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Maximizing dimension for Bernoulli measures and the Gauss map

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Abstract

We give a short proof that there exists a countable state Bernoulli measure maximizing the dimension of their images under the continued fraction expansion.

1 Introduction

Let $T : [0, 1) \to [0, 1)$ be the usual Gauss map defined by

$$T(x) = \begin{cases} \frac{1}{x} \pmod 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 0. \end{cases}$$

For each infinite probability vector in

$$\mathcal{P} = \left\{ \underline{p} = (p_k)_{k=1}^{\infty} \in [0, 1]^{\mathbb{N}} : \sum_{k=1}^{\infty} p_k = 1 \right\}$$

we can associate a natural T-invariant measure $\mu_{\underline{p}} := \nu_{\underline{p}} \pi^{-1}$, where $\nu_{\underline{p}}$ is the usual countable state Bernoulli measure on $\mathbb{N}^{\mathbb{Z}}$ and $\pi : \mathbb{N}^{\mathbb{N}} \to [0, 1)$ is the usual continued fraction expansion $\pi(x_n) = [x_1, x_2, x_3, \cdots]$. We can define the *dimension* of the measure $\mu_{\underline{p}}$ by

$$d(\mu_{\underline{p}}) := \inf \left\{ \dim_H(B) : B \text{ is a Borel set with } \mu_{\underline{p}}(B) = 1 \right\}$$

where $\dim_H(B)$ denotes the Hausdorff dimension of B (see [3], p. 229). We define the *entropy* and *Lyapunov exponents* of the measure $\mu_{\underline{p}}$ by

$$h(\mu_{\underline{p}}) = -\sum_{k=1}^{\infty} p_k \log p_k \text{ and } \lambda(\mu_{\underline{p}}) = \int \log |T'| d\mu_{\underline{p}}(x),$$

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whenever they are finite, and then we can write $d(\mu_{\underline{p}}) = \frac{h(\mu_{\underline{p}})}{\lambda(\mu_{\underline{p}})} > 0$. Kifer, Peres and Weiss [4] observed that $d(\mu_{\underline{p}})$ is uniformly bounded away from 1 (making use of a thermodynamic approach of Walters) [1] i.e.,

$$D := \sup\left\{ d(\mu_{\underline{p}}) : \underline{p} \in \mathcal{P} \right\} < 1. \tag{1.1}$$

We will give a simple proof of the following result.

Theorem 1.1 (Exact dimensionality)**.** *There exists $\underline{p}^\dagger \in \mathcal{P}$ such that:*

1. *$d(\mu_{\underline{p}^\dagger}) = D$, i.e., $\underline{p}^\dagger$ realises the supremum in (1.1);*
2. *$p_k^\dagger \asymp k^{-2D}$, i.e., $\exists c > 1$ such that $\frac{1}{ck^{2D}} \leq p_k^\dagger \leq \frac{c}{k^{2D}}$, for $k \geq 1$; and*
3. *$\mu_{\underline{p}^\dagger}$ is ergodic.*

The first part of the theorem answers a question the author was asked by K. Burns. [2] I posed the question to my graduate student N. Jurga who, in collaboration with my PDRA S. Baker, gave an elementary proof. Their proof is based on an iterative construction of a sequence of measures $\mu_{\underline{p}_n}$ with increasing dimension $d(\mu_{\underline{p}_n})$ by redistributing the weights in the probability vectors $\underline{p}_n$. In contrast, the proof presented below uses the classical method of Lagrange multipliers on finite dimensional subsets of $\mathcal{P}$, before taking a limit, and has the merits of being short and easy to generalize. Part 2. appears to be new.

2 Proof of Theorem 1.1

We can begin with the following standard lemma (see [2], Lemma 3.2, based on [5]).

