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*An extremality property of Szegő projections on Heisenberg groups*

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# AN EXTREMALITY PROPERTY OF SZEGŐ PROJECTIONS ON HEISENBERG GROUPS

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ABSTRACT. We prove that Heisenberg groups, a.k.a. the boundaries of Siegel domains, minimize the $L^p$ operator norm of the Szegő projection in a large class of weighted CR manifolds of hypersurface type.

## 1. INTRODUCTION

The classical Szegő projection associates to every $L^2$ function on the boundary of a domain in $\mathbb{C}^n$ its orthogonal projection onto the space of boundary values of holomorphic functions, that is, the Hardy space $H^2$. For a variety of pseudoconvex domains, the corresponding Szegő projection is known to be a singular integral operator (see, e.g., [PS77, NRSW89, MS97, CD06]), whose $L^p$ mapping behaviour is therefore of considerable interest from the point of view of harmonic and complex analysis alike.

In this note, we compare the $L^p$ mapping behaviour of the Szegő projection $\mathcal{S}$ on a real hypersurface $M \subset \mathbb{C}^{n+1}$ with the $L^p$ mapping behaviour of the Szegő projection $\mathcal{S}_n$ on the model strictly pseudoconvex hypersurface of the same dimension, namely the Heisenberg group

$$\mathbb{H}^n = \{z \in \mathbb{C}^{n+1} \colon \operatorname{Im} z_{n+1} = |\hat{z}|^2\}, \qquad \hat{z} = (z_1, \ldots, z_n).$$

The quantities of interest are the operator norms

$$\|\mathcal{S}\|_{p \to p} := \sup_f \frac{\|\mathcal{S}f\|_p}{\|f\|_p}, \quad \|\mathcal{S}_n\|_{p \to p} := \sup_f \frac{\|\mathcal{S}_n f\|_p}{\|f\|_p} \qquad (1 \leq p \leq \infty),$$

where the Szegő projections and the $L^p$ norms $\|\cdot\|_p$ are defined with respect to a fixed background measure, which we always assume to have a smooth positive density with respect to Lebesgue measure. The model hypersurface $\mathbb{H}^n$ is identified in this paper with $\mathbb{C}^n \times \mathbb{R}$ via the global coordinates $(\hat{z}, \operatorname{Re} z_{n+1})$. The background measure on $\mathbb{H}^n$ is then obtained by pulling back Lebesgue measure of $\mathbb{C}^n \times \mathbb{R}$ via this identification.

Our main result shows that the model Szegő projection $\mathcal{S}_n$ minimizes the $L^p$ operator norm in a large class of hypersurfaces $M$.

**Theorem 1.** *If $M$ is compact and pseudoconvex, then*

$$\|\mathcal{S}\|_{p \to p} \geq \|\mathcal{S}_n\|_{p \to p}.$$

*The same conclusion holds, more generally, if $M$ is a $(2n+1)$-dimensional abstract CR manifold of hypersurface type satisfying property C at some strongly pseudoconvex point $x_0$.*

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The definition of property C is given in Section 2. This property depends on both the CR structure and the choice of background measure on $M$, and it is automatically satisfied by every compact pseudoconvex CR submanifold of $\mathbb{C}^N$ (not necessarily a real hypersurface, see Remark 5 below) and by the model itself (see Proposition 8).

The idea of the proof is pretty straightforward: since $M$ is well-approximated by the model $\mathbb{H}^n$ near a strongly pseudoconvex point $x_0$, whenever $f \in L^2(M)$ is sharply localized around $x_0$ its Szegő projection $\mathcal{S}f$ can be well-approximated by $\mathcal{S}_n \tilde{f}$, where $\tilde{f}$ is a function obtained by "transplanting" $f$ on the model.

Yet, there is a difficulty. While the above sketch seems applicable to any abstract CR manifold of hypersurface type with at least a strongly pseudoconvex point $x_0$, we know a priori that *such an approach cannot work in this generality*. In fact, there are compact strongly pseudoconvex CR structures on the three-dimensional torus $\mathbb{T}^3$ with the property that the only square-integrable CR functions are the constants [Bar88]. For such a pathological non-embeddable CR manifold $M$, the associated Szegő projection is the averaging operator $\mathcal{S}f = \frac{1}{\nu(M)} \int_M f \, d\nu$, whose $L^p$ operator norm is $\|\mathcal{S}\|_{p\to p} = 1$ for every $p \in [1, \infty]$. Since $\mathcal{S}_n$ is unbounded on $L^1$ (see, e.g., [Ste93, Chapter 12]), the conclusion of Theorem 1 cannot hold for $M$. Thus, some additional hypothesis, like our property C, and the ensuing twists appearing in our argument, are indeed necessary.

A few more remarks may be of interest.

(1) We do not know whether a minimizer for the $L^p$ operator norm exists in the restricted class of compact pseudoconvex embeddable CR manifolds of hypersurface type (of a fixed dimension). *It is natural to ask whether the standard CR sphere, endowed with a rotation invariant measure, is such a minimizer.* Since $\mathbb{H}^n$ and the punctured sphere $\mathbb{S}^{2n+1} \setminus \{*\}$ are CR isomorphic (see, e.g., [Ste93, Chapter 12]) and points have zero capacity in dimension 3 or higher, the Szegő projection on the standard CR sphere with respect to a rotation invariant measure is equivalent to the Szegő projection $\widetilde{\mathcal{S}}_n$ on $\mathbb{H}^n$ with respect to a certain (finite) background measure, different from Lebesgue measure.

(2) To the best of our knowledge, the constants $\|\mathcal{S}_n\|_{p\to p}$ (except for the "trivial" cases $p = 1, \infty$) are unknown. C. Liu shows [Liu18] that $\|\mathcal{S}_n\|_{p\to p} \geq \frac{\Gamma\left(\frac{n+1}{p}\right)\Gamma\left(\frac{n+1}{q}\right)}{\Gamma\left(\frac{n+1}{2}\right)^2}$, where $q$ is the conjugate exponent to $p$, and conjectures that equality holds.

