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GENERALIZED q-FOCK SPACES AND STRUCTURAL IDENTITIES

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Abstract. Using q-calculus we study a family of reproducing kernel Hilbert spaces which interpolate between the Hardy space and the Fock space. We give characterizations of these spaces in terms of classical operators such as integration and backward-shift operators, and their q-calculus counterparts. Furthermore, these new spaces allow us to study intertwining operators between classic backward-shift operators and the q-Jackson derivative.

1. Introduction

1.1. **Prologue.** The Hardy space of the open unit disk $\mathbb{D}$, here denoted by $\mathbf{H}_2 = \mathbf{H}_2(\mathbb{D})$, is the reproducing kernel Hilbert space with reproducing kernel

$$\frac{1}{1 - z\overline{w}} = \sum_{n=0}^{\infty} z^n \overline{w}^n, \quad z, w \in \mathbb{D},$$

and plays a key role in operator theory, linear system theory and Schur analysis. On the other hand, the Bargmann-Fock-Segal space, here denoted by $\mathcal{F}$ and called Fock space for short, is the reproducing kernel Hilbert space with reproducing kernel

$$e^{z\overline{w}} = \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{n!}, \quad z, w \in \mathbb{C},$$

and plays a key role in quantum mechanics (and more recently in signal processing).

The Hardy space $\mathbf{H}_2$ can be characterized (up to a positive multiplicative factor for the inner product) as the only Hilbert space of power series converging at the origin and such that

$$R_0^* = M_z, \tag{1.1}$$

where M_z is the operator of multiplication by z and

$$R_0 f(z) = \frac{f(z) - f(0)}{z}. \tag{1.2}$$

Note that in $\mathbf{H}_2$ we have the identities

$$R_0 R_0^* = \mathcal{I}, \quad \text{and} \quad R_0 M_z - M_z R_0 = \mathcal{I} - R_0^* R_0 = C^* C, \tag{1.3}$$

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where $Cf = f(0)$ and $\mathcal{I}$ is the identity operator. We remark that

$$\mathcal{I} - R_0^* R_0 = C^* C, \tag{1.4}$$

which we will call *structural identity*, is the simplest of a family of identities characterizing de Branges spaces.

Similarly, and besides Bargmann celebrated characterization $\partial^* = M_z$ (see [10, 11]), the Fock space is (still up to a positive multiplicative factor for the inner product) the only Hilbert space of power series converging at the origin and such that

$$R_0^* = \mathrm{I}, \tag{1.5}$$

where I is the integration operator (see [3])

$$(\mathrm{I}f)(z) = \int_{[0,z]} f(s)ds. \tag{1.6}$$

1.2. **The paper.** The q-calculus allows to define a continuum of spaces between $\mathbf{H}_2$ and $\mathcal{F}$, namely the family of reproducing kernel Hilbert spaces $\mathbf{H}_{2,q}$ indexed by $q \in [0,1]$ and with reproducing kernel

$$K_q(z,w) = \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{[n]_q!}, \quad q \in [0,1], \quad z, w \in \mathbb{D}_{1/1-q},$$

where in the above expression

$$\mathbb{D}_{1/1-q} = \begin{cases} \mathbb{D}_\infty = \mathbb{C}, & q = 1, \\ \left\{ z \in \mathbb{C} : |z| < \frac{1}{1-q} \right\}, & q \in [0,1) \end{cases}.$$

Furthermore, $[0]_q! = 1$ and $[n]_q! = 1 \cdot (1+q) \cdot (1+q+q^2) \cdots (1+q+\cdots+q^{n-1})$, $\quad n \in \mathbb{N}$. Thus, in this notation, we have

$$\mathbf{H}_{2,0} = \mathbf{H}_2 \quad \text{and} \quad \mathbf{H}_{2,1} = \mathcal{F},$$

with

$$K_0(z,w) := k_{2,0}(z,w) = \frac{1}{1 - z\overline{w}} \quad \text{and} \quad K_1(z,w) := k_{2,1}(z,w) = e^{z\overline{w}}.$$

The q-calculus allows to gather into a common umbrella problems pertaining to the classical Hardy space $\mathbf{H}_2$ of the open unit disk and problems pertaining to the Fock space. Consider now

$$R_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad 0 \le q < 1, \tag{1.7}$$

while for $q = 1$, we consider $R_1 = \partial$. In this way we have a progression between two fundamental linear operators in analysis, namely the backward-shift and the differentiation operators. Then, one can introduce the q-Fock space $\mathbf{H}_{2,q}$ as the unique (up to a multiplicative positive constant) space of power series such that $R_q^* = M_z$. The case $q = 1$ corresponds to the classical Fock space (see [10]). It is important to note already at this stage that these operators satisfy a q-commutator relation (see also Lemma 2.2)

$$R_q M_z - q M_z R_q = \mathcal{I}. \tag{1.8}$$

2. q-CALCULUS

2.1. **Iterative powers of the operator R_q.** Recall that R_q was defined by (1.7).