Lemma 2.1. *If $d(\mu_{\underline{p}}) > \frac{1}{2}$ then $h(\mu_{\underline{p}}), \lambda(\mu_{\underline{p}}) < +\infty$.*

Since it is easy to exhibit $\underline{p} \in \mathcal{P}$ with $h(\mu_{\underline{p}}), \lambda(\mu_{\underline{p}}) < +\infty$ and $d(\mu_{\underline{p}}) > \frac{1}{2}$ we can deduce $D > \frac{1}{2}$ and use Lemma 2.1 to write

$$D = \sup\left\{ \frac{h(\mu_{\underline{p}})}{\lambda(\mu_{\underline{p}})} : \underline{p} \in \mathcal{P} \right\}.$$

Moreover, we can approximate any $\underline{p} \in \mathcal{P}$ with $h(\mu_{\underline{p}}), \lambda(\mu_{\underline{p}}) < +\infty$ by a probability vector $\underline{p}^*$ with:

$$p_k^* = \begin{cases} p_1 + \varepsilon_n & \text{if } k = 1 \text{ (where } \varepsilon_n := \sum_{l=n+1}^\infty p_l) \\ p_k & \text{if } 2 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

[1]In [4] they showed $D < 1 - 10^{-7}$, but in unpublished work Jenkinson and the author have improved this to $D < 1 - 5 \times 10^{-5}$

[2]At the *Workshop on Hyperbolic Dynamics* (Trieste, 19-23 June 2017)

so that $\frac{h(\mu_{\underline{p}^*})}{\lambda(\mu_{\underline{p}^*})}$ is arbitrarily close to $\frac{h(\mu_{\underline{p}})}{\lambda(\mu_{\underline{p}})}$ for n sufficiently large. For the entropy, we have

$$|h(\mu_{\underline{p}}) - h(\mu_{\underline{p}^*})| \leq |p_1 \log p_1 - (p_1 + \varepsilon_n) \log(p_1 + \varepsilon_n)| + \sum_{k=n+1}^{\infty} p_k |\log p_k| \to 0 \text{ as } n \to +\infty.$$

For the Lyapunov exponent, let $\varepsilon > 0$ and $\log_M |T'(x)| := \min\{\log |T'(x)|, 2\log M\}$ then

$$\left| \lambda(\mu_{\underline{p}}) - \int \log_M |T'(x)| d\mu_{\underline{p}}(x) \right| \leq 2 \int_0^{1/M} \log \left(\frac{1}{xM} \right) d\mu_{\underline{p}}(x) < \varepsilon$$

for $M \in \mathbb{N}$ sufficiently large (since $\sum_{n=1}^{\infty} p_n \log n \leq \lambda(\mu_{\underline{p}}) < +\infty$) and there is a corresponding inequality with $\underline{p}^*$ replacing $\underline{p}$. We can next bound

$$\left| \int \log_M |T'(x)| d\mu_{\underline{p}}(x) - \sum_{\underline{i} \in \mathbb{N}^N} p_{\underline{i}} \log_M |T'(x_{\underline{i}})| \right| < \varepsilon$$

for N sufficiently large, where $\underline{i} = (i_1, \cdots, i_N)$ gives the finite continued fraction $x_{\underline{i}} = [i_1, \cdots, i_N]$ and $p_{\underline{i}} = p_{i_1} \cdots p_{i_N}$, and again there is a corresponding inequality with $\underline{p}^*$ replacing $\underline{p}$. Finally, we can bound

$$\left| \sum_{|\underline{i}|=N} p_{\underline{i}} \log_M |T'(x_{\underline{i}})| - \sum_{|\underline{i}|=N} p^*_{\underline{i}} \log_M |T'(x_{\underline{i}})| \right| \leq \log M \sum_{|\underline{i}|=N} |p_{\underline{i}} - p^*_{\underline{i}}| < \varepsilon$$

for n sufficiently large. (For the last inequality first note that for those $\underline{i}$ with $2 \leq i_j \leq n$ for $1 \leq j \leq N$ then $p_{\underline{i}} = p^*_{\underline{i}}$ and there is no contribution. Furthermore, for those terms with $\underline{i}$ for which there exists $1 \leq j \leq N$ with $i_j > n$ the summation can be bounded $(\varepsilon_n + p_1 + \cdots + p_n)^N - (p_1 + \cdots + p_n)^N \to 0$ as $n \to +\infty$. Finally, the remaining part of the summation comes from $\underline{i}$ with $i_j \leq n$ for $1 \leq j \leq N$ and at least one term being equal 1, and this is $O(\varepsilon_n)$.) The triangle inequality gives $|\lambda(\mu_{\underline{p}}) - \lambda(\mu_{\underline{p}^*})| < 5\varepsilon$. Therefore, we can also write

