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Proceedings of the American Mathematical Society

DOI: 10.1090/proc/16837

Accepted Manuscript

ON WEAK SOLUTIONS TO THE KINETIC CUCKER–SMALE MODEL WITH SINGULAR COMMUNICATION WEIGHTS

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ABSTRACT. We establish the local-in-time existence of weak solutions to the kinetic Cucker–Smale model with singular communication weights $\phi(x) = |x|^{-\alpha}$ with $\alpha \in (0, d)$. In the case $\alpha \in (0, d-1]$, we also provide the uniqueness of weak solutions extending the work of Carrillo et al [MMCS, 17-35, ESAIM Proc. Surveys, 47, EDP Sci., Les Ulis, 2014] where the existence and uniqueness of weak solutions are studied for $\alpha \in (0, d-1)$.

1. Introduction

In the current work, we are concerned with the local-in-time existence and uniqueness of weak solutions to the kinetic Cucker–Smale model with singular communication weights. Specifically, let $f = f(t, x, v)$ denote the probability distribution function for a particle at position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$ at time t. Then f is governed by the following Vlasov-type kinetic equation:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [F_\phi(f) f] = 0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0 \tag{1.1}$$

subject to initial data:

$$f(0, x, v) =: f_0(x, v), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{1.2}$$

where $F_\phi(f) = F_\phi(f)(t, x, v)$ represents the velocity alignment given by

$$F_\phi(f) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x - y)(w - v) f(t, y, w)\, dydw = \phi \star m - v\phi \star \rho.$$

Here $\rho = \rho(t, x)$ and $m = m(t, x)$ denote the local density and momentum of particles, respectively:

$$\rho(t, x) := \int_{\mathbb{R}^d} f(t, x, v)\, dv, \quad m(t, x) := \int_{\mathbb{R}^d} v f(t, x, v)\, dv,$$

ϕ is the *singular* communication weight function given by

$$\phi_\alpha(x) := |x|^{-\alpha} \quad \text{with} \quad \alpha \in (0, d), \tag{1.3}$$

and $\star$ refers to convolution in the x variable unless specified otherwise.

Date: February 15, 2024.

2020 *Mathematics Subject Classification.* 2010 MSC: 35D30, 35Q92.

Key words and phrases. Kinetic Cucker–Smale model, singular communication weights, weak solutions, log-Lipschitz estimate.

The kinetic equation (1.1) can be derived from the following particle system by means of a mean-field limit:

$$\begin{aligned} &\frac{d}{dt}x_i = v_i, \quad i = 1, \ldots, N, \ t > 0, \\ &\frac{d}{dt}v_i = \frac{1}{N}\sum_{i \neq j}\phi(x_i - x_j)(v_j - v_i), \end{aligned} \tag{1.4}$$

where $x_i = x_i(t)$ and $v_i = v_i(t)$ are the position and velocity of ith particle at time $t > 0$, respectively. The system (1.4) with a regular communication weight, which is also known as the Cucker–Smale (CS) model, is proposed in [7]. Here the regular communication weight means that it is bounded and Lipschitz at least. The velocity alignment behavior of solutions showing $\max_{1 \le i,j \le N}|v_i(t) - v_j(t)| \to 0$ as $t \to \infty$ and $\sup_{t \ge 0}\max_{1 \le i,j \le N}|x_i(t) - x_j(t)| < \infty$ is obtained in [7] under suitable assumptions on the initial configurations, and later it is improved in [3, 9, 10]. The system (1.4) with the singular communication weight (1.3) is studied in [2, 6, 9, 15, 16] and it exhibits rich phenomena such as the unconditional collision-avoidance behavior or sticking of particles. We refer to [4, 14] and references therein for various results on the Cucker–Smale model and its variants.

Our main purpose is to establish the existence of weak solutions to the equation (1.1) for $\alpha \in (0, d)$. Moreover, we also study the uniqueness of weak solutions when $\alpha = d - 1$. In particular, our uniqueness result combined with [1] implies that the weak solution to (1.1) with $\alpha \in (0, d-1]$ exists uniquely.

To state our main result, we shall first introduce a notion of our weak solutions to the equation (1.1).

Definition 1.1. *For a given $T \in (0, \infty)$, f is a weak solution of* (1.1) *on the time interval* $[0, T]$ *if and only if the following holds:*

(i) $f \in L^\infty(0, T; L^1_+ \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d))$ *and*
(ii) *for all* $\Psi \in \mathcal{C}_c^\infty([0, T) \times \mathbb{R}^d \times \mathbb{R}^d)$,

$$\begin{aligned} &\iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, v)\Psi(0, x, v)\,dxdv \\ &\quad = -\int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(\partial_t \Psi + v \cdot \nabla_x \Psi + \phi \star m \cdot \nabla_v \Psi)\,dxdvdt \\ &\qquad + \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\phi \star \rho) f(v \cdot \nabla_v \Psi)\,dxdvdt. \end{aligned}$$

In order to show the uniqueness of weak solutions, we introduce the pth-order Wasserstein distance which is defined by

$$\mathrm{d}_p(\rho_1, \rho_2) := \left(\inf_\gamma \iint_{\mathbb{R}^k \times \mathbb{R}^k} |x - y|^p\,d\gamma(x, y)\right)^{\frac{1}{p}}$$

for $\rho_1, \rho_2 \in \mathcal{P}_p(\mathbb{R}^k)$, $k \in \mathbb{N}$, $p \in [1, \infty)$, where γ is any transference plan between ρ_1 and ρ_2, i.e. for any $\psi \in \mathcal{C}_b(\mathbb{R}^k)$,

$$\iint_{\mathbb{R}^k \times \mathbb{R}^k} \psi(x)\,d\gamma(x, y) = \int_{\mathbb{R}^k} \psi(x)\rho_1(x)\,dx$$

and

$$\iint_{\mathbb{R}^k \times \mathbb{R}^k} \psi(y)\,d\gamma(x, y) = \int_{\mathbb{R}^k} \psi(y)\rho_2(y)\,dy.$$

Here $\mathcal{P}_p(\mathbb{R}^k)$ stands for the set of probability measures with bounded pth-order moment. Note that $\mathcal{P}_p(\mathbb{R}^k)$ is a complete metric space endowed with the pth-order Wasserstein distance $\mathrm{d}_p(\cdot,\cdot)$ [17]. Throughout the paper, the notation for a probability measure and its probability density is often abused for notational simplicity.