Proposition 2.1. *Let $\Lambda_q f(z) = f(qz)$. We have*

$$R_q^n f(z) = \frac{\prod_{k=1}^n (1 - q^k \Lambda_q)}{(1-q)^n} R_0^n f(z), \qquad 0 \le q < 1, \quad n = 1, 2, \ldots \tag{2.9}$$

Proof. Firstly, we observe the intertwining between R_0 and Λ_q,

$$R_0 \Lambda_q f(z) = R_0 f(qz) = \frac{f(qz) - f(0)}{z} = q \frac{f(qz) - f(0)}{qz} = q \Lambda_q R_0 f(z).$$

Secondly,

$$\begin{aligned} R_q f(z) &= \frac{f(z) - f(qz)}{(1-q)z} = \frac{f(z) - f(0) - f(qz) + f(0)}{(1-q)z} \\ &= \frac{1}{1-q} \frac{f(z) - f(0)}{z} - \frac{q}{1-q} \frac{f(qz) - f(0)}{qz} \\ &= \frac{(1 - q\Lambda_q) R_0}{1-q} f(z). \end{aligned}$$

Hence,

$$\begin{aligned} R_q^2 f(z) &= \left(\frac{(1 - q\Lambda_q) R_0}{1-q} \right)^2 f(z) \\ &= \frac{(1 - q\Lambda_q) R_0 (1 - q\Lambda_q) R_0}{(1-q)^2} f(z) \\ &= \frac{(1 - q\Lambda_q)(1 - q^2 \Lambda_q) R_0^2}{(1-q)^2} f(z), \end{aligned}$$

and by induction the result holds:

$$\begin{aligned} R_q^n f(z) &= \left(\frac{(1 - q\Lambda_q) R_0}{1-q} \right)^n f(z) \\ &= \frac{(1 - q\Lambda_q) R_0 (1 - q\Lambda_q) R_0 \cdots (1 - q\Lambda_q) R_0}{(1-q)^n} f(z) \\ &= \frac{(1 - q\Lambda_q)(1 - q^2\Lambda_q) R_0^2 \cdots (1 - q\Lambda_q) R_0}{(1-q)^n} f(z) \\ &\;\vdots \\ &= \frac{(1 - q\Lambda_q)(1 - q^2\Lambda_q) \cdots (1 - q^n \Lambda_q) R_0^n}{(1-q)^n} f(z). \end{aligned}$$

□

As we will see later (see Theorems 4.4 and 4.5), R_q^* has completely different properties depending on which of the spaces at hand we compute the adjoint.

2.2. q−**Stirling numbers associated to higher commutation relations.** In this subsection let us recall some facts regarding higher-commutator relations in q-calculus. While they can be found, e.g., in [18] for the sake of self-sufficiency of the paper we present them with proofs.

For the q-commutator we have the following well-known formula.

Lemma 2.2 (q-commutator)**.** *For the q-commutator it holds the following identity:*

$$[R_q, M_z]_q := R_q M_z - q M_z R_q = \mathcal{I}. \tag{2.10}$$

Proof. We have

$$R_q M_z z^n = R_q z^{n+1} = (1 + q + \cdots + q^n) z^n, \quad n = 0, 1, 2, \ldots$$

while

$$q M_z R_q z^n = q M_z R_q 1 = 0, \quad n = 0,$$
$$q M_z R_q z^n = q M_z (1 + q + \cdots + q^{n-1}) z^{n-1} = (q + q + \cdots + q^n) z^n, \quad n = 1, 2, \ldots$$

so that it holds $(R_q M_z - q M_z R_q) z^n = z^n$, for all $n \in \mathbb{N}_0$. □

We define our q−Stirling numbers as coefficients $S(n,k)$ of the following commutation relation (see [7]):

$$(M_z R_q)^n := \sum_{k=1}^{n} S(n,k) M_z^k R_q^k, \qquad n \in \mathbb{N}. \tag{2.11}$$

This formula can also be found in [18] (Theorem 3.1) and indirectly also in [19]. Furthermore, in [18], Section 4.1. there is a general exposition on how to construct such higher order commutator relations including formulae for terms of the type $(M_z^r R_q^s)^n$ with r, s multi-indices.

Lemma 2.3. *We have for these q−Stirling numbers the following recursion formula*

$$S(1,1) = 1;$$
$$S(n,n) = S(n-1, n-1) q^{n-1}, \qquad n = 2, 3, \ldots;$$
$$S(n,k) = (1 + q + \cdots + q^{k-1}) S(n-1, k) + q^{k-1} S(n-1, k-1), \qquad k = 2, \ldots, n-1.$$

This recursion formula is known in the literature. One can find it in [19] formula (1.15) on page 93) or in the book [18] (Section 3.3, page 68 onwards).

Proof. In order to simplify notation, we write the expression for the q−Stirling numbers as

$$(ab)^n := \sum_{k=1}^{n} S(n,k) a^k b^k.$$

From the q−commutator we get $ba = 1 + qab$ so that

$$\begin{aligned} b^n a &= b^{n-1}(ba) = b^{n-1}(1 + qab) = b^{n-1} + q(b^{n-1}a)b \\ &= b^{n-1} + q\left[b^{n-2} + q(b^{n-2}a)b\right]b = (1+q)b^{n-1} + q^2(b^{n-2}a)b^2 \\ &\vdots \\ &= (1 + q + \cdots + q^{n-1}) b^{n-1} + q^n a b^n. \end{aligned}$$