$$D = \sup_n \sup \left\{ \frac{h(\mu_{\underline{p}^*})}{\lambda(\mu_{\underline{p}^*})} : \underline{p}^* \in \mathcal{P}_n \right\}, \tag{2.1}$$

where $\mathcal{P}_n$ ($n \geq 2$) is the finite dimensional simplex consisting of the probability vectors $\underline{p}^* = (p^*_k)_{k=1}^{\infty}$ satisfying $p^*_k = 0$, for $k > n$.

For each $n \geq 2$ we can extend the definition of $d(\mu_{\underline{p}^*}) := h(\mu_{\underline{p}^*})/\lambda(\mu_{\underline{p}^*})$ to a sufficiently small neighbourhood $U_n \supset \mathcal{P}_n$ so that the function $U_n \ni \underline{p}^* \mapsto d(\mu_{p^*})$ is well defined and smooth. We want to maximize this function subject to the additional restriction $\sum_{k=1}^{n} p^*_k = 1$ which makes it natural to use the method of Lagrange multipliers. This allows us to deduce that a critical point satisfies

$$\frac{\partial d(\mu_{\underline{p}^*})}{\partial p^*_i} = \frac{\partial d(\mu_{\underline{p}^*})}{\partial p^*_j} \text{ for } i \neq j. \tag{2.2}$$

The logarithmic derivative of $d(\mu_{\underline{p}^*})$ takes the form

$$\frac{1}{d(\mu_{\underline{p}^*})}\frac{\partial d(\mu_{\underline{p}^*})}{\partial p_i^*} = \frac{1}{h(\mu_{\underline{p}^*})}\frac{\partial h(\mu_{\underline{p}^*})}{\partial p_i^*} - \frac{1}{\lambda(\mu_{\underline{p}^*})}\frac{\partial \lambda(\mu_{\underline{p}^*})}{\partial p_i^*} \text{ for } 1 \leq i \leq n. \tag{2.3}$$

We can rewrite the right hand side of (2.3) using the following two lemmas. The first follows directly from the definition of $h(\mu_{\underline{p}^*})$.

Lemma 2.2. $\frac{\partial h(\mu_{\underline{p}^*})}{\partial p_i^*} = -(\log p_i^* + 1)$.

We denote the Cantor sets $E_n := \{[x_1, x_2, x_3, \ldots] : x_1, x_2, x_3, \cdots \leq n\} \subset [0, 1]$, for $n \geq 2$. For any Hölder continuous function $f : E_n \to \mathbb{R}$ we can define the *pressure* (restricted to E_n) by

$$P(f) = \sup \left\{ h(\mu) + \int f d\mu : \mu \text{ is a } T\text{-invariant probability measure supported on } E_n \right\},$$

where $h(\mu)$ is the entropy of the measure μ, and there is a unique measure μ_f realizing the supremum which is called the *equilibrium state for f*.

Example 2.3. *We denote the intervals $[i] := \left[\frac{1}{i+1}, \frac{1}{i}\right] \subset (0, 1]$, for $i \geq 1$. Then $\mu_{\underline{p}^*}$ is the equilibrium state for $f_{\underline{p}^*} = \sum_{j=1}^{n} \chi_{[j]} \log p_j^*$.*

For Hölder continuous functions $f, g : E_n \to \mathbb{R}$ we have that $\mathbb{R} \ni t \mapsto P(f + tg) \in \mathbb{R}$ is smooth and

$$\frac{\partial P(f + tg)}{\partial t}|_{t=0} = \int g d\mu_f \tag{2.4}$$

(see [7], Question 5 (a) p.96 and [6], Proposition 4.10). For Hölder continuous functions $f, g_1, g_2 : E_n \to \mathbb{R}$ we have that $\mathbb{R}^2 \ni (t, s) \mapsto P(f + tg_1 + sg_2) \in \mathbb{R}$ is smooth and

