—NOTES—

ON HERZBERGER'S DIRECT METHOD IN GEOMETRICAL OPTICS*

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1. Introduction. In recent papers M. Herzberger[1,2] has developed a "direct method" for analytical ray-tracing through an instrument of revolution. At the end of the first paper he refers to Hamilton's method, which he says "leads to an elimination problem, hitherto unsolved." Nevertheless the question arises: What is the connection between Herzberger's approach and that of Hamilton? This question is best answered by attacking Herzberger's problem by the method of Hamilton. As we shall see, this is quite feasible. Indeed, if we combine Herzberger's "direct method" with Hamilton's character function we obtain a very powerful technique.

Section 2 contains the formulation of the problem of determining the Herzberger transformation when Hamilton's angle-characteristic is known for the instrument in question. Herzberger's identity ($AD-BC=1$) is obtained immediately.

In Section 3 the case of a single surface (refracting or reflecting) is considered. It is found that the coefficients are connected by a new relation.

In Section 4 I show how the problem of the sphere may be treated, Herzberger's geometrical approach being replaced by a more systematic analytical method.

2. The Herzberger transformation. To facilitate comparison with Herzberger's work, I shall use his notation. The following table shows the correspondence between the notations of Herzberger and Hamilton:

	Herzberger	Hamilton
Coordinates of point on incident ray	$x,\ y,\ z$	$x',\ y',\ z'$
Components of incident ray	$\xi,\ \eta,\ \zeta$	$\sigma',\ \tau',\ \upsilon'$
Coordinates of point on final ray	$x',\ y',\ z'$	$x,\ y,\ z$
Component of final ray	$\xi',\ \eta',\ \zeta'$	$\sigma,\ \tau,\ \upsilon$

According to the method of Hamilton there exists an angle-characteristic T, a function of ξ, η, ξ', η', such that the equations of the incident and final rays are[3]

$$\begin{aligned} x - z\xi/\zeta &= T_{\xi}, & x' - z'\xi'/\zeta' &= -T_{\xi'}, \\ y - z\eta/\zeta &= T_{\eta}, & y' - z'\eta'/\zeta' &= -T_{\eta'}. \end{aligned} \tag{2.1}$$

The subscripts denote partial derivatives.

Now suppose that the instrument is of revolution and that the axes Oz, $O'z'$ lie along its axis. Then T is a function of the quantities

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[1] M. Herzberger, Trans. Amer. Math. Soc. **53**, 218–229 (1943).

[2] M. Herzberger, Quarterly of Applied Mathematics, **1**, 69–77 (1943).

[3] J. L. Synge, *Geometrical optics*, Cambridge, 1937, p. 31.

$$u_3 = \tfrac{1}{2}(\xi^2 + \eta^2), \qquad u_4 = \xi\xi' + \eta\eta', \qquad u_5 = \tfrac{1}{2}(\xi'^2 + \eta'^2). \tag{2.2}$$

Let us write $\partial T/\partial u_3 = T_3$, etc. Then, by (2.1), the intersections of the rays with the planes $z=0$, $z'=0$, satisfy

$$\begin{aligned} x &= T_3\xi + T_4\xi', & x' &= -T_4\xi - T_5\xi', \\ y &= T_3\eta + T_4\eta', & y' &= -T_4\eta - T_5\eta'. \end{aligned} \tag{2.3}$$

These equations involve the eight quantities

$$x',\ y',\ \xi',\ \eta'; \qquad x,\ y,\ \xi,\ \eta.$$

The basis of the Herzberger method is to express the first set in terms of the second set. To do this, we introduce

$$u_1 = \tfrac{1}{2}(x^2 + y^2), \qquad u_2 = x\xi + y\eta. \tag{2.4}$$

Let us multiply the x, y equations in (2.3) by ξ, η, respectively, and add; this gives

$$u_2 = 2T_3u_3 + T_4u_4. \tag{2.5}$$

Rearranging the x, y equations in (2.3), squaring and adding, we get

$$T_4^2u_5 = u_1 - T_3u_2 + T_3^2u_3. \tag{2.6}$$

Supposing T known as a function of u_3, u_4, u_5, we have in (2.5), (2.6) two equations to determine u_4, u_5 in terms of u_1, u_2, u_3; suppose the solutions are

$$u_4 = f(u_1, u_2, u_3), \qquad u_5 = g(u_1, u_2, u_3). \tag{2.7}$$

Making this substitution, we may express T_3, T_4, T_5 as functions of u_1, u_2, u_3.

