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IMPROVEMENTS UPON ALZER-RICHARDS' INEQUALITIES FOR THE RATIO OF ZERO-BALANCED HYPERGEOMETRIC FUNCTIONS

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Abstract: In this paper, the authors substantially improve H. Alzer and K.C. Richards' inequalities for the ratios $\mathscr{K}(r)/\mathscr{K}(\sqrt{r})$ of the complete elliptic integrals and $F(a,b;a+b;r^2)/F(a,b;a+b;r)$ of zero-balanced hypergeometric functions, including all bounds in their inequalities, and to give a complete answer to M.E.H. Ismail's question concerning the extensions of these inequalities to $F(a,b;a+b;r)$.

Key Words: Complete elliptic integrals, generalized complete elliptic integrals, hypergeometric function, inequality.

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1 Introduction

Throughout this paper, $\mathbb{N}$ ($\mathbb{R}$) denotes the set of positive integers (real numbers) as usual, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $r' = \sqrt{1-r^2}$ for $r \in [0,1]$. For $x, y \in (0,\infty)$, let

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,\ B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

be the classical Euler gamma, beta and psi functions, respectively [1, 6, 9, 12]. For complex numbers a, b and c with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, |x| < 1, \tag{1.1}$$

where the Pochhammer symbol $(a)_n$ denotes the shifted factorial defined as $(a)_0 = 1$ for $a \neq 0$, and$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(n+a)/\Gamma(a)$ for $n \in \mathbb{N}$. $F(a,b;c;x)$ is said to be zero-balanced if $c = a+b$. (See [1, 6, 13, 15, 21, 25].)

For $a \in (0,1/2]$ and $r \in (0,1)$, the generalized elliptic integrals of the first and second kinds are defined as

$$\begin{cases} \mathscr{K}_a = \mathscr{K}_a(r) = \frac{\pi}{2}F\left(a, 1-a; 1; r^2\right), \\ \mathscr{K}_a' = \mathscr{K}_a'(r) = \mathscr{K}_a(r'), \\ \mathscr{K}_a(0) = \pi/2, \mathscr{K}_a(1) = \infty, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \mathscr{E}_a = \mathscr{E}_a(r) = \frac{\pi}{2}F\left(a-1, 1-a; 1; r^2\right), \\ \mathscr{E}_a' = \mathscr{E}_a'(r) = \mathscr{E}_a(r'), \\ \mathscr{E}_a(0) = \pi/2, \mathscr{E}_a(1) = [\sin(\pi a)]/[2(1-a)], \end{cases} \tag{1.3}$$

respectively (cf. [4, 9, 15, 21, 25, 33]). For $a = 1/2$, the functions $\mathscr{K} = \mathscr{K}_{1/2}$ and $\mathscr{K}' = \mathscr{K}_{1/2}'$, $\mathscr{E} = \mathscr{E}_{1/2}$ and $\mathscr{E}' = \mathscr{E}_{1/2}'$ are the well-known complete elliptic integrals of the first and second kinds, respectively.

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It is well known that the special functions above-mentioned have wide and important applications in several fields of mathematics, as well as in physics and engineering. Numerous properties of these functions have been revealed (cf. [1–5, 7–11, 14, 16–18, 20, 22–24, 26–36] and the references therein), including functional inequalities among which are a kind of elegant inequalities stated below.

In 1990, G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen proved the following inequality [7]

$$\frac{1}{1+r} < \frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})} \ (0 < r < 1), \tag{1.4}$$

and in 1992, they proved in [8] that for $r \in (0,1)$,

$$\frac{1}{\sqrt[4]{1+r}} < \frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})} < \frac{\min\{\sqrt[4]{2}, 1/\sqrt{r'}\}}{\sqrt[4]{1+r}}. \tag{1.5}$$

In [24], it was proved that the function $r \mapsto \sqrt[4]{1+r}\mathscr{K}(r)/\mathscr{K}(\sqrt{r})$ is strictly increasing from $[0,1)$ onto $[1, \sqrt[4]{2})$. Inspired by these results, H. Alzer and K.C. Richards proved in [5, Theorems 3.1, 3.3 & 4.1] that for all $r \in (0,1)$, $a \in (0,1/2]$, and $\lambda, \mu, \lambda_a, \mu_a \in \mathbb{R}$,

$$\frac{1}{1+\lambda r} < \frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})} < \frac{1}{1+\mu r}, \tag{1.6}$$

$$\frac{1}{1+\lambda_a r} < \frac{\mathscr{K}_a(r)}{\mathscr{K}_a(\sqrt{r})} < \frac{1}{1+\mu_a r}, \tag{1.7}$$

with the best possible constants $\lambda = 1/4$, $\mu = 0$, $\lambda_a = a(1-a)$ and $\mu_a = 0$. At the end of [5], H. Alzer and K.C. Richards stated M.E.H. Ismail's question: Can the inequalities obtained in [5] including (1.6)–(1.7) be extended to the zero-balanced hypergeometric function $F(a,b;a+b;r^2)$? Part of the answer to this question was recently given by K.C. Richards in [28], and he proved that

$$\frac{1}{(1+r)^{\lambda(a,b)}} < \frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} < \frac{1}{(1+r)^{\mu(a,b)}} \tag{1.8}$$

for $a, b > 0$ with $a+b > ab$ and for $r \in (0,1)$, with the best possible exponents $\lambda(a,b) = ab/(a+b)$ and $\mu(a,b) = 0$. It was also indicated in [28, Remarks] that if $0 < \lambda = ab/(a+b) < 1$, then

$$\frac{1}{1+\lambda r} < \frac{1}{(1+r)^{\lambda}} < \frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} < 1. \tag{1.9}$$

H. Alzer and K.C. Richards' results are significant and beautiful. However, the bounds especially the upper bounds in (1.6)–(1.9) are not sharp enough. In fact, the inequality $F(a,b;a+b;r^2) < F(a,b;a+b;r)$ is obvious. In [19], the upper bound 1 in (1.6) was improved to $[4/(4+r)] + r^2/5$. On the other hand, the following problem is natural: For what values of $a, b \in (0,\infty)$, the first inequality in (1.6) or in (1.7) can not be directly extended to $F(a,b;a+b;r)$? This problem is actually contained in Ismail's question above-mentioned.

The main purpose of this paper is to improve H. Alzer and K.C. Richards' inequalities (1.6)–(1.9) (see Theorems 1.1 and 1.2), including all the lower and upper bounds in these inequalities, and to give the solution of the problem above-mentioned (see Theorem 1.3). Richards' results together with our Theorem 1.3 give a better answer to Ismail's question. We now state our main results below.

Theorem 1.1. *Let* $\rho = a(1-a)$, $\eta = \rho/(1+\rho)$, $\mu = \rho^2(\rho+2)/[4(\rho+1)]$,

$$D_1 = D_1(\rho, r) = 1 - \frac{\rho(\rho+1)(\rho+2)}{4(1+\rho r)(1-\rho+\rho r)}, D_2 = D_2(r) = D_1(1/4, r) = 1 - \frac{45}{16(3+r)(4+r)}.$$

Then for $a \in (0,1/2]$ *and* $r \in (0,1)$,

$$\frac{1}{1+\rho r} + \mu r^2 < \frac{1}{1+\rho r} + \eta(1-D_1)r^2 < \frac{\mathscr{K}_a(r)}{\mathscr{K}_a(\sqrt{r})} < \frac{1}{1+\rho r} + \eta r^2 \tag{1.10}$$

and the coefficient η in the upper bound is best possible. In particular, for $r \in (0,1)$,

$$\frac{4}{4+r}+\frac{9}{320}r^2 < \frac{4}{4+r}+\frac{1}{5}(1-D_2)r^2 < \frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})} < \frac{4}{4+r}+\frac{1}{5}r^2, \tag{1.11}$$

and the coefficient $1/5$ in the upper bound is best possible.