We now present our main result on the existence of weak solutions to (1.1).

Theorem 1.1. *Let $d \geq 1$ and $\alpha \in (0, d)$. Suppose that the initial data f_0 is compactly supported in velocity and satisfies*

$$f_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d) \quad \textit{and} \quad |x| f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d).$$

Then there exist $T > 0$ and at least one weak solution f to (1.1)-(1.2) in the sense of Definition 1.1. Moreover, if $\alpha \in (0, d-1]$, the solution obtained above exists uniquely. Indeed, if f_i, $i = 1, 2$ are two such solutions to (1.1), then we have the following stability estimate:

$$\frac{d}{dt}\mathrm{d}_1(f_1(t), f_2(t)) \leq C\mathrm{d}_1(f_1(t), f_2(t))\left(1 - \mathbf{1}_{\{\alpha=d-1\}}\log^- \mathrm{d}_1(f_1(t), f_2(t))\right) \tag{1.5}$$

for $t \in (0, T)$, where $\log^-(x) := 0 \wedge \log x$.

Remark 1.1. *Notice that when $\alpha \in (0, d-1)$ the negative part of log on the right-hand side of (1.5) does not appear. In the case $\alpha = d - 1$, the stability estimate of solutions resembles that in [12], where the uniqueness of weak solutions to the Vlasov–Poisson system with bounded density is discussed. More specifically, the following functional inequality is used in [12]:*

$$\|\nabla(-\Delta)^{-1}(\rho_1 - \rho_2)\|_{L^2} \leq (\|\rho_1\|_{L^\infty} \vee \|\rho_2\|_{L^\infty})^{\frac{1}{2}}\, \mathrm{d}_2(\rho_1, \rho_2)$$

for $\rho_1, \rho_2 \in \mathcal{P}_2 \cap L^\infty(\mathbb{R}^d)$. By employing the above inequality, the log-Lipschitz stability estimate in the second-order Wasserstein distance d_2 is obtained. However, we do not use any functional relations and directly provide the log-Lipschitz stability estimate for solutions in d_1.

Remark 1.2. *For the existence of solutions, the regularity of solutions can be relaxed to $f \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$ for some $p \in \left(\frac{d}{d-\alpha}, \infty\right)$, see Remark 2.2 for details. However, we need L^∞-regularity of solutions for the uniqueness.*

In the case $\alpha \in (0, d-1)$, the local-in-time existence and uniqueness of weak solutions to (1.1) in the sense of Definition 1.1 are obtained in [1]. Thus, the main contribution of this work corresponds to the case $\alpha \in [d-1, d)$ with respect to the existence and $\alpha = d-1$ in regard to the uniqueness. For the existence of solutions, we regularize the equation of (1.1) removing the singularity in the communication weight ϕ. We then discuss the uniform bound estimates for the velocity-support and $L^1 \cap L^\infty$-norm of regularized solutions. Here, we would like to point out that a Cauchy estimate of solutions with respect to the regularization parameter is obtained in [1], and from which the existence and uniqueness of solutions to (1.1) are shown. However, this strategy enforces the condition on the singular exponent α such that $\alpha < d - 1$. To cover the case $\alpha \in [d-1, d)$, unlike [1], we crucially employ appropriate weak and strong compactness theorems to obtain the existence of $L^1 \cap L^\infty$-solutions.

For the uniqueness of such solutions, we only deal with the case $\alpha = d-1$. Motivated from [12, 13], we show the force field $F_\phi(f)(x, v)$ is log-Lipschitz continuous and from which we establish the stability estimate (1.5). In the case $\alpha > d-1$, the uniqueness can be obtained by dealing with more regular solutions [5]. In particular, when $\alpha \geq d$, it is proved in [8] that suitable weak measure-valued solutions to (1.1) are monokinetic, rougly speaking, it has a Dirac delta distribution with respect to the velocity variable.

The rest of this paper is organized as follows. In Section 2, we introduce the regularized equation and provide some uniform bound estimates with respect to the regularization parameter. Section 3 is devoted to the proof of Theorem 1.1.

2. Regularized equation & uniform bound estimates

We first regularize the equation (1.1) as follows:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \nabla_v \cdot [(\phi^\varepsilon \star m^\varepsilon - v\phi^\varepsilon \star \rho^\varepsilon) f^\varepsilon] = 0, \tag{2.1}$$

subject to the initial data:

$$f^\varepsilon(0,x,v) = (f_0 \star_{x,v} \theta_\varepsilon)(x,v), \quad (x,v) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where $\theta_\varepsilon(x,v) = \varepsilon^{-2d}\theta(x/\varepsilon, v/\varepsilon)$ is the standard mollifier with

$$0 \leq \theta(x,v) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d), \quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} \theta(x,v)\,dxdv = 1,$$

and $\phi^\varepsilon = \phi^\varepsilon(x)$ is written as

$$\phi^\varepsilon(x) := \frac{1}{(\varepsilon + |x|^2)^{\frac{\alpha}{2}}}.$$

Here we suppose that

$$\mathrm{supp}_v(f_0) \subset B(0,R_0), \quad \mathrm{supp}_v(\theta) \subset B(0,R_1).$$

Then we have

$$\mathrm{supp}_v(f_0^\varepsilon) \subset B(0, R_0 + \varepsilon R_1),$$

which means that f_0^ε is also compactly supported in v for every $\varepsilon > 0$. From now on, without loss of generality, we assume that $\varepsilon < 1$. Then the standard results on the classical solutions of the kinetic Cucker–Smale equation implies the global existence of classical solutions f^ε to (2.1) for every $\varepsilon > 0$, see [10] for instance. First, the following result is direct from the estimates in [1, Lemma 3.1].