Replacing in the above formula for the q–Stirling numbers we obtain

$$
\begin{aligned}
(ab)^n &= \sum_{k=1}^{n} S(n,k)a^k b^k \\
&= (ab)^{n-1}(ab) = \Big[\sum_{k=1}^{n-1} S(n-1,k)a^k b^k\Big](ab) \\
&= \sum_{k=1}^{n-1} S(n-1,k)a^k(b^k a)b = \sum_{k=1}^{n-1} a^k S(n-1,k)\left[(1+q+\cdots+q^{k-1})b^{k-1} + q^k a b^k\right] b \\
&= \sum_{k=1}^{n-1}\left[(1+q+\cdots+q^{k-1})S(n-1,k)a^k b^k + q^k S(n-1,k)a^{k+1}b^{k+1}\right] \\
&= \sum_{k=1}^{n-1}(1+q+\cdots+q^{k-1})S(n-1,k)a^k b^k + \sum_{k=2}^{n} q^{k-1}S(n-1,k-1)a^k b^k \\
&= S(n-1,1)ab + \sum_{k=2}^{n-1}\left[(1+q+\cdots+q^{k-1})S(n-1,k) + q^{k-1}S(n-1,k-1)\right]a^k b^k + q^{n-1}S(n-1,n-1)a^n b^n,
\end{aligned}
$$

so that we have $S(1,1) = 1$,

$$S(n,1) = S(n-1,1), \quad S(n,n) = q^{n-1}S(n-1,n-1),$$

for $n = 2, 3, \ldots$ and

$$S(n,k) = (1+q+\cdots+q^{k-1})S(n-1,k) + q^{k-1}S(n-1,k-1),$$

for $k = 2, \ldots, n-1$. □

One can easily see the first q–Stirling numbers

$S(n,k)$	1	2	3	4
1	1			
2	1	q		
3	1	$2q+q^2$	q^3	
4	1	q^3+3q^2+3q	$q^5+2q^4+3q^3$	q^6

Remark 2.4. We need to point out that there are two types of q-Stirling numbers of the first or of the second kind in the literature. The more classic ones were obtained by studying the corresponding partition problems in q-calculus (for a review on this topic see [12, 13]). Here we have them as coefficients of the expansion of $(M_z R_q)^n$ in (2.11) in the same way as in [19] and [18]. Only in the classic case of $q = 1$ this type of coefficients coincides with classic Stirling numbers of the second kind, i.e. with the numbers of partitions of a set of n objects into k non-empty subsets.

3. The q-Fock space

Consider the positive definite function $E_q(z\overline{w})$ given by the q-exponential:

$$E_q(z) = \sum_{k=0}^{\infty}\frac{z^k}{[k]_q!} = \frac{1}{\prod_{j=0}^{\infty}(1-z(1-q)q^j)} =: \frac{1}{(z(1-q);q)_\infty}, \quad z \in \mathbb{D}_{1/1-q}, \tag{3.1}$$

evaluated at $z\overline{w}$, with $[0]_q = 1$ and $[k]_q = 1+q+\cdots+q^{k-1}$ for $k = 1, 2, \ldots,$ and $[k]_q! = \prod_{j=0}^{k}[j]_q$, i.e.

$$[k]_q! = [1]_q[2]_q\cdots[k]_q = 1\cdot(1+q)\cdot(1+q+q^2)\cdots(1+q+\cdots+q^{k-1}).$$

The term $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ denotes the q-Pochhammer symbol.

Definition 3.1. *We denote by* $\mathbf{H}_{2,q}$ *the reproducing kernel Hilbert space of functions analytic in* $|z| < \frac{1}{1-q}$ *with reproducing kernel* $E_q(z\overline{w})$.

As stated before when $q = 0$ we get back the classical Hardy space of the open unit disk, while $q \to 1$ leads to the classical Fock space; see e.g. [14, 15, 20] for the former, [22] for the latter.
For functions belonging to the q-Fock space we have the following characterization based on its power series expansion.

Lemma 3.2. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *belongs to* $\mathbf{H}_{2,q}$ *if and only if*

$$\sum_{n=0}^{\infty} [n]_q! \, |a_n|^2 < \infty. \tag{3.2}$$

Based on the q-Jackson integral (see [16], [17])

$$\int_0^a f(x) d_q x := (1-q)a \sum_{k=0}^{\infty} q^k f(q^k a),$$

we can define the following q-integral transform.

Definition 3.3. *Given a bounded function* $f : [0, -1+1/(1-q)] \to \mathbb{R}$ *we define its q-integral transform as*

$$\mathcal{M}_q f(z) = \int_0^{1/(1-q)} t^{z-1} f(qt) d_q t := \sum_{k=0}^{\infty} q^k \Big(\frac{q^k}{1-q}\Big)^{z-1} f\Big(\frac{q^{k+1}}{1-q}\Big).$$

With the help of this q-integral transform we get that the coefficients $\frac{1}{[n]_q!}$ satisfy the moment problem

$$\begin{aligned} [n]_q! = \mathcal{M}_q(E_q^{-1})(n+1) &= \int_0^{1/(1-q)} t^n E_q^{-1}(qt) d_q t \\ &= \frac{(q;q)_\infty}{(1-q)^n} \sum_{k=0}^{\infty} \frac{q^{(n+1)k}}{(q;q)_k}, \quad 0 < q < 1, \; n \in \mathbb{N}_0, \end{aligned}$$

where $(a;q)_n = \Pi_{j=0}^{n-1}(1-aq^j)$ denotes the q-Pochhammer symbol and $(q;q)_n = \frac{(q;q)_\infty}{(q^{n+1};q)_\infty}$.
For the disk $\mathbb{D}_{1/(1-q)}$ we have the measure (see [21])

$$d\mu_q(z) = (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} d\lambda_{r_k}(z),$$

where $r_k = \frac{q^{k/2}}{\sqrt{1-q}}$ while $d\lambda_{r_k}$ is the normalized Lebesgue measure in the circle of radius r_k.
This leads to the following characterization of the space $\mathbf{H}_{2,q}$.