Theorem 1.2. *For each $n \in \mathbb{N}_0$, $a,b \in (0,\infty)$ with $c=a+b$ and $\alpha = ab/c$, and for $r \in (0,1)$, let $a_n = (a)_n(b)_n/[(c)_n n!]$, $P_n(r) = \sum_{k=0}^n a_k r^k$ and $P(r) = 1-2a_{n+2}r^{n+3}(1-r)/F(a,b;c;r)$. Then*

$$\frac{P_n(r^2)}{P_n(r)} - \alpha r < \frac{F(a,b;c;r^2)}{F(a,b;c;r)} < \frac{P_n(r^2)}{P_n(r)}P(r) < \frac{P_n(r^2)}{P_n(r)} \tag{1.12}$$

for $n \in \mathbb{N}_0$, $a,b \in (0,\infty)$ and for $r \in (0,1)$. The first inequality is sharp as $r \to 0$, while the second and third inequalities are sharp as $r \to 0$ or $r \to 1$. In particular, for all $a,b \in (0,\infty)$ and $r \in (0,1)$,

$$\frac{1+\alpha r^2}{1+\alpha r} - \alpha r < \frac{F(a,b;c;r^2)}{F(a,b;c;r)} < \frac{1+\alpha r^2}{1+\alpha r}. \tag{1.13}$$

Moreover, if $ab \le c$, then for $n \in \mathbb{N}_0$ and $r \in (0,1)$,

$$\frac{P_n(r^2)}{P_n(r)}\left(1-r^{n+1}\right) < \frac{F(a,b;c;r^2)}{F(a,b;c;r)} < \frac{P_n(r^2)}{P_n(r)}. \tag{1.14}$$

Theorem 1.3. *Let $c=a+b$ and $\alpha = ab/c$ for $a,b \in (0,\infty)$, $\lambda \in (0,\infty)$, and $F(r) = F(a,b;c;r)$.*

(1) The function $f(r) \equiv (1+\alpha r)F(r^2)/F(r)$ ($g(r) \equiv (1+r)^\lambda F(r^2)/F(r)$) is strictly increasing from $(0,1)$ onto $(1,1+\alpha)$ ($(1,2^\lambda)$) if and only if $ab \le c+1$ ($\lambda \ge \alpha$, respectively). In particular,

$$\frac{1}{1+\alpha r} < \frac{F(a,b;c;r^2)}{F(a,b;c;r)} \quad \text{and} \quad \frac{1}{(1+r)^\lambda} < \frac{F(a,b;c;r^2)}{F(a,b;c;r)} \tag{1.15}$$

for all $r \in (0,1)$ if and only if $ab \le c+1$ for the first inequality and $\lambda \ge \alpha$ for the second inequality.

(2) For $\tau, \delta \in (0,\infty)$, if the inequalities

$$\frac{1}{1+\alpha r} + \tau r^2 < \frac{F(a,b;c;r^2)}{F(a,b;c;r)} < \frac{1}{1+\alpha r} + \delta r^2 \tag{1.16}$$

hold for all $r \in (0,1)$, then

$$\tau \le (\alpha/2)\min\{2/(1+\alpha), 1-ab/(c+1)\}, \tag{1.17}$$

$$\delta \ge (\alpha/2)\max\{2/(1+\alpha), 1-ab/(c+1)\}. \tag{1.18}$$

Furthermore, if the first (second) inequality in (1.16) holds for $\tau = \alpha/(1+\alpha)$ ($\delta = \alpha/(1+\alpha)$) and for all $r \in (0,1)$, then $(ab)^2 - ab + c(c+1) \le 0$ ($(ab)^2 - ab + c(c+1) \ge 0$, respectively). In other words, if $\tau = \alpha/(1+\alpha)$ ($\delta = \alpha/(1+\alpha)$) and $(ab)^2 - ab + c(c+1) > 0$ ($(ab)^2 - ab + c(c+1) < 0$), then the first (second, respectively) inequality in (1.16) does not hold.

In the sequel, for $a,b \in (0,\infty)$ and $r \in (0,1)$, we always let $c = a+b$,

$$a_n = (a)_n(b)_n/[(c)_n n!] \ (n \in \mathbb{N}_0), \ \alpha = a_1 = ab/c, \tag{1.19}$$

$$F(r) = F(a,b;c;r), \ G(r) = F(a,b;c+1;r), \tag{1.20}$$

and for $a \in (0,1/2]$ and $n \in \mathbb{N}_0$, let

$$\rho = a(1-a), \ \eta = \rho/(1+\rho), \tag{1.21}$$

$$b_n = \pi(a)_n(1-a)_n/[2(n!)^2], \ c_n = b_{2n}/b_n, \tag{1.22}$$

$$d_n = b_{2n+1}/b_n, \ p_n = c_n + d_n. \tag{1.23}$$

By the derivative and linear transformation formulas for $F(r)$ (see [6, 15.2.1 & 15.3.3], for example),

$$F'(r) = \alpha G(r)/(1-r). \tag{1.24}$$

Clearly, $\rho \le 1/4$. It is well known that for $a,b \in (0,\infty)$,

$$B(a,b)F(x) = R(a,b) - \log(1-x) + O((1-x)\log(1-x)) \tag{1.25}$$

as $x \to 1$, where $R(a,b) = -2\gamma - \psi(a) - \psi(b)$ and γ is the Euler-Mascheroni constant (see [6, 15.3.10]).

2 Preliminaries

In this section, we prove four lemmas needed in the proofs of our main results.

Lemma 2.1. *For $a \in (0, 1/2]$ and $n \in \mathbb{N}_0$, let ρ be as in (1.21), and b_n, c_n, d_n and p_n as in (1.22)–(1.23).*
(1) The sequence $\{b_{n+1}/b_n\}$ is strictly increasing in $n \in \mathbb{N}_0$, with $\lim_{n\to\infty} b_{n+1}/b_n = 1$.
(2) The sequence $\{c_n\}$ is strictly decreasing in $n \in \mathbb{N}_0$, with $c_0 = 1$, $c_1 = (\rho + 2)/4$ and $c_\infty = 1/2$.
(3) The sequence $\{d_n\}$ is strictly increasing in $n \in \mathbb{N}_0$, with $d_0 = \rho$, $d_1 = (\rho + 2)(\rho + 6)/36$ and $d_\infty = 1/2$.
(4) The sequence $\{p_n\}$ is strictly increasing in $n \in \mathbb{N}$, with $p_1 = (\rho + 2)(\rho + 15)/36$ and $p_\infty = 1$.

Proof. (1) It is easy to obtain the following relation

$$\frac{b_{n+1}}{b_n} = \frac{n^2 + n + \rho}{(n + 1)^2} = 1 - \frac{n + 1 - \rho}{(n + 1)^2}. \tag{2.1}$$

Since $\rho \in (0, 1/4]$, it follows from (2.1) that $\lim_{n\to\infty} b_{n+1}/b_n = 1$, and

$$\frac{b_{n+2}}{b_{n+1}} - \frac{b_{n+1}}{b_n} = \frac{n^2 + (3 - 2\rho)n + 2 - 3\rho}{(n + 1)^2(n + 2)^2} \geq \frac{4n^2 + 10n + 5}{4(n + 1)^2(n + 2)^2}.$$

Hence we obtain the conclusion in part (1).