Lemma 2.1. *Let $T > 0$ and f^ε be the classical solution to* (2.1) *on the time interval* $[0,T]$. *Then we have*

$$\mathrm{supp}_v(f^\varepsilon(t)) \subset \mathrm{supp}_v(f_0^\varepsilon) \subset B(0, R_0 + R_1)).$$

Next, we investigate uniform-in-ε estimate for $L^1 \cap L^\infty$-norms of f^ε.

Lemma 2.2. *Let $T > 0$ and f^ε be the classical solution to* (2.1) *on the time interval* $[0,T]$. *Then there exists $T^* \in (0,T]$ independent of ε such that*

$$\sup_{0 \leq t \leq T^*} \|f^\varepsilon(t)\|_{L^1\cap L^\infty} \leq 4\|f_0\|_{L^1\cap L^\infty}.$$

Proof. Straightforward computation gives that for any $p \in [1, \infty)$

$$\begin{aligned}
\frac{d}{dt}\|f^\varepsilon(t)\|_{L^p}^p &= -p \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f^\varepsilon)^{p-1} \nabla_v \cdot \left[(\phi^\varepsilon \star m^\varepsilon - v\phi^\varepsilon \star \rho^\varepsilon) f^\varepsilon \right] dxdv \\
&= (p-1) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v [(f^\varepsilon)^p] \cdot (\phi^\varepsilon \star m^\varepsilon - v\phi^\varepsilon \star \rho^\varepsilon)\, dxdv \qquad (2.2) \\
&= d(p-1) \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\phi^\varepsilon \star \rho^\varepsilon)(f^\varepsilon)^p \, dxdv.
\end{aligned}$$

Thus if $p = 1$, we first obtain

$$\|f^\varepsilon(t)\|_{L^1} = \|f^\varepsilon(0)\|_{L^1} = \|f_0\|_{L^1}.$$

We also find

$$\begin{aligned}
\phi^\varepsilon \star \rho^\varepsilon &= \int_{\mathbb{R}^d} \frac{1}{(\varepsilon + |x-y|^2)^{\frac{\alpha}{2}}} \rho^\varepsilon(y)\, dy \\
&\le \left(\int_{|x-y|\ge 1} + \int_{|x-y|\le 1} \right) \frac{1}{|x-y|^\alpha} \rho^\varepsilon(y)\, dy \\
&\le \|\rho^\varepsilon(t)\|_{L^1} + C\|\rho^\varepsilon(t)\|_{L^\infty} \\
&\le \|f_0\|_{L^1} + C\|f^\varepsilon(t)\|_{L^\infty},
\end{aligned}$$

where we used

$$\|\rho^\varepsilon(t)\|_{L^\infty} = \left\| \int_{\mathbb{R}^d} f^\varepsilon(\cdot, v, t)\, dv \right\|_{L^\infty} \le C(R_0 + R_1)^d \|f^\varepsilon(t)\|_{L^\infty},$$

due to Lemma 2.1. This together with (2.2) yields

$$\begin{aligned}
\frac{d}{dt}\|f^\varepsilon(t)\|_{L^p}^p &\le d(p-1)\|\phi^\varepsilon \star \rho^\varepsilon\|_{L^\infty} \|f^\varepsilon(t)\|_{L^p}^p \\
&\le C(p-1)(\|f_0\|_{L^1} + \|f^\varepsilon(t)\|_{L^\infty})\|f^\varepsilon(t)\|_{L^p}^p,
\end{aligned}$$

and from which, we obtain

$$\frac{d}{dt}\|f^\varepsilon(t)\|_{L^p} \le C(\|f_0\|_{L^1} + \|f^\varepsilon(t)\|_{L^\infty})\|f^\varepsilon(t)\|_{L^p},$$

where $C > 0$ is independent of p and $\varepsilon > 0$. We finally apply the Grönwall's lemma and pass to the limit $p \to \infty$ to conclude the desired result. □

Remark 2.1. *Since* $f^\varepsilon \in \mathcal{C}([0, T^*]; L^1(\mathbb{R}^d \times \mathbb{R}^d) \cap L^\infty((0, T^*) \times \mathbb{R}^d \times \mathbb{R}^d)$ *uniformly in* ε*, we also actually have*

$$f^\varepsilon \in L^\infty(0, T^*; L^p(\mathbb{R}^d \times \mathbb{R}^d)) \quad \forall p \in [1, \infty].$$

Moreover, since the support of f^ε *in velocity is uniformly bounded in* ε*, we can find a constant* $C > 0$ *independent of* ε *such that*

$$\sup_{0 \le t \le T^*} (\|\rho^\varepsilon(t)\|_{L^p} + \|m^\varepsilon(t)\|_{L^p}) \le C \sup_{0 \le t \le T^*} \|f^\varepsilon\|_{L^p} < \infty.$$

Remark 2.2. *For $p \in \left(\frac{d}{d-\alpha}, \infty\right]$, we get*

$$\begin{aligned}\phi^\varepsilon \star \rho^\varepsilon &= \int_{\mathbb{R}^d} \frac{1}{(\varepsilon + |x-y|^2)^{\frac{\alpha}{2}}} \rho^\varepsilon(y)\, dy \\ &\le \left(\int_{|x-y|\ge 1} + \int_{|x-y|\le 1} \right) \frac{1}{|x-y|^\alpha} \rho^\varepsilon(y)\, dy \\ &\le \|\rho^\varepsilon(t)\|_{L^1} + C \||x|^{-\alpha} \mathbb{1}_{\{|x|\le 1\}}\|_{L^{p'}} \|\rho^\varepsilon(t)\|_{L^p} \\ &\le \|f_0\|_{L^1} + C\|f^\varepsilon(t)\|_{L^p},\end{aligned}$$

where p' is the Hölder conjugate of p, i.e. $p' = \frac{p}{p-1} < \frac{d}{\alpha}$. Thus, one can relax the assumption $f_0 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ to $f_0 \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$ for some $p \in \left(\frac{d}{d-\alpha}, \infty\right)$ and proceed our strategy for the proof of existence of weak solutions.