Theorem 3.4. *The space* $\mathbf{H}_{2,q}$ *corresponds to the space of all analytic functions in the disk* $\mathbb{D}_{\frac{1}{1-q}} = \{z : |z| < \frac{1}{1-q}\}$ *satisfying the condition*

$$\iint_{\mathbb{D}_{\frac{1}{1-q}}} |f(z)|^2 d\mu_q(z) < \infty.$$

The inner product of $\mathbf{H}_{2,q}$ *is given by*

$$\frac{1}{2\pi} \iint_{\mathbb{D}_{\frac{1}{1-q}}} f(z)\overline{g(z)} d\mu_q(z) = \sum_{n=0}^{\infty} f_n \overline{g_n} \, [n]_q! \,.$$

Proof. We have

$$\begin{aligned}
\frac{1}{2\pi}\iint_{\mathbb{D}_{\frac{1}{1-q}}} z^n\overline{z}^m d\mu_q(z) &= \frac{(q;q)_\infty}{2\pi}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k} r_k^{n+m}\underbrace{\int_0^{2\pi} e^{i(n-m)\theta}d\theta}_{=2\pi\delta_{n,m}} \\
&= \delta_{n,m}(q;q)_\infty\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}r_k^{2n} \\
&= \delta_{n,m}\frac{(q;q)_\infty}{(1-q)^n}\sum_{k=0}^{\infty}\frac{q^{(n+1)k}}{(q;q)_k}.
\end{aligned}$$

□

Combining this result with our moment problem we obtain

$$[n]_q! = \mathcal{M}_q(E_q^{-1})(n+1) = \int_0^{1/(1-q)} t^n E_q^{-1}(qt)d_qt = \frac{\delta_{n,m}}{2\pi}\iint_{\mathbb{D}_{\frac{1}{1-q}}} z^n\overline{z}^m d\mu_q(z).$$

We observe that for $q\to 1$ we obtain $d\mu_q(z) = \frac{1}{2}e^{-|z|^2}dxdy$.

Also we get a convolution-type formula for our q-integral transform.

Lemma 3.5. *Given bounded functions $f_1, f_2 : [0, -1+1/(1-q)] \to \mathbb{R}$ it holds (pointwisely)*

$$\mathcal{M}_q(f_1)(z)\mathcal{M}_q(f_2)(z) = \left(\frac{1}{1-q}\right)^{z-1}\mathcal{M}_q(f_1\circ f_2)(z), \tag{3.3}$$

where

$$f_1\circ f_2\left(q\frac{q^m}{1-q}\right) := \sum_{k=0}^{m} f_1\left(\frac{q^{k+1}}{1-q}\right)f_2\left(\frac{q^{m+1-k}}{1-q}\right). \tag{3.4}$$

Proof. From Definition 3.3 we have

$$\begin{aligned}
\mathcal{M}_q(f_1)(z)\mathcal{M}_q(f_2)(z) &= \left(\sum_{k=0}^{\infty} q^k\Big(\frac{q^k}{1-q}\Big)^{z-1} f_1\Big(\frac{q^{k+1}}{1-q}\Big)\right)\left(\sum_{n=0}^{\infty} q^n\Big(\frac{q^n}{1-q}\Big)^{z-1} f_2\Big(\frac{q^{n+1}}{1-q}\Big)\right) \\
&= \sum_{k,n=0}^{\infty} q^{k+n}\Big(\frac{q^{k+n}}{(1-q)^2}\Big)^{z-1} f_1\Big(\frac{q^{k+1}}{1-q}\Big)f_2\Big(\frac{q^{n+1}}{1-q}\Big) \\
&= \Big(\frac{1}{1-q}\Big)^{z-1}\sum_{m=0}^{\infty} q^m\Big(\frac{q^m}{1-q}\Big)^{z-1}\underbrace{\left(\sum_{k=0}^{m} f_1\Big(\frac{q^{k+1}}{1-q}\Big)f_2\Big(\frac{q^{m+1-k}}{1-q}\Big)\right)}_{:=f_1\circ f_2\left(q\frac{q^m}{1-q}\right)}.
\end{aligned}$$

□

For the multiplication operator M_z we have the following fact.

Proposition 3.6. *M_z is bounded from $\mathbf{H}_{2,q}$ into itself with norm $\|M_z\| \le \frac{1}{1-q}$.*

Proof. This follows from

$$\frac{\frac{1}{1-q} - z\overline{w}}{\prod_{j=0}^{\infty}(1-z\overline{w}(1-q)q^j)} = \frac{1}{1-q}\frac{1}{\prod_{j=1}^{\infty}(1-z\overline{w}(1-q)q^j)}.$$

Since the kernel $\frac{1}{1-q}\frac{1}{\prod_{j=1}^{\infty}(1-z\overline{w}(1-q)q^j)}$ is positive definite in $\mathbb{D}_{1/1-q}$ so is the kernel

$$\frac{\frac{1}{1-q}-z\overline{w}}{\prod_{j=0}^{\infty}(1-z\overline{w}(1-q)q^j)},$$

and we conclude with the characterization of multipliers in a reproducing kernel Hilbert space. □

Lemma 3.7. *(see e.g.* [2, Exercise 4.2.25, pp. 165 and 185]*)*

$$(R_qf)(z)=\lambda f(z) \quad\Longleftrightarrow\quad f(z)=\frac{c}{\prod_{j=0}^{\infty}(1-\lambda(1-q)zq^j)}. \tag{3.5}$$

Proposition 3.8. *The q-exponential satisfy*

$$(R_qE_q(\cdot\overline{w}))(z)=\overline{w}E_q(z\overline{w}). \tag{3.6}$$

Proof. We note that $E_q(qz\overline{w})=(1-z\overline{w}(1-q))E_q(z\overline{w})$ and so

$$\begin{aligned}(R_qE_q(\cdot\overline{w}))(z)&=\frac{E_q(z\overline{w})-E_q(qz\overline{w})}{(1-q)z}\\&=\frac{E_q(z\overline{w})-(1-z\overline{w}(1-q))E_q(z\overline{w})}{(1-q)z}\\&=\overline{w}E_q(z\overline{w}).\end{aligned}$$

□

Theorem 3.9. *Let $q\in[0,1)$. The only Hilbert space of functions which is analytic in a neighborhood of the origin and for which*

$$R_q^*=M_z \tag{3.7}$$

is $\mathbf{H}_{2,q}$ (up to a multiplicative factor for the inner product).