(3) Since $\|\mathcal{S}_n\|_{1\to 1} = \infty$, Theorem 1 implies that the Szegő projection is unbounded on $L^1$ for a variety of CR structures and background measures. More flexible arguments are available to investigate the $L^1$ (un)boundedness of projection operators onto spaces of solutions of first-order PDEs, see [Dal22].

(4) On a more speculative note, it is reasonable to expect that if $M$ satisfies property C at some *weakly* pseudoconvex point $x_0$, then $\|\mathcal{S}\|_{p\to p} \geq \|\widetilde{\mathcal{S}}\|_{p\to p}$ for an appropriate model Szegő projection $\widetilde{\mathcal{S}}$ depending on the nature of the point $x_0$. E.g., if $M$ is three-dimensional, pseudoconvex, and $x_0$ has type $2m \geq 2$, then the appropriate model should be $\{\text{Im}\, z_2 = \varphi(z_1, \overline{z_1})\} \subset \mathbb{C}^2$, where $\varphi$ is a subharmonic (nonharmonic) homogeneous polynomial of degree $2m$. If $\mathcal{S}_\varphi$ is the projector on this model, a natural follow-up question is whether $\|\mathcal{S}_\varphi\|_{p\to p} > \|\mathcal{S}_{|z_1|^2}\|_{p\to p} = \|\mathcal{S}_1\|_{p\to p}$ whenever $\varphi$ has degree 4 or higher. If this were the case, equality in Theorem 1 (in an appropriate class of manifolds) could only be achieved in the strongly pseudoconvex case. This would be a first step in the understanding of possible extremizers, other than the model, of the inequality of Theorem 1.

The paper is organized as follows: in Section 2 we rigorously define Szegő projections on "weighted" CR manifolds and discuss property C, while in Section 4 we prove the main theorem exploiting a couple of preliminary lemmas presented in Section 3.

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## 2. Szegő projections on weighted CR manifolds

We assume that the reader is familiar with the basics of CR manifolds, for which we refer to [BER99, DT07]. We limit ourselves to recalling a few notions, mostly to establish notation.

A pair $(M, T_{1,0}M)$ is said to be a *CR manifold of hypersurface type* if

   i. $M$ is a $(2n + 1)$-dimensional connected and orientable real smooth manifold;
   ii. $T_{1,0}M$ is a rank $n$ vector subbundle of the complexified tangent bundle $\mathbb{C}TM$ such that $T_{1,0}M \cap \overline{T_{1,0}M} = 0$ and $[T_{1,0}M, T_{1,0}M] \subseteq T_{1,0}M$, i.e., the commutator of smooth sections of $T_{1,0}M$ is again a section of $T_{1,0}M$.

A $(2n + 1)$-dimensional real smooth submanifold $M$ of $\mathbb{C}^N$ with the property that $T_{1,0}M := \mathbb{C}TM \cap T_{1,0}\mathbb{C}^N$ has rank constantly equal to $n$ is called a *CR submanifold of hypersurface type*: the pair $(M, T_{1,0}M)$ is a CR manifold of hypersurface type.

In what follows, we will usually omit the specification "of hypersurface type" and the bundle $T_{1,0}M$ from the notation, and we will just say that $M$ is a CR manifold, or a CR submanifold of some $\mathbb{C}^N$.

The CR manifold $M$ is *pseudoconvex* if there exists a global nowhere vanishing purely imaginary one-form $\theta$ annihilating $T_{1,0}M \oplus \overline{T_{1,0}M}$ and such that $d\theta(L, \overline{L}) \geq 0$ for every section $L$ of $T_{1,0}M$. A point $x_0 \in M$ is said to be *strongly pseudoconvex* if $d\theta(L, \overline{L})_{x_0} > 0$ for every $L$ that does not vanish at $x_0$.

**Example.** *In this paper $\mathbb{H}^n$, which will be simply called the Heisenberg group, is the CR manifold obtained by endowing $\mathbb{C}_z^n \times \mathbb{R}_t$ with the CR structure bundle generated by the complex vector fields*

$$L_j := \partial_{z_j} + i\overline{z}_j \partial_t \qquad j = 1, \ldots, n.$$

*See [Ste93, Chapter 12] for details on the nilpotent Lie group structure of $\mathbb{H}^n$ and its ties with complex analysis. The Heisenberg group is everywhere strongly pseudoconvex. Our model weighted CR manifold will be $(\mathbb{H}^n, \sigma)$, where $\sigma = \left(\frac{i}{2}\right)^n dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n \wedge dt$ is Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$.*

If $f \in L_{\text{loc}}^1(M)$, we say that $f$ is a CR-function if $\overline{L}f = 0$ in the sense of distributions for every smooth section $L$ of $T_{1,0}M$. For the convenience of the reader, we recall what this means in our context.

If $L$ is a smooth vector field with complex coefficients, that is, a smooth section of the complexified tangent bundle $\mathbb{C}TM$, then we denote by $L^\dagger$ the adjoint of $L$ with respect to the natural pairing of smooth functions with top-degree forms. More precisely, there is a unique first order differential operator $L^\dagger : \Omega_c^{2n+1}M \to \Omega_c^{2n+1}M$ such that

$$\int_M Lf \cdot \omega = \int_M f L^\dagger \omega \qquad \forall f \in C^\infty(M), \quad \forall \omega \in \Omega_c^{2n+1}M,$$

where $\Omega_c^n M$ is the space of smooth compactly supported $(2n+1)$-forms on $M$. If $L = \sum_{j=1}^{2n+1} a_j(x)\partial_{x_j}$ and $\omega = g(x)dx_1 \wedge \ldots \wedge dx_{2n+1}$ in a local coordinate system, then $L^\dagger \omega = -\sum_{j=1}^{2n+1} a_j(x)\partial_{x_j} g - \left(\sum_{j=1}^{2n+1} \partial_{x_j} a_j(x)\right) g$. If $f \in L_{\text{loc}}^1(M)$, then one says that $Lf = 0$ in the sense of distributions if

$$\int_M f L^\dagger \omega = 0 \qquad \forall \omega \in \Omega_c^{2n+1}M.$$

A *weighted CR manifold* is a pair $(M, \nu)$, where $M$ is a CR manifold and $\nu$ is a smooth positive $(2n+1)$-form on $M$. Of course, $\nu$ may be thought of as a Borel measure on $M$ with smooth positive density with respect to Lebesgue measure in any coordinate chart.