$$\frac{\partial^2 P(f + tg_1 + sg_2)}{\partial t \partial s}|_{s=t=0} = \int (g_1 - \overline{g_1})(g_2 - \overline{g_2}) d\mu_f + 2\sum_{n=1}^{\infty} \int (g_1 - \overline{g_1})(g_2 - \overline{g_2}) \circ \sigma^n d\mu_f \tag{2.5}$$

where we denote $\overline{g_1} = \int g_1 d\mu$ and $\overline{g_2} = \int g_2 d\mu$ (see [7], Question 5 (b) p.96 and [6], Proposition 4.11).

Lemma 2.4. $\frac{1}{\lambda(\mu_{\underline{p}^*})}\frac{\partial \lambda(\mu_{\underline{p}^*})}{\partial p_i^*} = \frac{1}{p_i^*}\frac{\int_{[i]} \log |T'| d\mu_{\underline{p}^*}}{\int \log |T'| d\mu_{\underline{p}^*}} - 1$

Proof. Using (2.4) and the definition of $\lambda(\mu_{\underline{p}^*})$ we can first rewrite

$$\lambda(\mu_{\underline{p}^*}) = \frac{\partial P(f_{\underline{p}^*} + t \log |T'|)}{\partial t}|_{t=0} \text{ and } \frac{\partial \lambda(\mu_{\underline{p}^*})}{\partial p_i^*} = \frac{\partial^2 P(f_{\underline{p}^*} + s\chi_{[i]}/p_i^* + t \log |T'|)}{\partial s \partial t}|_{t=0, s=0}. \tag{2.6}$$

Next we can use (2.5) with $f = f_{\underline{p}^*}$, $g_1 = \chi_{[i]}/p_i^*$ and $g_2 = \log|T'|$ to write

$$\begin{aligned} &\frac{\partial^2 P(f_{\underline{p}^*} + s\chi_{[i]}/p_i^* + t\log|T'|)}{\partial s \partial t}\Big|_{t=0,s=0} \\ &= \frac{1}{p_i^*}\int \left(\chi_{[i]} - p_i^*\right)\left(\log|T'| - \int \log|T'| d\mu_{\underline{p}^*}\right) d\mu_{\underline{p}^*} \\ &\quad + \frac{2}{p_i^*}\sum_{n=1}^{\infty}\int \left(\chi_{[i]} - p_i^*\right)\left(\log|T'| - \int \log|T'| d\mu_{\underline{p}^*}\right)\circ \sigma^n d\mu_{\underline{p}^*}. \end{aligned} \tag{2.7}$$

If we consider the transfer operator $\mathcal{L}_{f_{\underline{p}^*}} : C^0(E_n) \to C^0(E_n)$ defined by

$$\mathcal{L}_{f_{\underline{p}^*}} w(x) = \sum_{k=1}^{n} p_k^* w\left(\frac{1}{k+x}\right) \tag{2.8}$$

which is the dual to the Koopman operator (see [7]) then since the dual to the transfer operator satisfies $\mathcal{L}^*_{f_{\underline{p}^*}}\mu_{\underline{p}^*} = \mu_{\underline{p}^*}$ we can rewrite (2.7) as

$$\begin{aligned} &\frac{\partial^2 P(f_{\underline{p}^*} + s\chi_{[i]}/p_i^* + t\log|T'|)}{\partial s \partial t}\Big|_{t=0,s=0} \\ &= \frac{1}{p_i^*}\int \left(\chi_{[i]} - p_i^*\right)\left(\log|T'| - \int \log|T'| d\mu_{\underline{p}^*}\right) d\mu_{\underline{p}^*} \\ &\quad + \frac{2}{p_i^*}\sum_{n=1}^{\infty}\int \mathcal{L}^n_{f_{\underline{p}^*}}\left(\chi_{[i]} - p_i^*\right)\left(\log|T'| - \int \log|T'| d\mu_{\underline{p}^*}\right) d\mu_{\underline{p}^*}. \end{aligned} \tag{2.9}$$