Proof. We have that $E_q(z\overline{w})=K_q(z,w)$ is the reproducing for $\mathbf{H}_{2,q}$, i.e. $f(z)=\langle f,E_q(\cdot\overline{z})\rangle_{\mathbf{H}_{2,q}}$. Using (3.6) we can write:

$$\begin{aligned}\left(R_q^*E_q(\cdot\overline{w})\right)(z)&=\langle R_q^*E_q(\cdot\overline{w}),E_q(\cdot\overline{z})\rangle_{\mathbf{H}_{2,q}}\\&=\langle E_q(\cdot\overline{w}),R_qE_q(\cdot\overline{z})\rangle_{\mathbf{H}_{2,q}}\\&=\langle E_q(\cdot\overline{w}),\overline{z}E_q(\cdot\overline{z})\rangle_{\mathbf{H}_{2,q}}\\&=zE_q(z\overline{w}).\end{aligned}$$

Therefore, we obtain $R_q^*=M_z$. □

Proposition 3.10. *The space $\mathbf{H}_{2,q}$ is a de Branges-Rovnyak space.*

Proof. This follows from [4, Theorem 2.1 p. 51], since the sequence $[k]_q!$, for $k=0,1,\ldots$ is an increasing sequence with initial term 1. □

We now compute the adjoint of R_q in $\mathbf{H}_{2,q}$. Since

$$\langle z^n,z^m\rangle_{\mathbf{H}_{2,q}}=[n]_q!\,\delta_{n,m}, \tag{3.8}$$

we have

$$\begin{aligned}\langle z^n, R_q z^m\rangle_{\mathbf{H}_{2,q}} &= \left\langle z^n, \frac{z^m - q^m z^m}{(1-q)z}\right\rangle_{\mathbf{H}_{2,q}} = (1+q+\cdots+q^{m-1})\left\langle z^n, z^{m-1}\right\rangle_{\mathbf{H}_{2,q}}\\ &= (1+q+\cdots+q^n)\,[n]_q! = [n+1]_q!\\ &= \left\langle z^{n+1}, z^m\right\rangle_{\mathbf{H}_{2,q}} =: \left\langle R_q^* z^n, z^m\right\rangle_{\mathbf{H}_{2,q}}.\end{aligned}$$

Therefore, we obtain $R_q^* = M_z$.

In the case $q = 1$ the Fock space can be characterized (up to a multiplicative positive factor in the inner product) as the only Hilbert space of power series converging in a convex neighborhood of the origin and such that

$$(R_0^* f)(z) = \int_{[0,z]} f(s)ds, \tag{3.9}$$

that is, R_0^* coincides with the integration operator. It is therefore natural to try and define the integral in $\mathbf{H}_{2,q}$ by R_q^* for $q \in (0,1)$.

Lemma 3.11. *The operator R_0 is bounded in $\mathbf{H}_{2,q}$ and it holds that (with $e_k(z) = z^k$)*

$$R_0^* e_k = \frac{e_{k+1}}{1+q+\cdots+q^k}, \quad k = 0, 1, \ldots \tag{3.10}$$

Proof. We have for $k \geq 1$ and $\ell \geq 0$

$$\begin{aligned}\langle R_0 e_k, e_\ell\rangle_{\mathbf{H}_{2,q}} &= \langle e_{k-1}, e_\ell\rangle_{\mathbf{H}_{2,q}}\\ &= \delta_{k-1,\ell}\,[\ell]_q!\\ &= \delta_{k-1,\ell}\langle e_k, e_k\rangle_{\mathbf{H}_{2,q}} \frac{[\ell]_q!}{[k]_q!}\\ &= \delta_{k-1,\ell}\langle e_k, e_k\rangle_{\mathbf{H}_{2,q}} \frac{1}{1+q+\cdots+q^\ell}\\ &= \langle e_k, R_0^* e_\ell\rangle_{\mathbf{H}_{2,q}},\end{aligned}$$

with

$$R_0^* e_\ell = \frac{e_{\ell+1}}{1+q+\cdots+q^\ell}. \tag{3.11}$$

□

Consider the q-Jackson integral

$$\int_0^a f(x)d_qx := (1-q)a\sum_{k=0}^\infty q^k f(q^k a),$$

which is said to converge provided that the sum on the right-hand-side converges absolutely.

Lemma 3.12.

$$\int_0^z x^\ell d_qx = z^{\ell+1}\frac{1}{1+q+\cdots+q^\ell}. \tag{3.12}$$

Proof. By definition we have

$$\begin{aligned}\int_0^z x^\ell d_q x &= z(1-q)\sum_{k=0}^\infty q^k(q^k z)^\ell = z^{\ell+1}(1-q)\left(\sum_{k=0}^\infty (q^{1+\ell})^k\right)\\ &= z^{\ell+1}(1-q)\frac{1}{1-q^{\ell+1}} = z^{\ell+1}\frac{1}{1+q+\cdots+q^\ell}.\end{aligned}$$

□

It is well known that

$$\partial^* = M_z \tag{3.13}$$

in the Fock space, and that in fact the Fock space is characterized (up to a positive multiplicative constant in the inner product) by this equality; see [10]. In [8] it is proved that in the Hardy space we have

$$\partial^* = M_z \partial M_z, \tag{3.14}$$

and that the above equality does characterize the Hardy space (as usual, up to a positive multiplicative constant in the inner product). We now prove a formula which is valid for $q \in [0,1]$ and englobes the two above formulas.