Given a weighted CR manifold $(M, \nu)$, we denote by $\text{CR}^2(M, \nu)$ the space of CR functions that are square-integrable with respect to $\nu$, with the usual identification of almost everywhere equal functions.

**Proposition 2.** $\text{CR}^2(M, \nu)$ *is a closed subspace of the Hilbert space $L^2(M, \nu)$.*

This follows immediately from the definition of CR functions and the fact that if a sequence of functions $\{f_k\}_k$ converges in $L^2(M, \nu)$ to a function $f$, then $\lim_{k \to +\infty} \int_M f_k \omega = \int_M f\omega$ for every $\omega \in \Omega_c^{2n+1}M$.

Thus, we may define the main object of study of the present paper.

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**Definition 3.** *The Szegő projection $\mathcal{S}_{M,\nu}$ of the weighted CR manifold $(M,\nu)$ is defined as the orthogonal projection operator mapping $L^2(M,\nu)$ onto $\mathrm{CR}^2(M,\nu)$. We denote by $\mathcal{S}_n$ the Szegő projection associated to the Heisenberg group $(\mathbb{H}^n, \sigma)$.*

We are interested in the $L^p$-operator norms of Szegő projections, namely the quantities defined by

$$N_p(M,\nu) := \sup\left\{ \left( \int_M |\mathcal{S}f|^p d\nu \right)^{\frac{1}{p}} : f \in L^2(M,\nu) \text{ and } \int_M |f|^p d\nu = 1 \right\},$$

for $p \in [1, +\infty)$, and by

$$N_\infty(M,\nu) := \sup\left\{ \|\mathcal{S}f\|_\infty : f \in L^2(M,\nu) \text{ and } \|f\|_\infty = 1 \right\}.$$

To formulate our main result, we need the notion of Property C (C is for "compactness"). Recall that $\overline{\partial}_b$ is the operator mapping $f \in C^\infty(M)$ into the smooth section of $B_{0,1}M$, the dual bundle of $T_{0,1}M$, defined by $\langle \overline{\partial}_b f, \overline{L} \rangle = \overline{L}f$ for every section $L$ of $T_{1,0}M$. Given a smooth positive measure $\nu$ on $M$ and a Hermitian metric $h$ on $B_{0,1}M$, we have the associated quadratic form

(1) $$\mathcal{E}(f) = \int_M |f|^2 \, d\nu + \int_M |\overline{\partial}_b f|_h^2 \, d\nu.$$

Given a precompact open set $B \subseteq M$, we denote by $\mathfrak{D}(B) \subseteq L^2(B)$ the completion of $C_c^\infty(B)$ with respect to $\mathcal{E}$. Notice that different choices of $\nu$ and $h$ produce isomorphic topological vector spaces, because the corresponding quadratic forms $\mathcal{E}$ are comparable on test functions supported on $B$.

**Definition 4** (Property C). *We say that the weighted CR manifold $(M,\nu)$ satisfies property C at $x_0 \in M$ if there exists a precompact open neighborhood $B$ of $x_0$ such that the operator*

$$\begin{aligned} \mathfrak{D}(B) &\longrightarrow L^2(B) \\ f &\longmapsto 1_B \left(1 - \mathcal{S}_{M,\nu}\right) f \end{aligned}$$

*is compact, where $1_B$ denotes the indicator function of $B$. By the observation right before this definition, compactness is independent of the choice of metric on $B_{0,1}M$.*

**Remark 5.** *Property C is a local compactness condition of the kind playing an important role in the theories of the $\overline{\partial}$ and the $\overline{\partial}_b$ problem (see, e.g., [Str10, Chapter 4]). By Sobolev embedding, it holds whenever a subelliptic estimate of the form*

$$\|u\|_{W^\epsilon(B)} \leq C \|\overline{\partial}_b u\|_{L^2(M)}$$

*holds for every $u \in L^2(M,\nu)$ orthogonal to $\mathrm{CR}^2(M,\nu)$, where $\epsilon > 0$. Such a subelliptic estimate with $\epsilon = \frac{1}{2}$ is known to hold in a small enough neighborhood $B$ of a strongly pseudoconvex point $x_0 \in M$, under the additional assumptions that $M$ is compact, pseudoconvex, and that the maximal $L^2$ extension of $\overline{\partial}_b$ has closed range. We refer to [Koh85] for this and far more general results on these matters.*

*We point out that, if $M$ is a compact pseudoconvex CR submanifold of $\mathbb{C}^N$, then the closed range property automatically holds [Bar12b, Bar12a]. Since such a CR submanifold necessarily has a point of strong pseudoconvexity (viz. any point at maximal distance from the origin of $\mathbb{C}^N$), we see that every compact pseudoconvex CR submanifold of $\mathbb{C}^N$ has a strongly pseudoconvex point at which property C holds, that is, satisfies the assumption of Theorem 1' below.*

We can now restate the main theorem in the more precise notation of this section.