(2) By (2.1), we obtain

$$\begin{aligned}\frac{c_{n+1}}{c_n} = \frac{b_{2n+2}}{b_{2n+1}} \cdot \frac{b_{2n+1}}{b_{2n}} \cdot \frac{b_n}{b_{n+1}} &= \frac{(4n^2 + 2n + \rho)(4n^2 + 6n + 2 + \rho)}{4(2n + 1)^2(n^2 + n + \rho)} \\ &= 1 - \rho\frac{8n^2 + 8n + 2 - \rho}{4(2n + 1)^2(n^2 + n + \rho)} < 1\end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}_0$. Hence the monotonicity property of the sequence $\{c_n\}$ follows.

Clearly, $c_0 = 1$ and $c_1 = (\rho + 2)/4$. By the asymptotic expansion for $\Gamma(x + a)/\Gamma(x + b)$ (cf. [6, 6.1.47]),

$$c_\infty = \lim_{n\to\infty} \frac{\Gamma(2n + a)\Gamma(2n + 1 - a)\Gamma(n + 1)^2}{\Gamma(n + a)\Gamma(n + 1 - a)\Gamma(2n + 1)^2} = \frac{1}{2}. \tag{2.3}$$

(3) Similarly, by (2.1) and the fact $\rho \in (0, 1/4]$, we have

$$\begin{aligned}\frac{d_{n+1}}{d_n} =& \frac{b_{2n+3}}{b_{2n+2}} \cdot \frac{b_{2n+2}}{b_{2n+1}} \cdot \frac{b_n}{b_{n+1}} = \frac{(4n^2 + 10n + 6 + \rho)(4n^2 + 6n + 2 + \rho)}{4(2n + 3)^2(n^2 + n + \rho)} \\ =&1 + \frac{8(1 - \rho)n^2 + 4(5 - 8\rho)n + \rho^2 - 28\rho + 12}{4(2n + 3)^2(n^2 + n + \rho)} > 1\end{aligned}$$

for all $n \in \mathbb{N}_0$. Hence the monotonicity property of $\{d_n\}$ follows. It is clear that $d_0 = b_1/b_0 = \rho$ and $d_1 = (\rho + 2)(\rho + 6)/36$. Similarly to (2.3), one can obtain the limiting value $d_\infty = 1/2$.

(4) For $n \in \mathbb{N}$, put

$$\begin{aligned}q_n =& 128n^6 + 608n^5 + (80\rho + 1104)n^4 + (272\rho + 952)n^3 \\ &+ (16\rho^2 + 332\rho + 388)n^2 + (30\rho^2 + 168\rho + 60)n + \rho^3 + 17\rho^2 + 30\rho, \\ r_n =& 128n^6 + 608n^5 + (144\rho + 1072)n^4 + (544\rho + 856)n^3 \\ &+ (16\rho^2 + 676\rho + 300)n^2 + (48\rho^2 + 300\rho + 36)n + 36\rho^2 + 36\rho.\end{aligned}$$

Then by (2.1) and (2.2), we obtain

$$\begin{aligned}\frac{p_{n+1}}{p_n} &= \frac{c_{n+1}}{c_n} \cdot \frac{1 + d_{n+1}/c_{n+1}}{1 + d_n/c_n} = \frac{c_{n+1}}{c_n} \cdot \frac{1 + b_{2n+3}/b_{2n+2}}{1 + b_{2n+1}/b_{2n}} \\ &= \frac{(4n^2 + 2n + \rho)(4n^2 + 6n + 2 + \rho)}{4(2n + 1)^2(n^2 + n + \rho)} \cdot \frac{1 + (4n^2 + 10n + 6 + \rho)(2n + 3)^{-2}}{1 + (4n^2 + 2n + \rho)(2n + 1)^{-2}} \\ &= \frac{(4n^2 + 2n + \rho)(4n^2 + 6n + 2 + \rho)(8n^2 + 22n + 15 + \rho)}{4(2n + 3)^2(n^2 + n + \rho)(8n^2 + 6n + 1 + \rho)} = \frac{q_n}{r_n}.\end{aligned} \tag{2.4}$$

It is easy to show that the function $\rho \mapsto \rho(\rho^2 - 19\rho - 6)$ is strictly decreasing on $(0, 1/4]$. Since $\rho \le 1/4$,

$$\begin{aligned} q_n - r_n =&(32 - 64\rho)n^4 + (96 - 272\rho)n^3 + (88 - 344\rho)n^2 \\ &+ \left(24 - 132\rho - 18\rho^2\right)n + \rho\left(\rho^2 - 19\rho - 6\right) \\ \ge&(q_n - r_n)|_{\rho=1/4} = 16n^4 + 28n^3 + 2n^2 - \frac{81}{8}n - \frac{171}{64} > 0, \end{aligned}$$

namely, $q_n/r_n > 1$. Hence by (2.4), $\{p_n\}$ is strictly increasing in $n \in \mathbb{N}$.

The limiting values of p_n are clear. □

Lemma 2.2. *For $n \in \mathbb{N}$, let $A_{2n+1} = (1+\rho)(b_{2n+1} - \rho b_n) + \rho(b_{2n-1} + \rho b_{2n-2})$ and $A_{2n+2} = (1+\rho)(b_{n+1} - b_{2n+2}) - \rho(b_{2n} + \rho b_{2n-1})$. Then for all $n \in \mathbb{N}$,*

$$A_{2n+1} > A_{2n+2} > 0. \tag{2.5}$$

Proof. First, we prove the second inequality in (2.5). It is clear that for all $n \in \mathbb{N}$,

$$A_{2n+2} >0 \Leftrightarrow B_1(n) \equiv \frac{(1+\rho)b_{n+1}}{\rho^2 b_{2n-1} + \rho b_{2n} + (1+\rho)b_{2n+2}} > 1. \tag{2.6}$$

Let $B_2(n) = (2n+1)^2(n^2+n+\rho)(4n^2-2n+\rho)$, $B_3(n) = 16(\rho+1)B_2(n) - (\rho+2)B_4(n)$,

$$\begin{aligned} B_4(n) =&(1+\rho)(4n^2+6n+2+\rho)(4n^2+2n+\rho)(4n^2-2n+\rho) \\ &+ [4\rho(n^2+n)(2n+1)]^2 + 4\rho(n+1)^2(2n+1)^2(4n^2-2n+\rho), \\ \varphi_1(\rho) =&128 - 64\rho - 256\rho^2 - 64\rho^3,\ \varphi_2(\rho) = 192 - 224\rho - 640\rho^2 - 192\rho^3, \\ \varphi_3(\rho) =&32 + 16\rho - 400\rho^2 - 272\rho^3,\ \varphi_4(\rho) = -48 + 152\rho - 144\rho^2 - 192\rho^3, \\ \varphi_5(\rho) =&-16 + 48\rho - 76\rho^2 - 52\rho^3 - 12\rho^4,\ \varphi_6(\rho) = -4\rho^2 + 22\rho^3 - 6\rho^4, \end{aligned}$$

and $\varphi_7(\rho) = 4 + 4\rho - 5\rho^2 - \rho^3$. Then it follows from (2.1) and Lemma 2.1(2) that for $n \in \mathbb{N}$,

$$\begin{aligned} B_1(n) =&\frac{1+\rho}{c_n} \cdot \frac{b_{n+1}/b_n}{\rho^2 b_{2n-1}/b_{2n} + (1+\rho)b_{2n+2}b_{2n+1}/[b_{2n+1}b_{2n}] + \rho} \\ \ge&\frac{4(\rho+1)}{\rho+2} \cdot \frac{b_{n+1}/b_n}{\rho^2 b_{2n-1}/b_{2n} + (1+\rho)b_{2n+2}b_{2n+1}/[b_{2n+1}b_{2n}] + \rho} \\ =&B_5(n) \equiv \frac{16(\rho+1)B_2(n)}{(\rho+2)B_4(n)} = 1 + \frac{B_3(n)}{(\rho+2)B_4(n)}, \end{aligned} \tag{2.7}$$

$$B_3(n) =\varphi_1(\rho)n^6 + \varphi_2(\rho)n^5 + \varphi_3(\rho)n^4 + \varphi_4(\rho)n^3 + \varphi_5(\rho)n^2 + \varphi_6(\rho)n + \rho^2\varphi_7(\rho). \tag{2.8}$$