Theorem 3.13. *Let $q \in [0,1]$. Then in $\mathbf{H}_{2,q}$ it holds that*

$$\partial^* = M_z \partial R_0^* \tag{3.15}$$

and this equality characterizes the space $\mathbf{H}_{2,q}$ up to a positive multiplicative constant in the inner product.

When $q = 0$ (Hardy space) we have R_0^* that $R_0^* = M_z$ and so (3.15) reduces to

$$M_z \partial M_z,$$

i.e. (3.14). When $q = 1$ (Fock space), we have $R_0^* = I$ (the integration operator) and $\partial I e_k = e_k$, $k = 0, 1, \ldots$. We thus get back (3.13).

Proof of Theorem 3.13. Let $k \in \mathbb{N}_0$. Let us set *a priori* $\partial^* e_k = a_{k,q} e_{k+1}$ for some $a_{k,q} \in \mathbb{C}$. We have on the one hand

$$\begin{aligned}\langle \partial^* e_k, e_{k+1}\rangle_{\mathbf{H}_{2,q}} &= \langle e_k, \partial e_{k+1}\rangle_{\mathbf{H}_{2,q}}\\ &= (k+1)\langle e_k, e_k\rangle_{\mathbf{H}_{2,q}}\\ &= (k+1)\,[k]_q!\end{aligned}$$

and on the other hand, with $\partial^* e_k = a_{k,q} e_{k+1}$ we have

$$\begin{aligned}\langle \partial^* e_k, e_{k+1}\rangle_{\mathbf{H}_{2,q}} &= a_{k,q}\langle e_{k+1}, e_{k+1}\rangle_{\mathbf{H}_{2,q}}\\ &= a_{k,q}\,[k+1]_q!\,.\end{aligned}$$

Thus

$$a_{k,q}\,[k+1]_q! = (k+1)\,[k]_q!$$

from which we get

$$a_{k,q} = \frac{k+1}{1+q+\cdots+q^k}. \tag{3.16}$$

In view of (3.11), we can write

$$\begin{aligned}\partial^* e_k &= \frac{(k+1)M_z e_k}{1+q+\cdots+q^k}\\ &= (k+1)R_0^* e_k\\ &= M_z\partial R_0^* e_k\end{aligned}$$

since

$$M_z\partial e_{k+1} = (k+1)e_{k+1},\quad k = 0,1,\dots$$

□

Note that in (3.16), we set $a_{k,0} = k+1$ and $a_{k,1} = 1$, as it should be.

Theorem 3.14. *We have*

$$M_z^* = R_q M_z R_0. \tag{3.17}$$

Proof. For $m = 1,2,\dots$ we get

$$\langle z^n, R_q M_z R_0 z^m\rangle_{\mathbf{H}_{2,q}} = \left\langle z^n, R_q M_z z^{m-1}\right\rangle_{\mathbf{H}_{2,q}} = \langle z^n, R_q z^m\rangle_{\mathbf{H}_{2,q}}.$$

By Proposition 3.10 we obtain

$$\langle z^n, R_q M_z R_0 z^m\rangle_{\mathbf{H}_{2,q}} = \langle z^n, R_q z^m\rangle_{\mathbf{H}_{2,q}} = \langle M_z z^n, z^m\rangle_{\mathbf{H}_{2,q}}.$$

We conclude our proof with the observation that $0 = \langle z^n, R_q M_z R_0 z^0\rangle_{\mathbf{H}_{2,q}} = \langle M_z z^n, z^0\rangle_{\mathbf{H}_{2,q}}$. □

4. The space $\mathcal{F}_{2,q}$

The space $\mathcal{F}_{2,q}$ appeared in [6] motivated by a study of discrete analytic functions.

Definition 4.1. *Consider the reproducing kernel*

$$K_{2,q}(z,w) = \sum_{n=0}^{\infty}\frac{z^n\overline{w}^n}{([n]_q!)^2}.$$

Then the corresponding reproducing kernel Hilbert space $\mathcal{F}_{2,q}$ is the space of all functions $f(z) = \sum_{n=0}^{\infty} f_n z^n$ such that $\sum_{n=0}^{\infty}|f_n|^2([n]_q!)^2 < \infty$.

In this way, we have $K_q =: K_{1,q}$ and $K_{2,q}$ as the reproducing kernels of $\mathbf{H}_{2,q}$ and $\mathcal{F}_{2,q}$, respectively. As both kernels are positive definite and the same holds for its difference $K_{1,q} - K_{2,q}$ we get that $\mathcal{F}_{2,q}$ is contractively included in $\mathbf{H}_{2,q}$ (see [1, 9]).