Next, we discuss the x-moment estimates for f^ε.

Lemma 2.3. *Let f^ε be the classical solution to* (2.1) *on the time interval* $[0, T^*]$. *Then we have*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle x \rangle f^\varepsilon(t,x,v)\, dxdv \le C \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0^\varepsilon(x,v)\, dxdv, \quad t \in (0, T^*].$$

Proof. Direct computation gives

$$\begin{aligned}\frac{d}{dt} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle x \rangle f^\varepsilon(t,x,v)\, dxdv &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(x \cdot v)}{\langle x \rangle} f^\varepsilon(t,x,v)\, dxdv \\ &\le C(R_0 + R_1) \iint_{\mathbb{R}^d \times \mathbb{R}^d} f^\varepsilon(t,x,v)\, dxdv \\ &\le C \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0^\varepsilon(t,x,v)\, dxdv,\end{aligned}$$

and this yields the desired estimate. □

3. Proof of Theorem 1.1

3.1. **Existence of weak solutions.** Now, we present the proof of Theorem 1.1. Before proceeding, we present a variant of velocity averaging lemma in [11] as follows.

Lemma 3.1. *For $p \in (1, \infty]$, let $\{f^m\}$ be compactly supported in velocity uniformly in m and $\{G^m\}$ be bounded in $L^p_{loc}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$. Suppose that f^m and G^m satisfy*

$$\partial_t f^m + v \cdot \nabla f^m = \nabla_v^\ell G^m, \quad f^m|_{t=0} = f_0 \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$$

for some multi-index ℓ and

$$\sup_{m \in \mathbb{N}} \|f^m\|_{L^\infty(0,T;L^1 \cap L^p(\mathbb{R}^d \times \mathbb{R}^d))} + \sup_{m \in \mathbb{N}} \||x| f^m\|_{L^\infty(0,T;L^1(\mathbb{R}^d \times \mathbb{R}^d))} < \infty,$$

then for every $\varphi \in \mathcal{C}^{|\ell|}(\mathbb{R}^{2d})$ satisfying $|\varphi(v)| \le c|v|$, the sequence

$$\left\{ \int_{\mathbb{R}^d} f^m \varphi \, dv \right\}$$

is relatively compact in $L^q((0,T) \times \mathbb{R}^d)$ for $q \in (1, p)$.

Proof. The proof proceeds similarly to the arguments in [11, Lemma 2.7], but we write here for readers' convenience.

First, we write

$$\varrho_\varphi^m := \int_{\mathbb{R}^d} f^m \varphi \, dv.$$

Due to the uniform compact support in velocity of f^m, we also have

$$\sup_{m\in\mathbb{N}} \|\varrho_\varphi^m\|_{L^1\cap L^p((0,T)\times\mathbb{R}^d)} + \sup_{m\in\mathbb{N}} \||x|\varrho_\varphi^m\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} < \infty. \tag{3.1}$$

Then we use [11, Proposition 2.5] to find ϱ_φ satisfying

$$\varrho_\varphi^m \to \varrho_\varphi \quad \text{strongly in} \quad L^q_{loc}((0,T)\times\mathbb{R}^d)$$

for $q \in (1,p)$, up to a subsequence. Due to the uniform bounds (3.1), we can use a diagonal extraction process to make ϱ_φ satisfy

$$\|\varrho_\varphi\|_{L^1\cap L^p((0,T)\times\mathbb{R}^d)} + \||x|\varrho_\varphi\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} < \infty.$$

For the compactness in $L^q((0,T)\times\mathbb{R}^d)$, we again use the compact support in velocity and choose any $k > 0$ so that

$$\begin{aligned}\int_{|x|\geq k} (\varrho_\varphi^m)^q \, dx &\leq \int_{|x|\geq k} \left(\frac{|x|}{k}\right)^{\frac{1}{\ell}} (\varrho_\varphi^m)^{q-\frac{1}{\ell}+\frac{1}{\ell}} \, dx \\ &\leq C\left(\frac{1}{k}\right)^{\frac{1}{\ell}} \left(\int_{\mathbb{R}^d} (\varrho_\varphi^m)^{\frac{q\ell-1}{\ell-1}} \, dx\right)^{\frac{\ell-1}{\ell}} \left(\int_{\mathbb{R}^d} |x|\varrho_\varphi^m \, dx\right)^{\frac{1}{\ell}},\end{aligned}$$

where $\ell > 1$ is a constant satisfying

$$q \leq \frac{p\ell - p + 1}{\ell}, \quad \text{so that} \quad \frac{q\ell-1}{\ell-1} \leq p.$$

Then due to (3.1), we get

$$\int_{|x|\geq k} (\varrho_\varphi^m)^q \, dx \to 0$$

as $k \to \infty$ uniformly in m. Since the above argument also holds for ϱ_φ, this combined with the strong convergence in $L^q_{loc}((0,T)\times\mathbb{R}^d)$ gives the desired result. □