**Theorem 1'.** *Let $(M,\nu)$ be a weighted CR manifold of dimension $2n+1$. If $(M,\nu)$ satisfies property C at a strongly pseudoconvex point $x_0 \in M$, then we have*

$$N_p(M,\nu) \geq N_p(\mathbb{H}^n, \sigma) \qquad \forall p \in [1, +\infty].$$

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## 3. Preliminaries

### 3.1. Folland–Stein coordinates.

Define the one-parameter group of *parabolic scalings* on $\mathbb{C}^n \times \mathbb{R}$ as follows:

$$\Phi_\lambda(z,t) := (\lambda z, \lambda^2 t) \qquad (\lambda > 0).$$

We say that a monomial $z^\alpha \overline{z}^\beta t^\gamma$, where $\alpha, \beta \in \mathbb{N}^n$ and $\gamma \in \mathbb{N}$, has *parabolic weight*

$$\sum_{j=1}^n (\alpha_j + \beta_j) + 2\gamma,$$

and that a smooth function defined in a neighborhood of $(0,0) \in \mathbb{C}^n \times \mathbb{R}$ has parabolic weight $\geq w$ if every monomial with nonzero coefficient in its Taylor expansion has weight $\geq w$. The parabolic weight of a vector field $E = \sum_{j=1}^n \left\{ a_j(z,t)\partial_{z_j} + b_j(z,t)\partial_{\overline{z}_j} \right\} + c(z,t)\partial_t$ is computed by assigning weight $-1$ to $\partial_{z_j}$ and $\partial_{\overline{z}_j}$ and weight $-2$ to $\partial_t$. In other words, $E$ has parabolic weight $\geq w$ if every $a_j$ and every $b_j$ has parabolic weight $\geq w+1$ and $c$ has parabolic weight $\geq w+2$.

We refer to [FS74] and [DT07, Theorem 3.5] for a proof of the following proposition.

**Lemma 6** (Folland–Stein coordinates). *Let $M$ be a CR manifold and let $x_0 \in M$ be a strictly pseudoconvex point. Then in a neighborhood of $x_0$ there exists a local system of coordinates $(z,t) \in \mathbb{C}^n \times \mathbb{R}$ such that $(z(x_0), t(x_0)) = 0$ and a system of local generators of $T_{1,0}M$ of the form*

$$(2) \qquad L_j = \partial_{z_j} + i\overline{z}_j \partial_t + E_j \qquad (j = 1, \ldots, n),$$

*where the error terms $E_j$ have parabolic weight $\geq 0$.*

### 3.2. A density lemma.

**Lemma 7.** *There exists a dense subspace $\mathcal{D}$ of $\mathrm{CR}^2(\mathbb{H}^n, \sigma)$ such that for every $h \in \mathcal{D}$, $\alpha, \beta \in \mathbb{N}^n$, $\gamma \in \mathbb{N}$, and $N \in \mathbb{N}$, we have*

$$|\partial_z^\alpha \partial_{\overline{z}}^\beta \partial_t^\gamma h(z,t)| \leq C_N (|z|^2 + |t|)^{-N}.$$

*Proof.* In view of the invariance under $t$-translations of the Heisenberg CR structure, we exploit the partial Fourier transform

$$\mathcal{F}f(z,\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z,t) e^{-i\xi t} dt$$

(cf. [Nag86,Has94,Has98] for similar arguments). The operator $\mathcal{F}$ is a unitary isometry of $L^2(\mathbb{H}^n, \sigma)$, and

$$(3) \qquad \mathcal{F}\left( (\partial_{\overline{z}_j} - iz_j \partial_t)f \right) = (\partial_{\overline{z}_j} + z_j \xi)\mathcal{F}f = e^{-\xi|z|^2} \partial_{\overline{z}_j}\left( e^{\xi|z|^2} \mathcal{F}f \right).$$

Thus a function $f \in L^2(\mathbb{H}^n, \sigma)$ is CR if and only if $g(z,\xi) = e^{\xi|z|^2}\mathcal{F}f$ is holomorphic for every $\xi \in \mathbb{R}$. Since $g \in L^2(\mathbb{C}^n, e^{-2\xi|z|^2})$ for almost every $\xi$ and there are no nonzero Lebesgue square-integrable holomorphic functions on $\mathbb{C}^n$, we must have $g(z,\xi) = 0$ for almost every $\xi \leq 0$. In conclusion, the operator $f \mapsto e^{\xi|z|^2}\mathcal{F}f$ establishes a unitary isomorphism between $\mathrm{CR}^2(\mathbb{H}^n, \sigma)$ and the Hilbert space

$$\mathcal{H} = \Bigg\{ g \colon \mathbb{C}^n \times (0, +\infty) \to \mathbb{C} \colon g(\cdot, \xi) \text{ is holomorphic } \forall \xi > 0 \text{ and}$$

$$\int_{\mathbb{C}^n \times (0,+\infty)} |g|^2 e^{-2\xi|z|^2} \, dV(z) \, d\xi < +\infty \Bigg\},$$

where $dV(z) = \left(\frac{i}{2}\right)^n dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n$ is Lebesgue measure on $\mathbb{C}^n$. A dense subspace $\mathcal{D}'$ of $\mathcal{H}$ is given by the linear span of functions of the form $g(z,\xi) = P(z)\varphi(\xi)$, where $P(z) = P(z_1, \ldots, z_n)$ is a holomorphic polynomial and $\varphi \in C_c^\infty(\mathbb{R}^+)$. To see this, notice that if $g \in \mathcal{H}$ is orthogonal to $\mathcal{D}'$, then $\varepsilon^{-1} \int_a^{a+\varepsilon} \int_{\mathbb{C}^n} g(z,\xi)\overline{P(z)} e^{-2\xi|z|^2} dV(z) \, d\xi = 0$ for every $a, \varepsilon > 0$. Letting $\varepsilon$ tend to zero, we see that, for almost every $a > 0$, $\int_{\mathbb{C}^n} g(z,a)P(z)e^{-2a|z|^2} dV(z) = 0$. By the arbitrariness of $P$ and the density of polynomials in the Fock space [Zhu12], we conclude that $g(z,a) = 0$ for almost every $a > 0$.