Remark 4.2. We observe that for $f_1(z) = f_2(z) = E_q^{-1}(z)$ we have

$$\begin{aligned}\left[\mathcal{M}_q(E_q^{-1})(n+1)\right]^2 &= \left(\frac{1}{1-q}\right)^n\sum_{m=0}^{\infty}q^m\left(\frac{q^m}{1-q}\right)^{z-1}\left[\sum_{k=0}^{m}E_q^{-1}\left(\frac{q^{k+1}}{1-q}\right)E_q^{-1}\left(\frac{q^{m+1-k}}{1-q}\right)\right]\\ ([n]_q!)^2 &= \left(\frac{1}{1-q}\right)^n\sum_{m=0}^{\infty}q^m\left(\frac{q^m}{1-q}\right)^n\sum_{k=0}^{m}(q^{k+1};q)_\infty(q^{m+1-k};q)_\infty.\end{aligned}$$

Hence, we get as density $\omega_{2,q}$ of our q-Fock space $\mathcal{F}_{2,q}$

$$\omega_{2,q}(|z|) := \left(\frac{1}{1-q}\right)^{|z|^2-1}(E_q^{-1}\circ E_q^{-1})(|z|^2), \tag{4.1}$$

and satisfying to

$$\mathcal{M}_q(\omega_{2,q})(n+1) = \left(\frac{1}{1-q}\right)^n \mathcal{M}_q(E_q^{-1} \circ E_q^{-1})(n+1) = ([n]_q!)^2. \tag{4.2}$$

Now, we can define $T_q : \mathbf{H}_2 \mapsto \mathcal{F}_{2,q}$ given as $z^n \to \frac{z^n}{[n]_q!}$.

Lemma 4.3. *In* $\mathbf{H}_2$ *it holds:*

$$R_q T_q = T_q R_0. \tag{4.3}$$

Proof. The case of $n = 0$ is immediate. For $n = 1, 2, \ldots$ we have

$$\begin{aligned} R_q T_q z^n &= R_q\left(\frac{z^n}{[n]_q!}\right) = \frac{1}{[n]_q!}(1 + q + \cdots + q^{n-1}) z^{n-1} \\ &= \frac{1}{[n-1]_q!} z^{n-1} = T_q z^{n-1} = T_q R_0 z^n. \end{aligned}$$

□

Theorem 4.4. *The map* T_q *is an isometry from* $\mathbf{H}_2$ *onto* $\mathcal{F}_{2,q}$.

Proof. We have $\langle e_n, e_m\rangle_{\mathcal{F}_{2,q}} = ([n]_q!)^2 \delta_{n,m}$. Hence, we get

$$\begin{aligned} \langle T_q e_n, T_q e_m\rangle_{\mathcal{F}_{2,q}} &= \frac{1}{[n]_q!}\frac{1}{[m]_q!}\langle e_n, e_m\rangle_{\mathcal{F}_{2,q}} \\ &= \delta_{n,m}\frac{([n]_q!)^2}{([n]_q!)^2} \\ &= \delta_{n,m} \\ &= \langle e_n, e_m\rangle_{\mathbf{H}_2}. \end{aligned}$$

□

Theorem 4.5. *In* $\mathcal{F}_{2,q}$ *it holds that*

$$R_q^* e_n = \frac{e_{n+1}}{[n]_q}, \quad n = 0, 1, \ldots \tag{4.4}$$

and

$$\mathcal{I} - R_q^* R_q = C^* C, \tag{4.5}$$

and this structural identity characterizes the space $\mathcal{F}_{2,q}$ *up to a multiplicative factor.*

Proof. To prove (4.4) we write

$$\begin{aligned} \langle R_q^* e_n, e_m\rangle_{\mathcal{F}_{q,2}} &= \langle e_n, R_q e_m\rangle_{\mathcal{F}_{2,q}} \\ &= [m]_q \langle e_n, e_{m-1}\rangle_{\mathcal{F}_{q,2}} \\ &= \delta_{m-1,n}([n]_q!)^2 [m]_q. \end{aligned}$$

On the other hand we show that one can assume that $R_q^* e_n = \alpha_n e_{n+1}$; we have

$$\begin{aligned} \langle R_q^* e_n, e_m\rangle &= \alpha_n \langle e_{n+1}, e_m\rangle_{\mathcal{F}_{2,q}} \\ &= \alpha_n \delta_{n+1,m}([n+1]_q!)^2. \end{aligned}$$

Comparing these equalities we obtain

$$\alpha_n([n+1]_q!)^2 = ([n]_q!)^2 [n]_q,$$

so that $\alpha_n = \frac{1}{[n]_q}$. It follows that

$$R_q^* R_q e_n = \begin{cases} 0, & n = 0, \\ e_n, & n = 1, 2, \dots \end{cases}$$

and hence the result. □

From the previous computations we also have:

Proposition 4.6. *R_0^* is an isometry in $\mathcal{F}_{2,q}$.*

Proof. This is a direct consequence of the fact that

$$R_0 R_0^* e_n = e_n,$$

for all $n \in \mathbb{N}_0$. □

From Lemma 3.12 we have:

Proposition 4.7. *In $\mathcal{F}_{2,q}$, it holds*

$$R_q^* = \mathrm{I},$$

where I *is the integration operator.*

We now use well a known method in characteristic function theory (see e.g. [5] in the case of Pontryagin spaces) and rewrite (4.5) as

$$\begin{pmatrix} R_q \\ C \end{pmatrix}^* \begin{pmatrix} R_q \\ C \end{pmatrix} = \mathcal{I}.$$

The operator

$$\begin{pmatrix} \mathcal{I} & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} R_q \\ C \end{pmatrix} \begin{pmatrix} R_q \\ C \end{pmatrix}^*$$

is therefore positive and for instance using its square root, one can find a Hilbert space $\tilde{\mathcal{H}}$ and operators B and D,

$$\begin{pmatrix} B \\ D \end{pmatrix} : \tilde{\mathcal{H}} \longrightarrow \mathcal{F}_{2,q} \oplus \mathbb{C},$$

such that

$$\begin{pmatrix} \mathcal{I} & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} R_q \\ C \end{pmatrix} \begin{pmatrix} R_q \\ C \end{pmatrix}^* = \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix}^*.$$