Now let us rearrange (2.3) into the Herzberger form:

$$\begin{aligned} x' &= Ax + B\xi, & \xi' &= Cx + D\xi, \\ y' &= Ay + B\eta, & \eta' &= Cy + D\eta. \end{aligned} \tag{2.8}$$

The coefficients are as follows:

$$A = -T_5T_4^{-1}, \qquad B = T_3T_5T_4^{-1} - T_4, \qquad C = T_4^{-1}, \qquad D = -T_3T_4^{-1}. \tag{2.9}$$

We immediately deduce Herzberger's identity

$$AD - BC = 1. \tag{2.10}$$

To sum up: *Given the angle-characteristic* $T(u_3, u_4, u_5)$ *of an instrument of revolution, we obtain the coefficients* A, B, C, D *of the Herzberger transformation in two steps:*

(i) *We solve* (2.5), (2.6) *for* u_4, u_5 *in terms of* u_1, u_2, u_3.

(ii) *We substitute these values in* (2.9), *and so obtain* A, B, C, D *in terms of* u_1, u_2, u_3.

For future reference, let us solve (2.9) for T_3, T_4, T_5:

$$T_3 = -DC^{-1}, \qquad T_4 = C^{-1}, \qquad T_5 = -AC^{-1}. \tag{2.11}$$

3. An identity satisfied by the Herzberger coefficients for a single surface. Consider a surface of revolution

$$z = f(r), \qquad r^2 = x^2 + y^2. \tag{3.1}$$

For refraction or reflection at this surface, the angle-characteristic is[4] (if we take the origins O, O' coincident)

$$T = (\xi - \xi')x + (\eta - \eta')y + (\zeta - \zeta')z, \tag{3.2}$$

from which x, y, z are to be eliminated by the relations

$$\frac{\xi - \xi'}{\zeta - \zeta'} = -\frac{\partial z}{\partial x} = -f'(r)\frac{x}{r}, \qquad \frac{\eta - \eta'}{\zeta - \zeta'} = -f'(r)\frac{y}{r}\cdot \tag{3.3}$$

It is clear that T will be a function of the two quantities

$$\phi = \tfrac{1}{2}[(\xi - \xi')^2 + (\eta - \eta')^2], \qquad \psi = \zeta - \zeta',$$

or, in the notation of (2.2),

$$\phi = u_3 - u_4 + u_5, \qquad \psi = \theta(n^2 - 2u_3)^{1/2} - \theta'(n'^2 - 2u_5)^{1/2}. \tag{3.4}$$

Here n, n' are the refractive indices of the initial and final media, and θ, θ' are ± 1; for refraction we have $\theta\theta' = 1$, and for reflection $\theta\theta' = -1$. If we take refraction with the rays proceeding in the positive sense, we have

$$\theta = \theta' = 1. \tag{3.5}$$

If we take reflection with the incident rays in the positive sense, we have

$$\theta = 1, \qquad \theta' = -1, \qquad n = n'. \tag{3.6}$$

By (3.4) we have

$$\begin{aligned} T_3 &= T_\phi - T_\psi\theta(n^2 - 2u_3)^{-1/2}, \\ T_4 &= -T_\phi, \\ T_5 &= T_\phi + T_\psi\theta'(n'^2 - 2u_5)^{-1/2}. \end{aligned} \tag{3.7}$$

Hence

$$\frac{T_3 + T_4}{T_4 + T_5} = -k\frac{(n'^2 - 2u_5)^{1/2}}{(n^2 - 2u_3)^{1/2}}, \tag{3.8}$$

where

$$\begin{aligned} k &= 1 \text{ for refraction,} \\ k &= -1 \text{ for reflection.} \end{aligned} \tag{3.9}$$

Let us substitute from (2.11) in (3.8); this gives

$$\frac{D - 1}{A - 1} = -k\frac{(n'^2 - 2u_5)^{1/2}}{(n^2 - 2u_3)^{1/2}}, \tag{3.10}$$

and so

$$u_5 = \tfrac{1}{2}n'^2 - \tfrac{1}{2}(n^2 - 2u_3)\left(\frac{D - 1}{A - 1}\right)^2. \tag{3.11}$$

When we substitute this value in (2.6), and at the same time substitute for T_3, T_4 from (2.11), we get

[4] Synge, op. cit., 33.

$$n'^2 - (n^2 - 2u_3)\left(\frac{D-1}{A-1}\right)^2 = 2(C^2u_1 + CDu_2 + D^2u_3). \tag{3.12}$$

To sum up: *For refraction or reflection at a surface of revolution, the coefficients* A, C, D *are connected by the identity* (3.12).