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Let $\mathcal{D} := \mathcal{F}^{-1}(\mathcal{D}')$. By what we just proved, $\mathcal{D}$ is dense in $\mathrm{CR}^2(\mathbb{H}^n)$. It is generated by elements of the form

$$P(z) \int_{\mathbb{R}^+} \varphi(\xi) e^{-\xi|z|^2} e^{i\xi t} d\xi$$

with $P$ and $\varphi$ as above. If $\varphi$ is supported on $[a, +\infty)$, then a standard integration by parts shows that

$$\left| (|z|^2 - it)^N \int \varphi(\xi) e^{-\xi|z|^2} e^{i\xi t} d\xi \right| \le e^{-a|z|^2} \int |\varphi^{(N)}(\xi)| d\xi.$$

Thus, every element of $\mathcal{D}$ decays faster than any negative power of $|z|^2 + |t|$. A similar argument proves that the same holds for every derivative. $\qquad\square$

3.3. **Models satisfy property C.** The next result is not strictly needed for the proof, but shows that the noncompact model $(\mathbb{H}^n, \sigma)$ is in the class of CR manifolds to which the theorem applies.

**Proposition 8.** *The model $(\mathbb{H}^n, \sigma)$ satisfies property C at every point.*

*Proof.* Let $\mathcal{F}$ be the partial Fourier transform as in the proof of Lemma 7. By (3), we have the identity

$$\mathcal{F}\left(\mathcal{S}_n f\right)(\cdot, \xi) = e^{-\xi|\cdot|^2} \mathcal{B}_\xi \left( e^{\xi|\cdot|^2} \mathcal{F}f(\cdot, \xi) \right),$$

where $\mathcal{B}_\xi$ is the Bergman projection of the Fock space $(\mathbb{C}^n, e^{-2\xi|z|^2})$, that is, the orthogonal projection onto the closed subspace of entire holomorphic functions that are square integrable with respect to the Gaussian measure with density $e^{-\xi|z|^2}$. In particular, $\mathcal{B}_\xi = 0$ when $\xi \le 0$. The following estimate is well-known:
(4)

$$\int_{\mathbb{C}^n} |(1 - B_\xi)u|^2 e^{-2\xi|z|^2} dV(z) \le 2\xi^{-1} \int_{\mathbb{C}^n} \left( \sum_j |\partial_{\overline{z}_j} u|^2 \right) e^{-2\xi|z|^2} dV(z) \quad \forall u \in C_c^\infty(\mathbb{C}^n), \ \xi > 0.$$

In fact, $g = (1 - B_\xi)u$ is the solution of the equation $\overline{\partial}g = \overline{\partial}u$ of minimal Gaussian $L^2$ norm, and one may apply [Hör90, Lemma 4.4.1].

Let $f \in C_c^\infty(\mathbb{H}^n)$. Define the projection operator $P_{\le T}$ ($T \in \mathbb{R}$) by the identity

$$\mathcal{F}\left(P_{\le T} f\right)(z, \xi) = 1_{(-\infty, T]}(\xi) \mathcal{F}f(z, \xi),$$

and put $P_{>T} := I - P_{\le T}$.

Let $T > 0$. Applying (4) to $e^{\xi|\cdot|^2}\mathcal{F}f(\cdot, \xi)$ and integrating in $\xi \in (T, +\infty)$, we find

(5) $$\int_{\mathbb{H}^n} |P_{>T}f - \mathcal{S}_n P_{>T}f|^2 d\sigma \le \frac{2}{T} \int_{\mathbb{H}^n} \left( \sum_j |\overline{L}_j f|^2 \right) d\sigma.$$

Notice that we used the fact that $\overline{L}_j$ and $\mathcal{S}_n$ commute with $P_{>T}$. We treat the low-frequency component using the standard commutator formula

$$\int_{\mathbb{H}^n} |L_j f|^2 d\sigma = \int_{\mathbb{H}^n} |\overline{L}_j f|^2 d\sigma + \int_{\mathbb{H}^n} [L_j, \overline{L}_j]f \cdot \overline{f} d\sigma.$$

Since $[L_j, \overline{L}_j] = -2i\partial_t$, this immediately yields

(6) $$\int_{\mathbb{H}^n} |L_j P_{\le T} f|^2 d\sigma \le \int_{\mathbb{H}^n} |\overline{L}_j P_{\le T} f|^2 d\sigma + 2T \int_{\mathbb{H}^n} |P_{\le T} f|^2 d\sigma.$$

The identity $[L_j, \overline{L}_j] = -2i\partial_t$ also implies that $\{\Re(L)_j, \Im(L)_j\}_j$ is a system of vector fields satisfying Hörmander's bracket condition of order 2. Hence (see, e.g., [Koh73]), we have

(7) $$\|f\|_{W^{1/2}(B)}^2 \le C_B \left( \|f\|_{L^2}^2 + \sum_j \left( \|L_j f\|_{L^2}^2 + \|\overline{L}_j f\|_{L^2}^2 \right) \right) \qquad \forall f \in C^\infty(\mathbb{H}^n),$$

where $B \subset \mathbb{C}^n$ is a ball (or any, say, smooth open set) and $W^{1/2}$ denotes the fractional Sobolev norm of order $1/2$. Putting (6) and (7) together (and using again $[\overline{L}_j, P_{\le T}] = 0$), we obtain

(8) $$\|P_{\le T} f\|_{W^{1/2}}^2 \le C_B(1 + 2T)\|f\|_{L^2}^2 + 2C_B \sum_j \|\overline{L}_j f\|_{L^2}^2.$$

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We can now complete the proof. Given $\{f_k\}_k \subset C_c^\infty(B)$ with bounded CR energy, the sequence $\{P_{\leq T} f_k\}_k$, and a fortiori $\{(1 - \mathcal{S}_n) P_{\leq T} f_k\}_k$, has a subsequence of $L^2$-diameter $\leq \varepsilon$. This follows from (8) and the compactness of the Sobolev embedding $W^{\frac{1}{2}}(B) \hookrightarrow L^2(B)$. If $T \geq 2\varepsilon^{-2}$, by virtue of (5) the whole sequence $\{(1 - \mathcal{S}_n) P_{>T} f_k\}_k$ has $L^2$-diameter $\leq \varepsilon$ too. The conclusion of the proposition follows immediately. $\qquad\square$

## 4. Proof of Theorem 1

Write $\mathcal{S} = \mathcal{S}_{M,\nu}$ for simplicity. Fix a coordinate patch $\Omega$ around $x_0$ equipped with a system of Folland–Stein coordinates $(z,t)$, and $p \in [1,+\infty)$ (notice that $\mathcal{S}$ is self-adjoint, so the case $p = \infty$ of the theorem follows by duality from the $p = 1$ case).