Proof of Theorem 1.1: existence. Now, we recall from Lemma 2.2 and Remark 2.1 that

$$\|f^\varepsilon\|_{L^\infty(0,T^*;L^p(\mathbb{R}^d\times\mathbb{R}^d))} + \|\rho^\varepsilon\|_{L^\infty(0,T^*;L^p(\mathbb{R}^d))} + \|m^\varepsilon\|_{L^\infty(0,T^*;L^p(\mathbb{R}^d))} \leq C,$$

for $p \in [1,\infty]$, where $C > 0$ is independent of ε. Then we can obtain the following weak convergence as $\varepsilon \to 0$ up to a subsequence: for any $p \in [1,\infty]$,

$$\begin{aligned} f^\varepsilon &\overset{*}{\rightharpoonup} f &&\text{in} \quad L^\infty(0,T^*;L^p(\mathbb{R}^d\times\mathbb{R}^d)),\\ \rho^\varepsilon &\overset{*}{\rightharpoonup} \rho &&\text{in} \quad L^\infty(0,T^*;L^p(\mathbb{R}^d)),\\ m^\varepsilon &\overset{*}{\rightharpoonup} m &&\text{in} \quad L^\infty(0,T^*;L^p(\mathbb{R}^d)).\end{aligned}$$

Once we write

$$G^\varepsilon := -\left(\phi^\varepsilon \star m^\varepsilon - v\phi^\varepsilon \star \rho^\varepsilon\right) f^\varepsilon,$$

it is not difficult to show that $G^\varepsilon \in L^p_{loc}([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ for some $p\in(1,\infty)$. Thus, we can use Lemma 3.1 to obtain, for every $p\in(1,\infty)$,

$$\begin{aligned} &\rho^\varepsilon \to \rho \quad \text{in} \quad L^p((0,T^*)\times\mathbb{R}^d)) \ \text{ and a.e.},\\ &m^\varepsilon \to m \quad \text{in} \quad L^p((0,T^*)\times\mathbb{R}^d)) \ \text{ and a.e.} \end{aligned}$$

as $\varepsilon\to 0$, up to a subsequence. Hence, it remains to show that the limit f satisfies (1.1) in the distributional sense. For this, it suffices to show that

$$(\phi^\varepsilon \star m^\varepsilon - v\phi^\varepsilon\star\rho^\varepsilon)\, f^\varepsilon \to (\phi\star m - v\phi\star\rho)\, f$$

in the distributional sense as $\varepsilon\to 0$. Since the argument is similar, we only show the convergence

$$(\phi^\varepsilon\star m^\varepsilon)\, f^\varepsilon \to (\phi\star m)\, f.$$

For this, we choose $\Psi\in\mathcal{C}^\infty_c([0,T^*]\times\mathbb{R}^d\times\mathbb{R}^d)$ arbitrarily and estimate

$$\begin{aligned} &\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left[(\phi^\varepsilon\star m^\varepsilon)\, f^\varepsilon - (\phi\star m)\, f\right]\cdot\Psi\, dxdvds\\ &\quad = \int_0^t\int_{\mathbb{R}^d} (\phi^\varepsilon\star(m^\varepsilon - m))\,\rho^\varepsilon_\Psi\, dxds + \int_0^t\int_{\mathbb{R}^d} ((\phi^\varepsilon-\phi)\star m)\,\rho^\varepsilon_\Psi\, dxds\\ &\qquad + \int_0^t\int_{\mathbb{R}^d} (\phi\star m)\cdot(\rho^\varepsilon_\Psi - \rho_\Psi)\, dxds\\ &\quad =: I^\varepsilon + II^\varepsilon + III^\varepsilon, \end{aligned}$$

where

$$\rho^\varepsilon_\Psi := \int_{\mathbb{R}^d} f^\varepsilon\Psi\, dv, \quad \text{and} \quad \rho_\Psi := \int_{\mathbb{R}^d} f\Psi\, dv.$$

Note that ρ^ε_Ψ and ρ_Ψ satisfy

$$\|\rho^\varepsilon_\Psi\|_{L^\infty(0,T^*;L^p(\mathbb{R}^d))} + \|\rho_\Psi\|_{L^\infty(0,T^*;L^p(\mathbb{R}^d))} \le C,$$

for any $p\in[1,\infty]$ and $C>0$ is a constant independent of ε.

For I^ε, we choose $p\in(\frac{d}{d-\alpha},\infty)$ and denote the Hölder conjugate of p by p'. Then the strong convergence of m^ε implies

$$\begin{aligned} I^\varepsilon &\le C\||x|^{-\alpha}\mathbb{1}_{\{|x|\le1\}}\|_{L^{p'}(\mathbb{R}^d)}\|m^\varepsilon - m\|_{L^p((0,T^*)\times\mathbb{R}^d)}\|\rho^\varepsilon_\Psi\|_{L^\infty(0,T^*;L^1(\mathbb{R}^d))}\\ &\quad + C\||x|^{-\alpha}\mathbb{1}_{\{|x|>1\}}\|_{L^{\frac{d+1}{\alpha}}(\mathbb{R}^d)}\|m^\varepsilon - m\|_{L^{\frac{d+1}{d+1-\alpha}}((0,T^*)\times\mathbb{R}^d)}\|\rho^\varepsilon_\Psi\|_{L^\infty(0,T^*;L^1(\mathbb{R}^d))}\\ &\le C\left(\|m^\varepsilon - m\|_{L^p((0,T^*)\times\mathbb{R}^d)} + \|m^\varepsilon - m\|_{L^{\frac{d+1}{d+1-\alpha}}((0,T^*)\times\mathbb{R}^d)}\right), \end{aligned}$$

and this implies $I^\varepsilon\to 0$ as $\varepsilon\to 0$.