From the definition of $\mathcal{L}_{f_{\underline{p}^*}}$ we see that $\mathcal{L}_{f_{\underline{p}^*}}\left(\chi_{[i]} - p_i^*\right) = 0$ and we can deduce that the series in (2.9) vanishes and then using (2.6) we can write

$$\frac{1}{\lambda(\mu_{\underline{p}^*})}\frac{\partial \lambda(\mu_{\underline{p}^*})}{\partial p_i} = \frac{1}{p_i^*}\int \left(\chi_{[i]} - p_i^*\right)\left(\frac{\log|T'|}{\int \log|T'| d\mu_{\underline{p}^*}} - 1\right) d\mu_{\underline{p}} = \frac{1}{p_i^*}\frac{\int_{[i]} \log|T'| d\mu_{\underline{p}}}{\int \log|T'| d\mu_{\underline{p}}} - 1.$$

$\square$

Using the formulae in Lemmas 2.2 and 2.4 and the equality (2.3) we can rewrite (2.2) as

$$-(\log p_i^* + 1) - d(\mu_{p^*})\left(\frac{1}{p_i^*}\frac{\int_{[i]} \log|T'| d\mu_{p^*}}{\int \log|T'| d\mu_{p^*}} - 1\right) = -(\log p_j^* + 1) - d(\mu_{p^*})\left(\frac{1}{p_j^*}\frac{\int_{[j]} \log|T'| d\mu_{p^*}}{\int \log|T'| d\mu_{p^*}} - 1\right)$$

for all $1 \le i, j \le n$. Moreover, since $2p_i^* \log i \le \int_{[i]} \log|T'| d\mu_{p^*} \le 2p_i^* \log(i+1)$ this implies that

$$2d(\mu_{p^*})\log\left(\frac{i}{j+1}\right) \le \log\left(\frac{p_j^*}{p_i^*}\right) \le 2d(\mu_{p^*})\log\left(\frac{i+1}{j}\right) \tag{2.10}$$

for any $n \ge 2$ and $n \ge i > j$.

To construct $p^\dagger \in \mathcal{P}$ we use a simple tightness argument. For each sufficiently large n, let $\underline{p}^{(n)} \in \mathcal{P}_n$ denote a measure maximizing $\mathcal{P}_n \ni \underline{p}^* \mapsto d(\mu_{p^*})$. It then follows from (2.1) that

$\lim_{n\to+\infty} d(\underline{p}^{(n)}) = D > \frac{1}{2}$. Since (2.10) applies to each of the $\underline{p}^{(n)}$, and $d(\mu_{\underline{p}^{(n)}})$ is arbitrarily close to D for n sufficiently large, one can choose $\varepsilon > 0$, $C > 0$ and $n_0 > 0$ such that

$$p_k^{(n)} \leq Ck^{-(D-\varepsilon)} \text{ for all } k \geq 1 \text{ and } n \geq n_0.$$

We can choose a subsequence $\underline{p}^{(n_r)}$ $(r \geq 1)$ converging (using the usual diagonal argument) to some $\underline{p}^{\dagger} \in \mathcal{P}$. To see that $d(\mu_{\underline{p}^{\dagger}}) = D$ we first observe that $d(\mu_{\underline{p}^{*(n_r)}}) > \frac{1}{2}$ for sufficiently large r. We can deduce from (2.1), and the definitions of the entropies and Lyapunov exponents, that $d(\mu_{\underline{p}^{\dagger}}) = \lim_{r\to+\infty} d(\mu_{\underline{p}^{*(n_r)}}) = D$. This completes the proof of part 1.

By applying (2.10) to $\underline{p}^{*(n_r)}$ and taking the limit $r \to +\infty$ shows that the same bounds apply for $\underline{p}^{\dagger}$ (with $d(\mu_{\underline{p}^{\dagger}}) = D$) and this completes the proof of part 2.