We identify $\Omega$ with an open neighborhood of $(0,0)$ in $\mathbb{H}^n \equiv \mathbb{C}^n \times \mathbb{R}$ via these coordinates.

Fix $f \in C_c^\infty(\mathbb{H}^n)$. If $\lambda > 0$ is large enough, depending on the support of $f$, $f \circ \Phi_\lambda$ is supported on $\Omega$, and hence it may be thought of as a test function on $M$. Without loss of generality, we may assume that $\frac{d\nu}{d\sigma}(x_0) = 1$, where $\sigma$ is Lebesgue measure in the Folland–Stein coordinates.

Thus $\mathcal{S}(f \circ \Phi_\lambda)$ is a well-defined element of $L^2(M,\nu)$. Let $\rho \in C_c^\infty(\mathbb{H}^n, [0,1])$ be a cut-off function identically equal to 1 in a neighborhood of the origin, i.e., of $x_0$. We set $\rho_\mu := \rho \circ \Phi_\mu$ ($\mu > 0$). Notice that the support of $\rho_\mu$ shrinks to $x_0$ as $\mu$ tends to $\infty$, and that $\mathcal{S}(f \circ \Phi_\lambda)\rho_\mu$ is supported on $\Omega$ for every large $\mu$. Thus, it can be thought of as a function on $\mathbb{H}^n$. Put

$$g_\lambda := \left( \mathcal{S}(f \circ \Phi_\lambda)\rho_{\sqrt{\lambda}} \right) \circ \Phi_{\lambda^{-1}} \in L^2(\mathbb{H}^n, \sigma).$$

**Remark 9.** *It will be clear soon that $\sqrt{\lambda}$ could be replaced by any function $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\lim_{\lambda \to +\infty} F(\lambda) = +\infty \quad and \quad \lim_{\lambda \to +\infty} \frac{F(\lambda)}{\lambda} = 0.$$

*In other words, what turns out to be crucial is localizing the Szegő projection of $f \circ \Phi_\lambda$, which is supported at the parabolic infinitesimal scale $\lambda^{-1}$, at an infinitesimal scale much larger than $\lambda^{-1}$.*

It is clear that for every $\lambda$ large

$$
\begin{aligned}
\int_{\mathbb{H}^n} |g_\lambda|^p d\sigma &= \lambda^{2n+2} \int_\Omega |\mathcal{S}(f \circ \Phi_\lambda)\rho_{\sqrt{\lambda}}|^p d\sigma \\
&= (1 + o(1))\lambda^{2n+2} \int_\Omega |\mathcal{S}(f \circ \Phi_\lambda)\rho_{\sqrt{\lambda}}|^p d\nu \\
&\leq (1 + o(1))\lambda^{2n+2} N_p(M,\nu)^p \int_\Omega |f \circ \Phi_\lambda|^p d\nu \\
&= (1 + o(1))\lambda^{2n+2} N_p(M,\nu)^p \int_\Omega |f \circ \Phi_\lambda|^p d\sigma \\
&= (1 + o(1)) N_p(M,\nu)^p \int_{\mathbb{H}^n} |f|^p d\sigma,
\end{aligned}
$$

where we used a couple of times the fact that $\frac{d\nu}{d\sigma}(x_0) = 1$.

By Banach–Alaoglu, along a diverging subsequence of $\lambda$'s, $g_\lambda$ has a weak limit both in $L^2(\mathbb{H}^n, \sigma)$ and $L^p(\mathbb{H}^n, \sigma)$, for any fixed $p > 1$. Denoting this limit by $g \in L^2 \cap L^p(\mathbb{H}^n, \sigma)$, we clearly have

$$\tag{9} \int_{\mathbb{H}^n} |g|^p d\sigma \leq N_p(M,\nu)^p \int_{\mathbb{C}^n \times \mathbb{R}} |f|^p d\sigma.$$

From now on, limits in $\lambda$ are always along appropriate diverging subsequences.

*We claim that the limit $g$ is* CR *with respect to the model structure*, i.e., that $g \in \mathrm{CR}^2(\mathbb{H}^n, \sigma)$. To see this, we start by noticing that by (2),

$$\tag{10} \overline{L}_j \left( \mathcal{S}(f \circ \Phi_\lambda)\rho_{\sqrt{\lambda}} \right) \circ \Phi_{\lambda^{-1}} = \lambda(\partial_{z_j} + i\overline{z}_j \partial_t) g_\lambda + \lambda E_j^\lambda g_\lambda,$$

in the sense of distributions. If $E_j = \sum_{k=1}^n \{a_{j,k}\partial_{z_k} + b_{j,k}\partial_{\overline{z}_k}\} + c_j \partial_t$, the rescaled error terms above are given by

$$E_j^\lambda = \sum_{k=1}^n \left\{ a_{j,k} \circ \Phi_\lambda^{-1} \cdot \partial_{z_k} + b_{j,k} \circ \Phi_\lambda^{-1} \cdot \partial_{\overline{z}_k} \right\} + \lambda c_j \circ \Phi_\lambda^{-1} \cdot \partial_t.$$

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Notice that the formal adjoint of $E_j^\lambda$ (with respect to Lebesgue measure $d\sigma$) is $-E_j^\lambda - \lambda^{-1} e_j \circ \Phi_\lambda^{-1}$, where $e_j^\lambda(z, t)$ is a smooth function.