For II^ε, we have

$$\begin{aligned} II^\varepsilon &\le C\|(\phi^\varepsilon-\phi)\mathbb{1}_{\{|x|\le1\}}\|_{L^1(\mathbb{R}^d)}\|m\|_{L^\infty((0,T^*)\times\mathbb{R}^d)}\|\rho^\varepsilon_\Psi\|_{L^\infty(0,T^*;L^1(\mathbb{R}^d))}\\ &\quad + C\|(\phi^\varepsilon-\phi)\mathbb{1}_{\{|x|>1\}}\|_{L^{\frac{d+1}{\alpha}}(\mathbb{R}^d)}\|m\|_{L^{\frac{d+1}{d+1-\alpha}}((0,T^*)\times\mathbb{R}^d)}\|\rho^\varepsilon_\Psi\|_{L^\infty(0,T^*;L^1(\mathbb{R}^d))}, \end{aligned}$$

and the dominated convergence theorem implies $II^\varepsilon\to 0$ as $\varepsilon\to 0$. Since the weak convergence of f^ε toward f directly implies $III^\varepsilon\to 0$ as $\varepsilon\to 0$, we have shown the desired convergence and hence, this concludes the proof for the existence result in Theorem 1.1. $\square$

3.2. **Uniqueness of weak solutions.** As mentioned before, the case $\alpha \in (0, d-1)$ is already studied in [1], thus in this section, we investigate the uniqueness of weak solutions when $\alpha = d-1$.

For this, motivated from [13, Lemma 8.1, Chapter 8], we first provide the log-Lipschitz estimate concerned with the communication weight $\phi(x) = |x|^{1-d}$.

Lemma 3.2. *Suppose that $h \in L^1 \cap L^\infty(\mathbb{R}^d)$. Then we have*

$$\int_{\mathbb{R}^d} |\phi(x_1 - y) - \phi(x_2 - y)||h(y)|\, dy \le C\|h\|_{L^1 \cap L^\infty} |x_1 - x_2|(1 - \log^- |x_1 - x_2|),$$

for any $x_1, x_2 \in \mathbb{R}^d$, where $C > 0$ is independent of x_1 and x_2.

Remark 3.1. *Lemma 3.2 implies $\phi \star h$ satisfies the log-Lipschitz estimate since*

$$|\phi \star h(x_1) - \phi \star h(x_2)| \le \int_{\mathbb{R}^d} |\phi(x_1 - y) - \phi(x_2 - y)||h(y)|\, dy$$

for any $x_1, x_2 \in \mathbb{R}^d$.

Proof of Lemma 3.2. We split the proof into two cases.

- (Case A: $|x_1 - x_2| \ge 1$) In this case, we observe

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\phi(x_1 - y) - \phi(x_2 - y)||h(y)|\, dy \\
&\quad = \int_{\mathbb{R}^d} \left| \left(\frac{|x_2 - y|^{d-1} - |x_1 - y|^{d-1}}{|x_1 - y|^{d-1}|x_2 - y|^{d-1}} \right| |h(y)|\, dy \right) \\
&\quad \le C \int_{\mathbb{R}^d} \left(\frac{|x_1 - x_2| \left(|x_1 - y|^{d-2} + |x_2 - y|^{d-2}\right)}{|x_1 - y|^{d-1}|x_2 - y|^{d-1}} |h(y)|\, dy \right) \\
&\quad = C|x_1 - x_2| \int_{\mathbb{R}^d} \left(\frac{1}{|x_1 - y|^{d-1}|x_2 - y|} + \frac{1}{|x_2 - y|^{d-1}|x_1 - y|} \right) |h(y)|\, dy.
\end{aligned}$$

Since $|x_1 - x_2| \ge 1$, for y with $|x_1 - y| < \frac{1}{4}$, we obtain $|x_2 - y| \ge \frac{3}{4}$. Conversely, for y with $|x_2 - y| < \frac{1}{4}$, we have $|x_1 - y| \ge \frac{3}{4}$. This yields

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{|x_1 - y|^{d-1}|x_2 - y|} |h(y)|\, dy \\
&\quad = \left(\int_{\{|x_1 - y| < \frac{1}{4}\}} + \int_{\{|x_2 - y| < \frac{1}{4}\}} + \int_{\{|x_1 - y|, |x_2 - y| \ge \frac{1}{4}\}} \frac{1}{|x_1 - y|^{d-1}|x_2 - y|} |h(y)|\, dy \right) \\
&\quad \le C\|h\|_{L^1 \cap L^\infty},
\end{aligned}$$

and this gives the desired result since the other term can be handled similarly.

• (Case B: $|x_1 - x_2| < 1$) We write $r := |x_1 - x_2|$ for notational simplicity. Then, we estimate the difference similarly as in [13]:

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\phi(x_1 - y) - \phi(x_2 - y)||h(y)|\,dx \\
&\quad = \left(\int_{\{|x_1-y|\geq 2\}} + \int_{\{2r<|x_1-y|<2\}} + \int_{\{|x_1-y|\leq 2r\}}\right) \\
&\qquad\qquad\qquad\qquad \left|\frac{1}{|x_1-y|^{d-1}} - \frac{1}{|x_2-y|^{d-1}}\right| |h(y)|\,dy \\
&\quad =: I + II + III.
\end{aligned}$$

For I, since $|x_1 - x_2| < 1$, for y with $|x_1 - y| \geq 2$, we get $|x_2 - y| \geq 1$. Thus, analogously to the Case A, we deduce

$$I \leq C|x_1 - x_2|\|h\|_{L^1\cap L^\infty}.$$

For II, note that two functions $|x_1 - y|^{-d+1}$ and $|x_2 - y|^{-d+1}$ are smooth in y in this region. Thus we use the mean value theorem to yield

$$\begin{aligned}
II &= -(d-1)\int_{\{2r<|x_1-y|<2\}} \left|\int_0^1 \frac{((x_2-y)+t(x_1-x_2))}{|(x_2-y)+t(x_1-x_2)|^{d+1}} \cdot (x_1-x_2)\,dt\right| |h(y)|\,dy \\
&\leq C\int_{\{2r<|x_1-y|<2\}} \frac{|x_1-x_2|}{|x_1-y|^d}|h(y)|\,dy \\
&\leq C|x_1-x_2|\|h\|_{L^\infty}\int_{2r}^2 \frac{1}{s}\,ds \\
&= -C|x_1-x_2|\log|x_1-x_2|,
\end{aligned}$$

where we used

$$\begin{aligned}
|(x_2-y)+t(x_1-x_2)| &= |(x_1-y)-(1-t)(x_1-x_2)| \\
&\geq ||x_1-y| - |(1-t)|x_1-x_2|| \\
&\geq \frac{|x_1-y|}{2}.
\end{aligned}$$