It is easy to verify that φ_1 (φ_2) is strictly decreasing on $(0, 1/4]$ with $\varphi_1(1/4) = 95$ ($\varphi_2(1/4) = 93$, respectively), φ_3 is strictly increasing and then decreasing on $(0, 1/4]$ with $\varphi_3(0) = 32$ and $\varphi_3(1/4) = 27/4$, φ_4 is strictly increasing on $(0, 1/4]$ with $\varphi_4(0) = -48$, φ_5 is strictly increasing and then decreasing on $(0, 1/4]$ with $\varphi_5(0) = -16$ and $\varphi_5(1/4) = -615/64$, φ_6 is strictly decreasing and then increasing on $(0, 1/4]$ with the minimum $\varphi_6((33 - \sqrt{897})/24) = -0.02101246\cdots > -0.02102$, and φ_7 is strictly increasing on $(0, 1/4]$ with $\varphi_7(0) = 4$. Hence it follows from (2.8) that for $n \in \mathbb{N}$,

$$\begin{aligned} B_3(n) >&95n^6 + 93n^5 + \frac{27}{4}n^4 - 48n^3 - 16n^2 - 0.02102n \\ >&48\left(n^6 - n^3\right) + 16\left(n^6 - n^2\right) + \left(n^6 - n\right) + 30n^6 + 93n^5 + \frac{27}{4}n^4 \\ >&30n^6 + 93n^5 + 6n^4 \ge 129. \end{aligned} \tag{2.9}$$

By (2.7) and (2.9), $B_1(n) \ge B_5(n) > 1$. Hence by (2.6), $A_{2n+2} > 0$ for all $n \in \mathbb{N}$.

Next, we prove the first inequality in (2.5). Clearly, this inequality holds if and only if

$$
\begin{aligned}
&(1+\rho)(b_{2n+2}+b_{2n+1})+\rho b_{2n}+(\rho+\rho^2)b_{2n-1}+\rho^2 b_{2n-2}>(1+\rho)(b_{n+1}+\rho b_n)\\
&\Leftrightarrow 1>Q_1(n)\equiv\frac{(\rho+1)(b_{n+1}+\rho b_n)}{(\rho+1)(b_{2n+2}+b_{2n+1})+\rho b_{2n}+(\rho+\rho^2)b_{2n-1}+\rho^2 b_{2n-2}}\\
&=\frac{\rho+1}{c_n}\frac{\rho+b_{n+1}/b_n}{(\rho+1)(b_{2n+2}+b_{2n+1})/b_{2n}+\rho(\rho+1)b_{2n-1}/b_{2n}+\rho^2 b_{2n-2}/b_{2n}+\rho},
\end{aligned}
\tag{2.10}
$$

for $n\in\mathbb{N}$ and $\rho\in(0,1/4]$. By Lemma 2.1(2) and (2.1), the following inequality holds

$$
Q_1(n)<Q_2(n)\equiv Q_3(n)/Q_4(n),
\tag{2.11}
$$

where

$$
\begin{aligned}
Q_3(n)=&2(\rho+1)\left(\rho+\frac{b_{n+1}}{b_n}\right)=2(\rho+1)\frac{(\rho+1)n^2+(2\rho+1)n+2\rho}{(n+1)^2},\\
Q_4(n)=&(\rho+1)\frac{b_{2n+1}}{b_{2n}}\left(\frac{b_{2n+2}}{b_{2n+1}}+1\right)+\rho(\rho+1)\frac{b_{2n-1}}{b_{2n}}+\rho^2\frac{b_{2n-2}}{b_{2n}}+\rho\\
=&\frac{Q_5(n)}{4(n+1)^2(2n+1)^2(4n^2-2n+\rho)(4n^2-6n+\rho+2)},\\
Q_5(n)=&\rho^2\left(\rho^3+9\rho^2+24\rho+20\right)+\left(512\rho^2+1024\rho+512\right)n^8+\left(896\rho^2+768\rho+128\right)n^7\\
&+\left(64\rho^3+576\rho^2-960\rho-832\right)n^6+\left(192\rho^3-96\rho^2-832\rho-160\right)n^5\\
&+\left(368\rho^3-64\rho^2+144\rho+368\right)n^4+\left(216\rho^3+72\rho^2+176\rho+32\right)n^3\\
&+\left(20\rho^4+104\rho^3+88\rho^2+32\rho-48\right)n^2+\left(8\rho^4+24\rho^3+8\rho^2-16\rho\right)n
\end{aligned}
$$

for $n\in\mathbb{N}$ and $\rho\in(0,1/4]$. By (2.11) and by computation, we obtain

$$
\begin{aligned}
Q_2(n)=&\frac{8(\rho+1)}{Q_5(n)}(2n+1)^2\left(4n^2-2n+\rho\right)\left(4n^2-6n+\rho+2\right)\left[(\rho+1)n^2+(2\rho+1)n+2\rho\right]\\
=&1-Q_6(\rho,n)/Q_5(n),
\end{aligned}
\tag{2.12}
$$

where

$$
\begin{aligned}
Q_6(\rho,n)=&Q_5(n)-8(\rho+1)(2n+1)^2\left(4n^2-2n+\rho\right)\left(4n^2-6n+\rho+2\right)\left[(\rho+1)n^2+(2\rho+1)n+2\rho\right]\\
=&Q_5(n)-\Big[\left(512\rho^2+1024\rho+512\right)n^8+\left(512\rho^2+512\rho\right)n^7\\
&+\left(256\rho^3+256\rho^2-768\rho-768\right)n^6+\left(512\rho^3-512\rho^2-1024\rho\right)n^5\\
&+\left(32\rho^4+448\rho^3+320\rho^2+192\rho+288\right)n^4+\left(96\rho^4-96\rho^3+224\rho^2+416\rho\right)n^3\\
&+\left(136\rho^4-64\rho^3-184\rho^2-16\rho-32\right)n^2+\left(80\rho^4+120\rho^3-8\rho^2-48\rho\right)n\\
&+16\rho^2\left(\rho^2+3\rho+2\right)\Big]\\
=&\left(384\rho^2+256\rho+128\right)n^7-\left(192\rho^3-320\rho^2+192\rho+64\right)n^6\\
&-\left(320\rho^3-416\rho^2-192\rho+160\right)n^5-\left(32\rho^4+80\rho^3+384\rho^2+48\rho-80\right)n^4\\
&-\left(96\rho^4-312\rho^3+152\rho^2+240\rho-32\right)n^3-\left(116\rho^4-168\rho^3-272\rho^2-48\rho+16\right)n^2\\
&-\left(72\rho^4+96\rho^3-16\rho^2-32\rho\right)n+\rho^5-7\rho^4-24\rho^3-12\rho^2.
\end{aligned}
\tag{2.13}
$$