Since $\rho$ is identically 1 on a neighborhood $V$ of the origin, the left-hand side of (10) vanishes on $\Phi_{\sqrt{\lambda}}(V)$. Since $\bigcup_\lambda \Phi_{\sqrt{\lambda}}(V) = \mathbb{C}^n \times \mathbb{R}$, for any $\varphi \in C_c^\infty(\mathbb{H}^n)$ and $\lambda$ large, we have

$$\int_{\mathbb{C}^n \times \mathbb{R}} (\partial_{z_j} + i \overline{z}_j \partial_t) g_\lambda \cdot \varphi \, d\sigma = -\int_{\mathbb{C}^n \times \mathbb{R}} E_j^\lambda g_\lambda \cdot \varphi \, d\sigma$$

$$= \int_{\mathbb{C}^n \times \mathbb{R}} g_\lambda \cdot (E_j^\lambda + \lambda^{-1} e_j \circ \Phi_\lambda^{-1}) \varphi \, d\sigma.$$

By Lemma 6, $a_{j,k}$ and $b_{j,k}$ have parabolic weight $\geq 1$ and $c$ has weight $\geq 2$, and therefore $\|(E_j^\lambda + \lambda^{-1} e_j \circ \Phi_\lambda^{-1}) \varphi\|_\infty = O(\lambda^{-1})$. Since $g_\lambda$ are uniformly in $L^2(\mathbb{C}^n \times \mathbb{R})$, we conclude that $(\partial_{z_j} + i \overline{z}_j \partial_t) g_\lambda$ tends to zero in the sense of distributions. Thus, $(\partial_{z_j} + i \overline{z}_j \partial_t) g = 0$ and the claim is proved.

We are left with the proof that $g = \mathcal{S}_n(f)$. In fact, if we prove this then (9) and the arbitrariness of $f$ show that $N_p(\mathbb{H}^n, \sigma) \leq N_p(M, \nu)$, as we wanted.

Since we already know that $g$ is CR, to prove that $g = \mathcal{S}_n(f)$ what we need to show is that $f - g$ is orthogonal to $\mathrm{CR}^2(\mathbb{H}^n, \sigma)$. This is clearly equivalent to

$$\lim_{\lambda \to +\infty} \int_{\mathbb{H}^n} g_\lambda \overline{h} d\sigma = \int_{\mathbb{H}^n} f \overline{h} d\sigma \qquad \forall h \in \mathrm{CR}^2(\mathbb{H}^n, \sigma).$$

In fact, it is enough to verify this identity for $h$ in the dense subspace $\mathcal{D}$ of Lemma 7. By the self-adjointness of $\mathcal{S}$ in $L^2(M, \nu)$, we have

$$\int_{\mathbb{H}^n} g_\lambda \overline{h} d\sigma = \int_{\mathbb{H}^n} \left( \mathcal{S}(f \circ \Phi_\lambda) \rho_{\sqrt{\lambda}} \right) \circ \Phi_{\lambda^{-1}} \cdot \overline{h} d\sigma$$

$$= \lambda^{2n+2} \int_M \mathcal{S}(f \circ \Phi_\lambda) \rho_{\sqrt{\lambda}} \cdot \overline{h \circ \Phi_\lambda} \frac{d\sigma}{d\nu} d\nu$$

$$= \lambda^{2n+2} \int_\Omega f \circ \Phi_\lambda \cdot \overline{\mathcal{S}\left( \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \cdot h \circ \Phi_\lambda \right)} \frac{d\nu}{d\sigma} d\sigma$$

$$= \int_{\mathbb{H}^n} f \cdot \overline{\left\{ \mathcal{S}\left( \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \cdot h \circ \Phi_\lambda \right) \frac{d\nu}{d\sigma} \right\} \circ \Phi_\lambda^{-1}} d\sigma.$$

Notice that the various passages from $d\sigma$ to $d\nu$ are meaningful, because for $\lambda$ large, $\rho_\lambda$ and $f \circ \Phi_\lambda$ are supported in $\Omega$.

Thus, our task is reduced to proving that

$$\lim_{\lambda \to +\infty} \left\{ \mathcal{S}\left( \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \cdot h \circ \Phi_\lambda \right) \frac{d\nu}{d\sigma} \right\} \circ \Phi_\lambda^{-1} = h \qquad \forall h \in \mathcal{D}$$

in the sense of distributions.

Notice that if we remove $\mathcal{S}$ from the expression in the limit, we get $h \rho_{\lambda^{-\frac{1}{2}}}$, which clearly $L^2$-converges to $h$. Thus, setting

$$u_\lambda := (I - \mathcal{S}) \left( \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \cdot h \circ \Phi_\lambda \right),$$

it is enough to see that $\lim_{\lambda \to +\infty} \left( u_\lambda \frac{d\nu}{d\sigma} \right) \circ \Phi_\lambda^{-1} = 0$ in the sense of distributions, i.e.,

$$\int_{\mathbb{H}^n} u_\lambda \frac{d\nu}{d\sigma} \lambda^{2n+2} \varphi \circ \Phi_\lambda d\sigma = 0 \qquad \forall \varphi \in C_c^\infty(\mathbb{H}^n).$$

Let $B$ be the compact neighborhood of $x_0$ appearing in the definition of Property C. We claim that:

(11) $\qquad 1_B \lambda^{n+1} u_\lambda$ has a strong $L^2$ limit $v$, along an appropriate subsequence.

Notice that $\nu$ and $\sigma$ are comparable on $B$ and we do not need to specify in which $L^2$ norm the convergence happens.