To prove the final part of the theorem, we use another standard argument (cf. [6], Chapter 2, for the case of Hölder functions). We can consider the associated transfer operator $\mathcal{L}_{p^{\dagger}} : C^1([0,1]) \to C^1([0,1])$ defined (by analogy with (2.8)) as

$$\mathcal{L}_{f_{\underline{p}^{\dagger}}} w(x) = \sum_{k=1}^{\infty} p_k^{\dagger} w\left(\frac{1}{k+x}\right) \text{ for } w \in C^1([0,1]).$$

First observe that $\mathcal{L}_{p^{\dagger}} 1 = 1$, and thus $\|\mathcal{L}_{p^{\dagger}} w\|_{\infty} \leq \|w\|_{\infty}$, and $\left\| \frac{d}{dx}\left(\mathcal{L}^2_{f_{\underline{p}^{\dagger}}} w\right)(x)\right\|_{\infty} \leq \frac{1}{4}\|w\|_{\infty}$ (cf. [6], Proposition 2.1). This is sufficient to show that for any $w \in C^1([0,1])$ one has $\|\mathcal{L}^n_{p^{\dagger}} w - \int w d\mu_{\underline{p}^{\dagger}}\|_{\infty} \to 0$ as $n \to +\infty$ (compare [6], Theorem 2.2, (iv)). We can deduce that for $v, w \in C^1([0,1])$ with $\int v d\mu_{\underline{p}^{\dagger}} = 0 = \int v d\mu_{\underline{p}^{\dagger}}$ then since $\mathcal{L}^*_{\underline{p}^{\dagger}} \mu_{\underline{p}^{\dagger}} = \mu_{\underline{p}^{\dagger}}$ we have

$$\left|\int v \circ T^n w d\mu_{\underline{p}^{\dagger}}\right| = \left|\int v \mathcal{L}^n_{p^{\dagger}} w d\mu_{\underline{p}^{\dagger}}\right| \leq \|v\|_{\infty}.\|\mathcal{L}^n_{\underline{p}^{\dagger}} w\|_{\infty} \to 0 \text{ as } n \to +\infty.$$

In particular, this implies that $\mu_{\underline{p}^{\dagger}}$ is strong mixing, and thus ergodic (cf. [6], Proposition 2.4).

3 Additional remarks

Remark 3.1. The approach we have described should apply beyond the Gauss map. One of the best known classes are the Rényi f-expansions, where in the definition of T the function $1/x$ is replaced by a more general (continuously differentiable) monotone decreasing function $f : (0,1] \to [1,+\infty)$ and π is replaced by a map $\pi_f : \mathbb{N}^{\mathbb{N}} \to [0,1]$ (see [8], chapter 10). We can then set $s_0 = \inf\{s > 0 : \sum_{n=1}^{\infty} |(f^{-1})'(n)|^s < +\infty\}$ and define D_f by analogy with (2.1). We would need to assume that $D_f > s_0$ and then one expects that a version of Theorem 1.1 holds in this setting, where part 2. would now take the form $\underline{p}_k^{\dagger} \asymp |(f^{-1})'(k)|^D$, for $k \geq 1$.

Remark 3.2. The proof of Theorem 1.1 also generalizes in the following way. Let $f : (0,1] \to \mathbb{R}$ be a locally constant function of the form $f(x) = b_i$, say, if for $\frac{1}{i+1} < x \leq \frac{1}{i}$, with super-polynomial decay (i.e., for any $\beta > 0$, $|b_i| = O(i^{-\beta})$). Given $\alpha \in \mathrm{int}\{\int f d\mu_{\underline{p}} : \underline{p} \in \mathcal{P}\}$ we can consider an expression common in multifractal analysis:

$$D_f := \sup\left\{\frac{h(\mu)}{\int \log x d\mu} : h(\mu_{\underline{p}}), \lambda(\mu_{\underline{p}}) < +\infty \text{ and } \int f d\mu = \alpha\right\}. \tag{3.1}$$

Then there exists $\underline{p}^f = (\underline{p}_k^f)_{k=1}^{\infty} \in \mathcal{P}$ such that: $\mu_{\underline{p}^f}$ realises the supremum in (3.1); $p_k^f \asymp k^{-2D_f}$ for $k \geq 1$; and $\mu_{\underline{p}^f}$ is ergodic.

Remark 3.3. We can compute higher derivatives of $d(\mu_{\underline{p}})$ using higher derivatives of the pressure function. However, these do not seem particularly useful.

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