Differentiation gives

$$
\begin{aligned}
\frac{\partial Q_6}{\partial\rho}=&(256+768\rho)n^7-\left(192-640\rho+576\rho^2\right)n^6+\left(192+832\rho-960\rho^2\right)n^5\\
&-\left(48+768\rho+240\rho^2+128\rho^3\right)n^4-\left(240+304\rho-936\rho^2+384\rho^3\right)n^3\\
&+\left(48+544\rho+504\rho^2-464\rho^3\right)n^2+\left(32+32\rho-288\rho^2-288\rho^3\right)n\\
&+5\rho^4-28\rho^3-72\rho^2-24\rho,
\end{aligned}
\tag{2.14}
$$

$$\begin{aligned}\frac{\partial^2 Q_6}{\partial\rho^2} =&768n^7+(640-1152\rho)n^6+(832-1920\rho)n^5-\left(768+480\rho+384\rho^2\right)n^4\\&-\left(304-1872\rho+1152\rho^2\right)n^3+\left(544+1008\rho-1392\rho^2\right)n^2\\&+\left(32-576\rho-864\rho^2\right)n+20\rho^3-84\rho^2-144\rho-24,\end{aligned}\tag{2.15}$$

$$\begin{aligned}\frac{\partial^3 Q_6}{\partial\rho^3} =&-1152n^6-1920n^5-(480+768\rho)n^4+(1872-2304\rho)n^3\\&+(1008-2784\rho)n^2-(576+1728\rho)n+60\rho^2-168\rho-144.\end{aligned}\tag{2.16}$$

Clearly, the function $\rho\mapsto 60\rho^2-168\rho-144$ is strictly decreasing on $(0,1/4]$, and $\frac{\partial^3 Q_6}{\partial\rho^3}$ is strictly decreasing in $\rho\in(0,1/4]$. It is easy to show that for $n\in\mathbb{N}$, the function $Q_7(n)\equiv 16n^6-12n^5+12n^4-3n^3-15n^2+3n+2$ is strictly increasing with $Q_7(1)=3$. Hence for all $n\in\mathbb{N}$ and $\rho\in(0,1/4]$,

$$\begin{aligned}\frac{\partial^3 Q_6}{\partial\rho^3} < \frac{\partial^3 Q_6}{\partial\rho^3}\Big|_{\rho=0} &=\left(-1152n^6-1920n^5-480n^4\right)+\left(1872n^3+1008n^2\right)-576n-144\\&\le -3552n^4+2880n^3-576n-144\\&\le -48\left(14n^3+12n+3\right)\le -1392,\\ \frac{\partial^2 Q_6}{\partial\rho^2} \ge \frac{\partial^2 Q_6}{\partial\rho^2}\Big|_{\rho=1/4} &=560n^7+92n^3+478n^2+\left(208n^7+352n^6+352n^5-912n^4\right)\\&\quad+166\left(n^2-n\right)+\left(65n^2-\frac{1039}{16}\right)\\&\ge 2n^2\left(280n^5+46n+239\right)\ge 1130,\\ \frac{\partial Q_6}{\partial\rho} > \frac{\partial Q_6}{\partial\rho}\Big|_{\rho=0} &=256n^7-192n^6+192n^5-48n^4-240n^3+48n^2+32n\\&=16nQ_7(n)\ge 48n\ge 48\end{aligned}$$

from which we see that Q_6 is strictly increasing in $\rho\in(0,1/4]$ and

$$Q_6(\rho,n)>Q_6(0,n)=16n^2\left(n^2-1\right)(2n-1)^2(2n+1)\ge 0.\tag{2.17}$$

It follows from (2.11), (2.12) and (2.17) that $Q_1(n)<1$. Consequently, $A_{2n+1}>A_{2n+2}$ by (2.10). □

Lemma 2.3. *Let* $f_1(r)=[(1+r)F(r)-(1+\alpha r)G(r)]/r$, $f_2(r)=F(r)/G(r)$ *and* $f_3(r)=(1+r)^\alpha F(r^2)/F(r)$.

(1) Suppose that the Maclaurin series of f_1 *is* $\sum_{n=0}^\infty C_n r^n$. *Then* $C_n\ge 0$ *if and only if* $ab\le c+1$.

(2) f_2 *is strictly increasing from* $(0,1)$ *onto* $(1,\infty)$.

(3) f_3 *is strictly increasing from* $(0,1)$ *onto* $(1,2^\alpha)$.

Proof. (1) Put $C_0=(c+1-ab)/(c+1)$, and for $n\in\mathbb{N}$, let

$$C_n=\frac{2n^2+(3c+1-ab)n+c[(c+1)-ab]}{(n+c)(n+c+1)}a_n.$$

It is easy to obtain the following relation

$$\frac{a_{n+1}}{a_n}=\frac{n^2+cn+ab}{n^2+(c+1)n+c}=1-\frac{n+c-ab}{(n+1)(n+c)}.\tag{2.18}$$

It follows from (1.1) and (2.18) that

$$\begin{aligned}f_1(r)=&\frac{1}{r}\left(\sum_{n=0}^\infty a_nr^n+\sum_{n=1}^\infty a_{n-1}r^n-c\sum_{n=0}^\infty\frac{a_n}{n+c}r^n-ab\sum_{n=0}^\infty\frac{a_n}{n+c}r^{n+1}\right)\\=&\sum_{n=0}^\infty\left[\frac{(n+1)a_{n+1}}{(n+c+1)a_n}+\frac{n+c-ab}{n+c}\right]a_nr^n\\=&\sum_{n=0}^\infty\frac{2n^2+(3c+1-ab)n+c[(c+1)-ab]}{(n+c)(n+c+1)}a_nr^n=\sum_{n=0}^\infty C_nr^n.\end{aligned}\tag{2.19}$$

If $ab \le c+1$, then by (2.19), we see that $C_0 \ge 0$ and $C_n > 0$ for all $n \in \mathbb{N}$. Conversely, if $C_n \ge 0$ for all $n \in \mathbb{N}_0$, then $C_0 \ge 0$, that is, $ab \le c+1$. Hence part (1) follows.

(2) Let $\tilde{a}_n = ca_n/(n+c)$. Then by (1.1), $f_2(r) = \left(\sum_{n=0}^{\infty} a_n r^n\right)\Big/\left(\sum_{n=0}^{\infty} \tilde{a}_n r^n\right)$, and $a_n/\tilde{a}_n = 1 + n/c$. Hence the monotonicity of f_2 follows from [22, Lemma 2.1]. The limiting values of f_2 are clear.

(3) By differentiation and by part (2),

$$\frac{(1+r)f_2(r)}{\alpha f_3(r)} f_3'(r) = f_2(r) + \frac{2rf_2(r)/f_2(r^2) - (1+r)}{1-r} > f_2(r) - 1$$

for all $r \in (0,1)$, yielding the monotonicity of f_3. The limiting values of f_3 are clear. □

Lemma 2.4. *For each $n \in \mathbb{N}_0$, $a, b \in (0,\infty)$ with $c = a+b$, and for $r \in (0,1)$, let $P_n(r) = \sum_{k=0}^{n} a_k r^k$. Then the function $f_4(r) \equiv F(r)/P_n(r)$ is strictly increasing from $[0,1)$ onto $[1,\infty)$.*

Proof. Let $f_5(r) = r^{n+1}/P_n(r)$. Then we have

$$f_4(r) = 1 + f_5(r) \sum_{k=0}^{\infty} a_{k+n+1} r^k, \tag{2.20}$$

$$f_5'(r) = \frac{n+1+\sum_{k=1}^{n}(n+1-k)a_k r^k}{P_n(r)^2} r^n. \tag{2.21}$$

From (2.21) we see that f_5 is strictly increasing on $(0,1)$. Hence the monotonicity of f_4 follows from (2.20).