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Let us show that the claim allows us to conclude the proof of the theorem. We write

$$\int_{\mathbb{H}^n} u_\lambda \frac{d\nu}{d\sigma} \lambda^{2n+2} \varphi \circ \Phi_\lambda \, d\sigma = \int_{\mathbb{H}^n} (\lambda^{n+1} u_\lambda - v) \frac{d\nu}{d\sigma} \lambda^{n+1} \varphi \circ \Phi_\lambda \, d\sigma$$
$$+ \int_{\mathbb{H}^n} (v - v_0) \frac{d\nu}{d\sigma} \lambda^{n+1} \varphi \circ \Phi_\lambda \, d\sigma + \lambda^{-n-1} \int_{\mathbb{H}^n} v_0 \frac{d\nu}{d\sigma} \lambda^{2n+2} \varphi \circ \Phi_\lambda \, d\sigma,$$

where $v_0 \in C_c(\mathbb{H}^n)$ is such that $\|v - v_0\|_{L^2} \le \varepsilon$, with $\varepsilon$ small. Notice that $\varphi \circ \Phi_\lambda$ is supported on $B$ for $\lambda$ large. Thus, the first term vanishes in the limit, simply because $\lambda^{n+1} \varphi \circ \Phi_\lambda$ is uniformly bounded in $L^2$ norm, and the second is $\mathcal{O}(\varepsilon)$ for the same reason. Since the last term equals $\lambda^{-n-1} v_0(0,0) \int \varphi d\sigma + o(\lambda^{-n-1})$, the conclusion follows by the arbitrariness of $\varepsilon$.

We now prove claim (11). In virtue of property C, it is enough to prove that the family of test functions $\{\lambda^{n+1} \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \cdot h \circ \Phi_\lambda\}_\lambda$ is bounded with respect to the "CR energy" (1). The $L^2$ norm is clearly bounded uniformly in $\lambda$. We compute, for $L_j$ as in Lemma 6,

$$\overline{L}_j \left( \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \cdot h \circ \Phi_\lambda \right) = \left[ \overline{L}_j \left( \rho \circ \Phi_{\sqrt{\lambda}} \right) \frac{d\sigma}{d\nu} + \rho_{\sqrt{\lambda}} \overline{L}_j \left( \frac{d\sigma}{d\nu} \right) \right] h \circ \Phi_\lambda + \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \overline{L}_j \left( h \circ \Phi_\lambda \right)$$

$$= \left[ \left( \sqrt{\lambda} \left( (\partial_{z_j} + i \overline{z}_j \partial_t) \rho + E_j^{\sqrt{\lambda}} \rho \right) \circ \Phi_{\sqrt{\lambda}} \right) \frac{d\sigma}{d\nu} + \rho_{\sqrt{\lambda}} \overline{L}_j \left( \frac{d\sigma}{d\nu} \right) \right] h \circ \Phi_\lambda$$

$$+ \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \lambda \left\{ (\partial_{z_j} + i \overline{z}_j \partial_t) h + E_j^\lambda h \right\} \circ \Phi_\lambda$$

$$= \left[ \left( \sqrt{\lambda} \left( (\partial_{z_j} + i \overline{z}_j \partial_t) \rho + E_j^{\sqrt{\lambda}} \rho \right) \circ \Phi_{\sqrt{\lambda}} \right) \frac{d\sigma}{d\nu} + \rho_{\sqrt{\lambda}} \overline{L}_j \left( \frac{d\sigma}{d\nu} \right) \right] h \circ \Phi_\lambda$$

$$+ \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \lambda \left( E_j^\lambda h \right) \circ \Phi_\lambda,$$

where we used the fact that $h$ is CR with respect to the Heisenberg structure. The easiest term to deal with is

$$\int_M \left| h \circ \Phi_\lambda \rho_{\sqrt{\lambda}} \overline{L}_j \left( \frac{d\sigma}{d\nu} \right) \right|^2 d\nu = (1 + o(1)) \left| \overline{L} \left( \frac{d\sigma}{d\nu} \right) (0) \right|^2 \lambda^{-2n-2} \int_{\mathbb{H}^n} |h|^2 d\sigma.$$

Next,

$$\int_M \left| \sqrt{\lambda} \left( (\partial_{z_j} + i \overline{z}_j \partial_t) \rho + E_j^{\sqrt{\lambda}} \rho \right) \circ \Phi_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \right|^2 |h \circ \Phi_\lambda|^2 d\nu$$

$$= (1 + o(1)) \lambda^{-n-1} \lambda \int_{\mathbb{H}^n} \left| \left( (\partial_{z_j} + i \overline{z}_j \partial_t) \rho + E_j^{\sqrt{\lambda}} \rho \right) \right|^2 |h \circ \Phi_{\sqrt{\lambda}}|^2 d\sigma$$

$$\le C (1 + o(1)) \lambda^{-n} \int_{\mathbb{H}^n \setminus V} |h \circ \Phi_{\sqrt{\lambda}}|^2 d\sigma,$$

where we used the fact that $\rho$ is identically equal to 1 in the neighborhood $V$ of the origin and that the coefficients of $E_j^{\sqrt{\lambda}}$ are bounded uniformly in $\lambda$. Here we take advantage of the assumption that $h$ is in $\mathcal{D}$. By Lemma 7, the quantity above may be estimated by $|h \circ \Phi_{\sqrt{\lambda}}| \le C_N \lambda^{-N} (|z|^2 + |t|)^{-N}$ for any $N \in \mathbb{N}$. Choosing $N$ large enough, we obtain the desired estimate.
Finally,

$$\int_M \left| \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \lambda \left( E_j^\lambda h \right) \circ \Phi_\lambda \right|^2 d\nu = (1 + o(1)) \lambda^{-2n-2} \int_{\mathbb{H}^n} |\lambda \left( E_j^\lambda h \right) \rho \circ \Phi_{\lambda^{-\frac{1}{2}}}|^2 d\sigma.$$

The vector field $\lambda E_j^\lambda$ has coefficients bounded uniformly in $\lambda$. Therefore, to bound the last term we need $\int_{\mathbb{H}^n} \sum_{j=1}^n (|\partial_{z_j} h|^2 + |\partial_{\overline{z}_j} h|^2) + |\partial_t h|^2 < +\infty$, which certainly holds for $h \in \mathcal{D}$. This completes the proof of the boundedness of the CR energy of $\{\lambda^{n+1} \rho_{\sqrt{\lambda}} \frac{d\sigma}{d\nu} \cdot h \circ \Phi_\lambda\}_\lambda$, and hence of the theorem.

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