For III, note that $\{y \,:\, |x_1 - y| \leq 2r\} \subseteq \{y \,:\, |x_2 - y| \leq 3r\}$. This gives

$$\begin{aligned}
III &\leq \int_{\{|x_1-y|\leq 2r\}} \frac{1}{|x_1-y|^{d-1}}|h(y)|\,dy + \int_{\{|x_2-y|\leq 3r\}} \frac{1}{|x_2-y|^{d-1}}|h(y)|\,dy \\
&\leq C\|h\|_{L^\infty}\left(\int_0^{2r} + \int_0^{3r}\right) 1\,ds \\
&\leq C\|h\|_{L^\infty}|x_1-x_2|.
\end{aligned}$$

Combining all the above estimates concludes the desired inequality. □

Proof of Theorem 1.1: uniqueness in the case $\alpha = d - 1$. For two solutions f_1 and f_2 corresponding to initial data f_1^{in} and f_2^{in}, respectively, we separately consider their characteristics: for $i = 1, 2$, $Z_i(t; s, z) := (X_i(t; s, z), V_i(t; s, z))$, $z := (x, v)$ satisfies

$$\begin{aligned}
&\frac{d}{dt}X_i(t;s,z) = V_i(t;s,z), \quad t, s \in (0,T), \\
&\frac{d}{dt}V_i(t;s,z) = (\phi\star m_i)(X_i(t;s,z)) - V_i(t;s,z)(\phi\star\rho_i)(X_i(t;s,z))
\end{aligned}$$

subject to $Z_i(s; s, z) = z$. To ensure that the above characteristics are well-defined, one needs to check Osgood criterion, which tells that for a differential equation $\dot{y} = f(y)$ where f is continuous and

$$|f(x) - f(y)| \le \omega(|x - y|),\ \omega:\ \text{continuous and nondecreasing},\ \int_0^1 \frac{1}{\omega(s)}\,ds = \infty,$$

$\omega(r) > 0$ for all $r > 0$, $\omega(0) = 0$, then it has a unique solution. In our case, $\phi \star m_i$ and $\phi \star \rho_i$ are log-Lipschitz continuous thanks to Lemma 3.2, and hence the characteristics are well-defined. Note that $f_i, i = 1, 2$ are compactly supported in velocity, thus $|\phi \star m_i| \le C\phi \star \rho_i$ for some $C > 0$ which only depends on f_i^{in} and d. Then, at some time $t_0 \in (0, T)$, we choose an optimal transport $\mathcal{T}^0(z) := (\mathcal{T}_x^0(z), \mathcal{T}_v^0(z))$, i.e.

$$f_2(t_0, \cdot) = \mathcal{T}^0_{\#} f_1(t_0, \cdot).$$

Here $\mathcal{T}^0_{\#} f_1(t_0, \cdot)$ denotes the push-forward of f_1 by a measurable map $\mathcal{T}^0$. Then we have

$$f_2(t, \cdot) = \mathcal{T}^t_{\#} f_1(t, \cdot), \quad \mathcal{T}^t := Z_2(t; t_0, z)_{\#} \mathcal{T}^0_{\#} Z_1(t_0; t, z). \tag{3.2}$$

Here, we know that

$$\mathrm{d}_1(f_1(t), f_2(t)) \le \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| Z_1(t; t_0, z) - Z_2(t; t_0, \mathcal{T}^0(z)) \right| f_1(t_0, z)\,dz.$$

Set

$$Q(t) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| Z_1(t; t_0, z) - Z_2(t; t_0, \mathcal{T}^0(z)) \right| f_1(t_0, z)\,dz.$$

Then, we obtain

$$\begin{aligned}
&\frac{d}{dt} Q(t)\bigg|_{t=t_0^+} \\
&\quad \le \iint_{\mathbb{R}^d \times \mathbb{R}^d} |V_1(t; t_0, z) - V_2(t; t_0, z)|\, f_1(t, z)\,dz \bigg|_{t=t_0^+} \\
&\qquad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |(\phi \star m_1)(X_1(t; t_0, z)) - (\phi \star m_2)(X_2(t; t_0, z))|\, f_1(t, z)\,dz \bigg|_{t=t_0^+} \\
&\qquad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |V_1(t; t_0, z)(\phi \star \rho_1)(X_1(t; t_0, z)) \\
&\qquad\qquad\qquad - V_2(t; t_0, z)(\phi \star \rho_2)(X_2(t; t_0, z))|\, f_1(t, z)\,dz \bigg|_{t=t_0^+} \\
&\quad =: I + II + III,
\end{aligned}$$

where I can be easily estimated as

$$I = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| v - \mathcal{T}_v^0(z) \right| f_1(t_0, z)\,dz \le \mathrm{d}_1(f_1(t_0), f_2(t_0)).$$

For II, we get

$$\begin{aligned}
II &\le \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| (\phi \star m_1)(x) - (\phi \star m_1)(\mathcal{T}_x^0(z)) \right| f_1(t_0, z)\,dz \\
&\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| (\phi \star m_1)(\mathcal{T}_x^0(z)) - (\phi \star m_2)(\mathcal{T}_x^0(z)) \right| f_1(t_0, z)\,dz \\
&=: II_1 + II_2.
\end{aligned}$$