The limiting values of f_4 are clear. □

3 Proofs of Main Results

3.1 Proof of Theorem 1.1

For $r \in (0,1)$, let $f_6(r) = (1+\rho r)\mathscr{K}_a(r) - \mathscr{K}_a(\sqrt{r})$, $f_7(r) = \left[1+\rho+\rho r^2(1+\rho r)\right]\mathscr{K}_a(\sqrt{r}) - (1+\rho)(1+\rho r)\mathscr{K}_a(r)$, A_{2n+1} and A_{2n+2} be as in Lemma 2.2. Then by (1.2) and (1.22),

$$\begin{aligned}
f_6(r) =&(1+\rho r)\sum_{n=0}^{\infty} b_n r^{2n} - \sum_{n=0}^{\infty} b_n r^n \\
=&r^2\left(\sum_{n=0}^{\infty} b_{n+1} r^{2n} + \rho\sum_{n=0}^{\infty} b_{n+1} r^{2n+1} - \sum_{n=0}^{\infty} b_{n+2} r^n\right) \\
=&r^2\left(\sum_{n=0}^{\infty} b_{n+1} r^{2n} + \rho\sum_{n=0}^{\infty} b_{n+1} r^{2n+1} - \sum_{n=0}^{\infty} b_{2n+2} r^{2n} - \sum_{n=0}^{\infty} b_{2n+3} r^{2n+1}\right) \\
=&r^2\left[\sum_{n=0}^{\infty}(1-c_{n+1})b_{n+1} r^{2n} + \sum_{n=0}^{\infty}(\rho - d_{n+1})b_{n+1} r^{2n+1}\right],
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
f_7(r) =&(1+\rho)\sum_{n=0}^{\infty} b_n r^n + \rho\sum_{n=0}^{\infty} b_n r^{n+2} + \rho^2\sum_{n=0}^{\infty} b_n r^{n+3} \\
&- (1+\rho)\sum_{n=0}^{\infty} b_n r^{2n} - \rho(1+\rho)\sum_{n=0}^{\infty} b_n r^{2n+1} \\
=&\frac{\pi\rho}{8}\left(2+\rho^2-\rho\right) r^2 + \sum_{n=3}^{\infty}\left[(1+\rho)b_n + \rho b_{n-2} + \rho^2 b_{n-3}\right] r^n \\
&- (1+\rho)\sum_{n=2}^{\infty} b_n r^{2n} - \rho(1+\rho)\sum_{n=1}^{\infty} b_n r^{2n+1} \\
=&\frac{\pi\rho}{8}\left(2+\rho^2-\rho\right) r^2 + \sum_{n=1}^{\infty} A_{2n+1} r^{2n+1} - \sum_{n=1}^{\infty} A_{2n+2} r^{2n+2}.
\end{aligned} \tag{3.2}$$

By Lemma 2.1(3), $\rho - d_{n+1} \le (28\rho - \rho^2 - 12)/36 \le -9/64$. Hence by (3.1) and Lemma 2.1(4),

$$\begin{aligned} f_6(r) >& r^2 \left[\sum_{n=0}^{\infty} (1 - c_{n+1}) b_{n+1} r^{2n} + \sum_{n=0}^{\infty} (\rho - d_{n+1}) b_{n+1} r^{2n} \right] \\ =& \sum_{n=1}^{\infty} (1 + \rho - p_n) b_n r^{2n} > \rho \sum_{n=1}^{\infty} b_n r^{2n} = \rho \left[\mathscr{K}_a(r) - \frac{\pi}{2} \right], \end{aligned}$$

from which we obtain

$$(1 + \rho r)\mathscr{K}_a(r) > \mathscr{K}_a(\sqrt{r}) + \rho \left[\mathscr{K}_a(r) - \frac{\pi}{2} \right], \tag{3.3}$$

$$\mathscr{K}_a(r) > \frac{\mathscr{K}_a(\sqrt{r}) - \pi\rho/2}{1 - \rho + \rho r}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned} (1 + \rho r)\frac{\mathscr{K}_a(r)}{\mathscr{K}_a(\sqrt{r})} &> 1 + \rho \frac{\mathscr{K}_a(r) - \pi/2}{\mathscr{K}_a(\sqrt{r})} \\ &> 1 + \frac{\rho}{1 - \rho + \rho r} \cdot \frac{\mathscr{K}_a(\sqrt{r}) - \pi(1 + \rho r)/2}{\mathscr{K}_a(\sqrt{r})} \\ &= 1 + \frac{\rho r^2}{1 - \rho + \rho r} \cdot \frac{\sum_{n=0}^{\infty} b_{n+2} r^n}{\sum_{n=0}^{\infty} b_n r^n}. \end{aligned} \tag{3.5}$$

By Lemma 2.1(1), $b_{n+2}/b_n = (b_{n+2}/b_{n+1}) \cdot (b_{n+1}/b_n)$ is strictly increasing in $n \in \mathbb{N}_0$. Hence by [22, Lemma 2.1], the function $r \mapsto \left(\sum_{n=0}^{\infty} b_{n+2} r^n\right)\left(\sum_{n=0}^{\infty} b_n r^n\right)^{-1}$ is strictly increasing on $(0, 1)$, so that by (3.5),

$$\frac{\mathscr{K}_a(r)}{\mathscr{K}_a(\sqrt{r})} > \frac{1}{1 + \rho r} \left[1 + \frac{b_2 \rho r^2}{b_0(1 - \rho + \rho r)} \right] = \frac{1}{1 + \rho r} + \eta\,(1 - D_1)\, r^2, \tag{3.6}$$

yielding the second inequality in (1.10).

It is clear that $\eta(1 - D_1) > \eta[1 - D_1(\rho, 1)] = \mu$. Hence the first inequality in (1.10) holds.

