mathcal{M}(S')$ by restriction of the valuation on K. Since the kernel of this map contains $u - \tau(t)$ and hence is nontrivial, by Lemma 2.6 α must be a point of type 1. In particular, we can define an inclusion of $\tau(K)$ into S' taking $\tau(t)$ to u; this inclusion induces an isomorphism $\tau(K) \cong L$; and the type 1 property implies that the composition $\tau(K) \to L \to L\{t/u\} \to S' \to K$, which coincides with the inclusion $\tau(K) \to K$, must be an isomorphism. □

Proof of Theorem 1.3. We use [10, Theorem 1.5.6] as our blanket reference concerning the perfectoid correspondence. By [10, Example 1.3.5], there is an algebraic extension of $\mathbb{Q}_p$ whose completion is perfectoid with tilt isomorphic to the completed perfect closure of $\mathbb{F}_p((t))$; hence $\mathbb{C}_p$ is perfectoid and $\mathbb{C}_p^\flat$ is isomorphic to the completed algebraic closure of $\mathbb{F}_p((t))$. Suppose now that K is a perfectoid field admitting an isomorphism $\iota_1 : K^\flat \to \mathbb{C}_p^\flat$. Since $K^\flat$ is algebraically closed, so is K; in particular, K itself contains a copy of $\mathbb{C}_p$ (though not in a canonical way). By tilting, we obtain a homomorphism $\iota_2 : \mathbb{C}_p^\flat \to K^\flat$. By Theorem 1.2, $\iota_1 \circ \iota_2$ is an isomorphism, as then is ι_2; this implies that the inclusion $\mathbb{C}_p \subseteq K$ is an equality, proving the claim. □

Lemma 3.3. *Every complete subfield K of $\mathbb{C}_p$ is the completion of an algebraic extension of $\mathbb{Q}_p$ (namely, the integral closure of $\mathbb{Q}_p$ in K).*

Proof. Let $\overline{\mathbb{Q}}_p, \overline{K}$ be the algebraic closures of $\mathbb{Q}_p, K$ in $\mathbb{C}_p$, and put $G = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $H = \mathrm{Gal}(\overline{K}/K)$; then G and H both act faithfully on $\mathbb{C}_p$ via continuous automorphisms, via which we may identify H with a closed subgroup of G. Since $\overline{K}$ contains $\overline{\mathbb{Q}}_p$, $\mathbb{C}_p$ is the completion of $\overline{K}$, so the Ax-Sen-Tate theorem [1] implies that $\mathbb{C}_p^H = K$. On the other hand, the field $F = \overline{\mathbb{Q}}_p^H$ is henselian (being a union of complete fields), so Ax-Sen-Tate also implies that $\mathbb{C}_p^H$ is the completion of F. This proves the claim. □

Proof of Theorem 1.4. By Theorem 1.3, a completed algebraic closure of K must be isomorphic to $\mathbb{C}_p$. The claim thus follows from Lemma 3.3. □

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