Here, Lemma 3.2 directly implies

$$II_1 \le C \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-\mathcal{T}_x^0(z)|(1-\log^-|x-\mathcal{T}_x^0(z)|)f_1(t_0,z)\,dz,$$

where we used that f_1 is compactly supported in velocity. We write $\varphi_1(r) := r(1-\log^- r)$ and $\varphi_2(r) = -r\log r$. Note that $\varphi_1(r) \ge \varphi_2(r)$ on $r\in(0,1)$ and $\varphi_1(r)$ is nondecreasing on $r>0$. Moreover, $\varphi_2(r)$ is concave on $r<1$. Thus, by using Jensen's inequality,

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-\mathcal{T}_x^0(z)|(1-\log^-|x-\mathcal{T}_x^0(z)|)f_1(t_0,z)\,dz\\
&\quad= \left(\iint_{\{|x-\mathcal{T}_x^0(z)|\ge 1\}} + \iint_{\{|x-\mathcal{T}_x^0(z)|<1\}}\right)\\
&\qquad\qquad\qquad |x-\mathcal{T}_x^0(z)|(1-\log^-|x-\mathcal{T}_x^0(z)|)f_1(t_0,z)\,dz\\
&\quad\le \iint_{\{|x-\mathcal{T}_x^0(z)|\ge 1\}} |x-\mathcal{T}_x^0(z)|f_1(t_0,z)\,dz\\
&\qquad + \varphi_2\left(\iint_{\{|x-\mathcal{T}_x^0(z)|<1\}} |x-\mathcal{T}_x^0(z)|f_1(t_0,z)\,dz\right)\\
&\quad\le \iint_{\{|x-\mathcal{T}_x^0(z)|\ge 1\}} |x-\mathcal{T}_x^0(z)|f_1(t_0,z)\,dz\\
&\qquad + \varphi_1\left(\int_{\{|x-\mathcal{T}_x^0(z)|<1\}} |x-\mathcal{T}_x^0(z)|f_1(t_0,z)\,dz\right)\\
&\quad\le \mathrm{d}_1(f_1(t_0),f_2(t_0)) + \varphi_1(\mathrm{d}_1(f_1(t_0),f_2(t_0)))\\
&\quad\le C\mathrm{d}_1(f_1(t_0),f_2(t_0))(1-\log^-(\mathrm{d}_1(f_1(t_0),f_2(t_0))).
\end{aligned}$$

This implies

$$II_1 \le C\mathrm{d}_1(f_1(t_0),f_2(t_0))(1-\log^-(\mathrm{d}_1(f_1(t_0),f_2(t_0))).$$

For II_2, we use (3.2) and a simplified notation $\hat z := (\hat x,\hat v)$ to estimate

$$\begin{aligned}
II_2 &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left| \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(\phi(\mathcal{T}_x^0(z)-\hat x)\hat v - \phi(\mathcal{T}_x^0(z)-\mathcal{T}_x^0(\hat z))\mathcal{T}_v^0(\hat z)\right)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\times f_1(t_0,\hat z)\,d\hat z\right| f_1(t_0,z)\,dz\\
&\le \iiiint_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |\phi(x-\hat x)\hat v - \phi(x-\mathcal{T}_x^0(\hat z))\mathcal{T}_v^0(\hat z)| f_1(t_0,\hat z)f_2(t_0,z)\,d\hat z dz\\
&\le \iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |\phi(\hat x - x)-\phi(\mathcal{T}_x^0(\hat z)-x)|\rho_2(t_0,x)\,dx\right)|\hat v| f_1(t_0,\hat z)\,d\hat z\\
&\quad + \iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \phi(\mathcal{T}_x^0(\hat z)-x)\rho_2(t_0,x)\,dx\right)|\hat v - \mathcal{T}_v^0(\hat z)| f_1(t_0,\hat z)\,d\hat z.
\end{aligned}$$

We then use the similar arguments employed in Lemma 3.2 to yield

$$II_2 \le C\iint_{\mathbb{R}^d\times\mathbb{R}^d} |\hat x-\mathcal{T}_x^0(\hat z)|(1-\log^-|\hat x-\mathcal{T}_x^0(\hat z)|)f_1(t_0,\hat z)\,d\hat z$$

$$
\begin{aligned}
&+ C\|\phi \star \rho_2\|_{L^\infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\hat{v} - \mathcal{T}_v^0(\hat{z})| f_1(t_0, \hat{z})\, d\hat{z} \\
&\le C \mathrm{d}_1(f_1(t_0), f_2(t_0))(1 - \log^-(\mathrm{d}_1(f_1(t_0), f_2(t_0)))).
\end{aligned}
$$

For III, we estimate

$$
\begin{aligned}
III &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| v\phi \star \rho_1(x) - \mathcal{T}_v^0(z)(\phi \star \rho_2)(\mathcal{T}_x^0(z)) \right| f_1(t_0, z)\, dz \\
&\le \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| \phi \star \rho_1(x) - (\phi \star \rho_2)(\mathcal{T}_x^0(z)) \right| |v| f_1(t_0, z)\, dxdv \\
&\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\phi \star \rho_2)(\mathcal{T}_x^0(z)) |v - \mathcal{T}_v^0(z)| f_1(t_0, z)\, dz \\
&=: III_1 + III_2.
\end{aligned}
$$

We can use the similar arguments for II_2 to yield

$$
III_1 \le C \mathrm{d}_1(f_1(t_0), f_2(t_0))(1 - \log^-(\mathrm{d}_1(f_1(t_0), f_2(t_0)))).
$$

We also directly have

$$
III_2 \le C \mathrm{d}_1(f_1(t_0), f_2(t_0)).
$$

Therefore, we combine all the estimates for I, II, and III to get

$$
\frac{d}{dt} Q(t) \Big|_{t = t_0^+} \le C \mathrm{d}_1(f_1(t_0), f_2(t_0)) \left(1 - \log^-(\mathrm{d}_1(f_1(t_0), f_2(t_0)))\right).
$$

This completes the proof. □

Acknowledgments

The first author was supported by National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No. 2022R1A2C1002820). The second author was supported by NRF grant (No. RS-2022-00165600).

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