By Lemma 2.2 and (3.2), we obtain

$$f_7(r) > \frac{\pi\rho}{8}\left(2 + \rho^2 - \rho\right) r^2 + \sum_{n=1}^{\infty} (A_{2n+1} - A_{2n+2})\, r^{2n+1} > \frac{\pi\rho}{8}\left(2 + \rho^2 - \rho\right) r^2,$$

from which it follows that

$$\frac{\mathscr{K}_a(r)}{\mathscr{K}_a(\sqrt{r})} < \frac{1}{1 + \rho r} + \frac{\rho}{1 + \rho} r^2 - \frac{\pi\rho(2 + \rho^2 - \rho) r^2}{8(1 + \rho)(1 + \rho r)\mathscr{K}_a(\sqrt{r})} < \frac{1}{1 + \rho r} + \eta r^2,$$

and hence the third inequality in (1.10) holds for all $r \in (0, 1)$. By (1.25), we obtain

$$\lim_{r \to 1} \frac{1}{r^2} \left[\frac{\mathscr{K}_a(r)}{\mathscr{K}_a(\sqrt{r})} - \frac{1}{1 + \rho r} \right] = \eta.$$

Hence the coefficient η in the upper bound in (1.10) is best possible. The remaining conclusion is clear. □

3.2 Proof of Theorem 1.2

Let f_4 be as in Lemma 2.4, and $h_1(r) = P_n(r^2)F(r) - P_n(r)F(r^2)$. Then $h_1(r) = P_n(r^2)P_n(r)[f_4(r) - f_4(r^2)] > 0$ for all $r \in (0, 1)$ by Lemma 2.4, and by (1.1),

$$\begin{aligned} h_1(r) =& P_n\left(r^2\right) \left[P_n(r) + \sum_{k=n+1}^{\infty} a_k r^k \right] - P_n(r) \left[P_n\left(r^2\right) + \sum_{k=n+1}^{\infty} a_k r^{2k} \right] \\ =& r^{n+1} \left[P_n\left(r^2\right) \sum_{k=0}^{\infty} a_{k+n+1} r^k - r^{n+1} P_n(r) \sum_{k=0}^{\infty} a_{k+n+1} r^{2k} \right]. \end{aligned} \tag{3.7}$$

It is easy to see that

$$r^{n+1}P_n(r) = P_n\left(r^2\right) - \sum_{k=0}^{n} a_k r^{2k}\left(1 - r^{n+1-k}\right),$$

and hence by (3.7), $h_1(r)$ can be rewritten as

$$\begin{aligned} h_1(r) =& r^{n+1}\left\{P_n\left(r^2\right)\sum_{k=1}^{\infty} a_{k+n+1}r^k\left(1-r^k\right) + \left[\sum_{k=0}^{n} a_k r^{2k}\left(1-r^{n+1-k}\right)\right]\sum_{k=0}^{\infty} a_{k+n+1}r^{2k}\right\} \\ =& r^{n+1}(1-r)\left\{P_n\left(r^2\right)\sum_{k=1}^{\infty} a_{k+n+1}r^k\sum_{i=0}^{k-1} r^i + \left[\sum_{k=0}^{n}\left(a_k r^{2k}\sum_{i=0}^{n-k} r^i\right)\right]\sum_{k=0}^{\infty} a_{k+n+1}r^{2k}\right\}. \end{aligned} \tag{3.8}$$

By (3.7) and (1.24),

$$\begin{aligned} h_1(r) <& r^{n+1}P_n(r)\left(\sum_{k=0}^{\infty} a_{k+n+1}r^k - r^{n+1}\sum_{k=0}^{\infty} a_{k+n+1}r^{2k}\right) \\ =& r^{n+1}(1-r)P_n(r)\sum_{k=0}^{\infty}\left(a_{k+n+1}r^k\sum_{i=0}^{k+n} r^i\right) < r(1-r)P_n(r)\sum_{k=n+1}^{\infty} k a_k r^{k-1} \\ \le& r(1-r)P_n(r)F'(r) = \alpha r P_n(r)G(r), \end{aligned}$$

from which it follows that

$$\frac{F(r^2)}{F(r)} > \frac{P_n(r^2)}{P_n(r)} - \alpha r\frac{G(r)}{F(r)}.$$

This, together with Lemma 2.3(2), yields the first inequality in (1.12).

Next, from (3.8) we obtain

$$\begin{aligned} h_1(r) >& r^{n+1}(1-r)\left[P_n\left(r^2\right)\sum_{k=1}^{\infty} a_{k+n+1}r^k + P_n\left(r^2\right)\sum_{k=0}^{\infty} a_{k+n+1}r^{2k}\right] \\ >& 2r^{n+1}(1-r)P_n\left(r^2\right)\sum_{k=1}^{\infty} a_{k+n+1}r^{2k} > 2a_{n+2}r^{n+3}(1-r)P_n\left(r^2\right), \end{aligned}$$

from which it follows that

$$\frac{F(r^2)}{F(r)} < \frac{P_n(r^2)}{P_n(r)}P(r) < \frac{P_n(r^2)}{P_n(r)}.$$

If $\alpha \le 1$, namely, $ab \le c$, then by (2.18), a_n is strictly decreasing in $n \in \mathbb{N}_0$. Hence by (3.7),

$$h_1(r) < r^{n+1}P_n(r^2)\sum_{k=0}^{\infty} a_{k+n+1}r^k < r^{n+1}P_n(r^2)\sum_{k=0}^{\infty} a_k r^k = r^{n+1}P_n(r^2)F(r),$$

from which the first inequality in (1.14) follows.

The remaining conclusions in Theorem 1.2 are clear. □

3.3 Proof of Theorem 1.3

Clearly, $f(0) = g(0) = 1$. By (1.25), we can obtain the limiting values $f(1^-) = 1 + \alpha$ and $g(1^-) = 2^\lambda$. Let f_1, f_2 and f_3 be as in Lemma 2.3. Then by differentiation,

$$\frac{r'^2F(r)^2}{\alpha(1+\alpha r)F(r^2)G(r)}f'(r) = f_8(r) \equiv f_2(r)\left[\frac{r'^2}{1+\alpha r} + \frac{2r}{f_2(r^2)}\right] - r - 1, \tag{3.9}$$

$$\frac{1+r}{g(r)}g'(r) = f_9(r) \equiv \lambda - \frac{\alpha}{1-r}\left[\frac{1+r}{f_2(r)} - \frac{2r}{f_2(r^2)}\right]. \tag{3.10}$$

By Lemma 2.3(2), we obtain

$$\begin{aligned}f_8(r) >&f_2(r)\left[\frac{r'^2}{1+\alpha r}+\frac{2r}{f_2(r)}\right]-r-1=r-1+\frac{r'^2}{1+\alpha r}f_2(r)\\=&(1-r)\left[\frac{1+r}{1+\alpha r}f_2(r)-1\right]=\frac{r(1-r)}{(1+\alpha r)G(r)}f_1(r).\end{aligned}\tag{3.11}$$

If $ab \le c+1$, then by Lemma 2.3(1) and (3.11), $f_8(r) > 0$ for all $r \in (0,1)$. Hence by (3.9), f is strictly increasing on $(0,1)$. Conversely, if f is strictly increasing from $(0,1)$ onto $(1,1+\alpha)$, then $f_{10}(r) \equiv F(r)[f(r)-1] > 0$ for all $r \in (0,1)$, so that by l'Hôpital's rule and (1.24),

$$\begin{aligned}0 \le \lim_{r\to 0}\frac{f_{10}(r)}{r^2}&=\frac{1}{2}\lim_{r\to 0}\frac{1}{r}\left[\alpha F(r^2)+\frac{2\alpha r(1+\alpha r)}{1-r^2}G(r^2)-\frac{\alpha}{1-r}G(r)\right]\\=&\alpha+\frac{\alpha}{2}\lim_{r\to 0}\frac{(1-r)F(r^2)-G(r)}{r}\\=&\alpha+\frac{\alpha}{2}\lim_{r\to 0}\left[\frac{2\alpha r}{1+r}G\left(r^2\right)-F\left(r^2\right)-\frac{ab}{c+1}F(a+1,b+1;c+2;r)\right]\\=&\frac{\alpha}{2(c+1)}(c+1-ab),\end{aligned}$$

and hence $ab \le c+1$, thus completing the proof of the assertion on f.

It follows from (3.10) that $f_9(0) = \lambda-\alpha$, so that if g is strictly increasing on $(0,1)$, then $\lambda \ge \alpha$. Conversely, if $\lambda \ge \alpha$, then $g(r) = (1+r)^{\lambda-\alpha}f_3(r)$, which is strictly increasing on $(0,1)$ by Lemma 2.3(3).