The operator matrix

$$\begin{pmatrix} R_q & B \\ C & D \end{pmatrix} \tag{4.6}$$

is co-isometric. We set

$$S_q(z) = D + zC(\mathcal{I} - zR_q)^{-1} B. \tag{4.7}$$

We now look into the properties of the matrix (4.6). We observe that

$$\begin{pmatrix} \mathcal{I} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_q & B \\ C & D \end{pmatrix} \begin{pmatrix} R_q & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} R_q R_q^* + BB^* & R_q C^* + BD^* \\ C R_q^* + DB^* & CC^* + DD^* \end{pmatrix},$$

so that we get

$$DD^* = 1 - CC^*, \quad BB^* = \mathcal{I} - R_q R_q^*, \quad BD^* = -R_q C^*.$$

Hence, from (4.7) we get

$$\begin{aligned} S_q(z)[S_q(w)]^* &= [D\mathcal{I} + zC(\mathcal{I} - zR_q)^{-1}B][D^*\mathcal{I} + \overline{w}B^*(\mathcal{I} - \overline{w}R_q^*)^{-1}C^*] \\ &= DD^*\mathcal{I} + zC(\mathcal{I} - zR_q)^{-1}BD^* + \overline{w}DB^*(\mathcal{I} - \overline{w}R_q^*)^{-1}C^* + z\overline{w}C(\mathcal{I} - zR_q)^{-1}BB^*(\mathcal{I} - \overline{w}R_q^*)^{-1}C^* \\ &= (1 - CC^*)\mathcal{I} + zC(\mathcal{I} - zR_q)^{-1}BD^* + \overline{w}DB^*(\mathcal{I} - \overline{w}R_q^*)^{-1}C^* + z\overline{w}C(\mathcal{I} - zR_q)^{-1}BB^*(\mathcal{I} - \overline{w}R_q^*)^{-1}C^* \end{aligned}$$

so that

$$\begin{aligned} &\mathcal{I} - S_q(z)[S_q(w)]^* \\ &= CC^*\mathcal{I} - zC(\mathcal{I} - zR_q)^{-1}BD^* - \overline{w}DB^*(\mathcal{I} - \overline{w}R_q^*)^{-1}C^* - z\overline{w}C(\mathcal{I} - zR_q)^{-1}BB^*(\mathcal{I} - \overline{w}R_q^*)^{-1}C^* \\ &= CC^*\mathcal{I} + zC(\mathcal{I} - zR_q)^{-1}R_qC^* + \overline{w}CR_q^*(\mathcal{I} - \overline{w}R_q^*)^{-1}C^* - z\overline{w}C(\mathcal{I} - zR_q)^{-1}(\mathcal{I} - R_qR_q^*)(\mathcal{I} - \overline{w}R_q^*)^{-1}C^* \\ &= C(\mathcal{I} - zR_q)^{-1}\Big[\underbrace{(\mathcal{I} - zR_q)(1 - \overline{w}R_q^*) + zR_q(\mathcal{I} - \overline{w}R_q^*) + \overline{w}(\mathcal{I} - zR_q)R_q^* - z\overline{w}(\mathcal{I} - R_qR_q^*)}_{(A)}\Big](\mathcal{I} - \overline{w}R_q^*)^{-1}C^*. \end{aligned}$$

Easy calculations give now

$$\begin{aligned} (A) &= (\mathcal{I} - zR_q)(\mathcal{I} - \overline{w}R_q^*) + zR_q(\mathcal{I} - \overline{w}R_q^*) + \overline{w}(\mathcal{I} - zR_q)R_q^* - z\overline{w}(\mathcal{I} - R_qR_q^*) \\ &= \mathcal{I} - zR_q - \overline{w}R_q^* + z\overline{w}R_qR_q^* + zR_q - z\overline{w}R_qR_q^* + \overline{w}R_q^* - z\overline{w}R_qR_q^* - z\overline{w}\mathcal{I} + z\overline{w}R_qR_q^* \\ &= (1 - z\overline{w})\mathcal{I}, \end{aligned}$$

Hence, it holds that

$$\frac{\mathcal{I} - S_q(z)S_q(w)^*}{1 - z\overline{w}} = C(\mathcal{I} - zR_q)^{-1}[(\mathcal{I} - wR_q)^*]^{-1}C^*, \quad z, w \in \mathbb{D}. \tag{4.8}$$

The operator S_q bears various names in operator theory; it is the characteristic operator function, or the transfer function, or the scattering function, associated to the operator matrix (4.6). From (4.8) one sees that S_q is analytic and contractive in the open unit disk, i.e. is a Schur function.

When $q = 0$ we have for $f(z) = \sum_{n=0}^{\infty} c_n z^n$ that

$$C(\mathcal{I} - zR_0)^{-1}f = f(z), \quad z \in \mathbb{D}.$$

Here, for $0 < q \le 1$ we define

$$f_q(z) = C(\mathcal{I} - zR_q)^{-1}f, \quad z \in \mathbb{D}.$$

As $CR_q^n f = [n]_q!\, c_n$ we get that the coefficients

$$c_n = \frac{CR_q^n f}{[n]_q!} \tag{4.9}$$

are independent of q and one has $f_q(z) = f(z)$, that is, we obtain $f(z) = C(\mathcal{I} - zR_q)^{-1}f$ for all $z \in \mathbb{D}$.

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