The remaining conclusions in part (1) are clear.

(2) Let $f_{11}(r) = f(r)-1-\delta r^2(1+\alpha r)$. Then by part (1), $f_{11}(0) = 0$ and $f_{11}(1^-) = \alpha-\delta(1+\alpha)$.

If the second inequality in (1.16) holds for all $r \in (0,1)$, then $f_{11}(r) < 0$. In particular, $f_{11}(1^-) \le 0$, namely, $\delta \ge \alpha/(1+\alpha)$. On the other hand, $f_{11}'(r)$ must be nonpositive for sufficiently small r, since $f_{11}(0) = 0$. By differentiation,

$$f_{11}'(r)/r = f'(r)/r-\delta(2+3\alpha r),$$

and hence by (3.9),

$$\begin{aligned}\lim_{r\to 0}\frac{f'(r)}{r}&=\lim_{r\to 0}\frac{\alpha(1+\alpha r)F(r^2)G(r)}{r'^2F(r)^2}\left\{\frac{1}{r}\left[\frac{r'^2f_2(r)}{1+\alpha r}-1\right]+\frac{2f_2(r)}{f_2(r^2)}-1\right\}\\&=\alpha+\alpha\lim_{r\to 0}\frac{1}{r}\left[\frac{r'^2f_2(r)}{1+\alpha r}-1\right]=\alpha+\alpha\lim_{r\to 0}\frac{f_2(r)-1-\alpha r}{r}\\&=\alpha(1-\alpha)+\alpha\lim_{r\to 0}\frac{F(r)-G(r)}{r}=\alpha\left(1-\frac{ab}{c+1}\right),\\\lim_{r\to 0}\frac{f_{11}'(r)}{r}&=\lim_{r\to 0}\left[\frac{f'(r)}{r}-\delta(2+3\alpha r)\right]=\frac{c+1-ab}{c+1}\alpha-2\delta\le 0,\end{aligned}$$

yielding $\delta \ge (c+1-ab)\alpha/[2(c+1)]$. Consequently (1.18) follows. The proof of the condition (1.17) is similar.

Finally, if the first inequality in (1.16) holds for all $r \in (0,1)$ and $\tau = \alpha/(1+\alpha)$, then

$$0 < \frac{1}{r^2}\left\{(1+\alpha)(1+\alpha r)F\left(r^2\right)-\left[1+\alpha+\alpha r^2(1+\alpha r)\right]F(r)\right\}$$

so that by l'Hôpital's rule,

$$\begin{aligned}0 \le& \lim_{r\to 0}\frac{(1+\alpha)(1+\alpha r)F(r^2)-\left[1+\alpha+\alpha r^2(1+\alpha r)\right]F(r)}{r^2}\\=&-\frac{\alpha}{2c(c+1)}\left[(ab)^2-ab+c(c+1)\right].\end{aligned}$$

This implies that $(ab)^2-ab+c(c+1) \le 0$. Similarly, if the second inequality in (1.16) holds for all $r \in (0,1)$ and $\delta = \alpha/(1+\alpha)$, then $(ab)^2-ab+c(c+1) \ge 0$. The remaining conclusion is clear. □

4 Concluding Remark

1. In (1.11), the coefficient $9/320$ is not best possible. For $\delta \in (0,\infty)$ and $r \in (0,1)$, let

$$G_1(r) = (4+r)\mathscr{K}(r) - [4+\delta(4+r)r^2]\mathscr{K}(\sqrt{r}),$$
$$h(r) = 2\frac{(4+r)\mathscr{E}(r) - 4r'^2\mathscr{K}(r) - 2(1+r)[\mathscr{E}(\sqrt{r}) - (1-r)\mathscr{K}(\sqrt{r})]}{r^2(1+r)[(4+r)\mathscr{E}(\sqrt{r}) + (1-r)(12+5r)\mathscr{K}(\sqrt{r})]}.$$

Clearly, $h(1^-) = 1/5$. By (1.3) and [4, Lemma 5.2(1)], we obtain

$$\begin{aligned}
h(0^+) =& \frac{1}{4\pi}\lim_{r\to 0}\left\{4\frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{r^2} - 2\frac{\mathscr{E}(\sqrt{r}) - (1-r)\mathscr{K}(\sqrt{r})}{r}\right.\\
&\left. + \frac{r\mathscr{E}(r) - 2\left[\mathscr{E}(\sqrt{r}) - (1-r)\mathscr{K}(\sqrt{r})\right]}{r^2}\right\}\\
=& \frac{1}{8} + \frac{1}{4\pi}\lim_{r\to 0}\frac{1}{r}\left[\mathscr{E}(r) - 2\frac{\mathscr{E}(\sqrt{r}) - (1-r)\mathscr{K}(\sqrt{r})}{r}\right]\\
=& \frac{1}{8} - \frac{1}{8}\lim_{r\to 0}\left\{\sum_{n=1}^{\infty}\frac{1}{2n-1}\left[\frac{(1/2)_n}{n!}\right]^2 r^{2n-1} + \sum_{n=1}^{\infty}\frac{1}{n+1}\left[\frac{(1/2)_n}{n!}\right]^2 r^{n-1}\right\} = \frac{7}{64}.
\end{aligned}$$

By differentiation,

$$\frac{2(1-r)G_1'(r)}{r[(4+r)\mathscr{E}(\sqrt{r}) + (1-r)(12+5r)\mathscr{K}(\sqrt{r})]} = h(r) - \delta. \tag{4.1}$$

Computation supports us to raise the following *conjecture*: There exists a number $r_0 \in (0.56177187, 0.56177188)$ such that h is strictly decreasing on $(0, r_0]$ and increasing on $[r_0, 1)$, so that $\sup_{0<r<1} h(r) = 1/5$ and

$$0.07481609685 < \sigma \equiv \inf_{0<r<1} h(r) = h(r_0) < 0.07481609978. \tag{4.2}$$

If this conjecture is true, then by (4.1), G_1 is strictly increasing (decreasing) on $(0,1)$ if and only if $\delta \le \sigma$ ($\delta \ge 1/5$, respectively), and (1.11) can be improved to the following one

$$\frac{4}{4+r} + \sigma r^2 < \frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})} < \frac{4}{4+r} + \frac{1}{5}r^2, \tag{4.3}$$

with the best possible coefficients σ and $1/5$. More generally, we raise the following *open problem*: What is the best possible value of μ depending only on $a \in (0,1/2]$ such that for all $a \in (0,1/2]$ and $r \in (0,1)$,

$$\frac{1}{1+\rho r} + \mu r^2 < \frac{\mathscr{K}_a(r)}{\mathscr{K}_a(\sqrt{r})} < \frac{1}{1+\rho r} + \eta r^2?$$

2. Clearly, the upper bounds in (1.10)–(1.14) are all less than 1, and improve those given in (1.6)–(1.9).

3. Based on the third inequality in (1.10) or in (1.11), it is natural to ask whether the inequality

$$\frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} < \frac{1}{1+\alpha r} + \frac{\alpha}{1+\alpha}r^2 \tag{4.4}$$

holds for all $a, b \in (0,\infty)$ and $r \in (0,1)$. Our Theorem 1.3(2) gives a kind of necessary conditions with which the inequality (4.4) holds. We raise the following *open problem*: Find the necessary and sufficient condition(s) under which (4.4) is valid, as we did in Theorem 1.3(1).

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