If $B=0$, then $A=D^{-1}$, and (3.12) simplifies to

$$n'^2 - n^2D^2 = 2C(Cu_1 + Du_2). \tag{3.13}$$

As an alternative procedure we may use the fact that T is of the form

$$T = (\zeta - \zeta')F(\chi), \tag{3.14}$$

where

$$\chi = [(\xi - \xi')^2 + (\eta - \eta')^2]/(\zeta - \zeta')^2. \tag{3.15}$$

This is evident from (3.2) and (3.3); the form of the function F depends on the form of the surface. On differentiating (3.14) we obtain three equations analogous to (3.7), but containing F and its derivative on the right hand sides. If we eliminate these quantities we obtain (3.8) and hence the identity (3.12).

4. The Herzberger transformation for a sphere. Let us take the origins O, O' at the center of a sphere of radius $|r|$. The angle characteristic for refraction or reflection at the sphere is[5]

$$T = \pm |r| [(\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2]^{1/2}. \tag{4.1}$$

If we suppose the rays incident in the positive sense, all ambiguities of sign are removed by writing

$$T = r(kn' - n)\rho^{1/2}, \tag{4.2}$$

where

$$\rho = 1 + \frac{2}{(kn' - n)^2}[knn' - u_4 - k(n^2 - 2u_3)^{1/2}(n'^2 - 2u_5)^{1/2}]. \tag{4.3}$$

Here r is positive if the rays are incident on the convex side, and negative if they are incident on the concave side; $k=1$ for refraction and $k=-1$ for reflection. All roots are positive.

We have then

$$\begin{aligned} T_3 &= kr(kn' - n)^{-1}\rho^{-1/2}\frac{(n'^2 - 2u_5)^{1/2}}{(n^2 - 2u_3)^{1/2}}, \\ T_4 &= -r(kn' - n)^{-1}\rho^{-1/2}, \\ T_5 &= kr(kn' - n)^{-1}\rho^{-1/2}\frac{(n^2 - 2u_3)^{1/2}}{(n'^2 - 2u_5)^{1/2}}. \end{aligned} \tag{4.4}$$

It is evident that

$$T_4^2 = T_3T_5, \tag{4.5}$$

and so, by (2.9) and (2.10),

$$B = 0, \qquad A = D^{-1}. \tag{4.6}$$

[5] Synge, op. cit., p. 36.

We now solve (4.4) for u_4, u_5 in terms of T_3, T_4, obtaining

$$u_4 = k(n^2 - 2u_3)\frac{T_3}{T_4} - \frac{1}{2}\frac{r^2}{T_4^2} + \frac{1}{2}(n'^2 + n^2),$$

$$u_5 = \frac{1}{2}n'^2 - \frac{1}{2}(n^2 - 2u_3)\frac{T_3^2}{T_4^2}. \tag{4.7}$$

Substitution of these values into (2.5), (2.6) gives

$$\begin{aligned} u_2T_4 &= \tfrac{1}{2}(n^2 + n'^2)T_4^2 + n^2T_3T_4 - \tfrac{1}{2}r^2, \\ u_1 &= u_2T_3 + \tfrac{1}{2}n'^2T_4^2 - \tfrac{1}{2}n^2T_3^2. \end{aligned} \tag{4.8}$$

These are two equations for T_3, T_4; they may be written

$$T_3 = n^{-2}[u_2 - \tfrac{1}{2}(n^2 + n'^2)T_4 + \tfrac{1}{2}r^2T_4^{-1}], \tag{4.9}$$

$$T_4^4(n^2 - n'^2)^2 + 4T_4^2[p^2 - \tfrac{1}{2}r^2(n^2 + n'^2)] + r^4 = 0, \tag{4.10}$$

where (in Herzberger's notation)

$$p^2 = 2n^2u_1 - u_2^2. \tag{4.11}$$

Solving (4.10) we get, after some simple reductions,

$$C = T_4^{-1} = r^{-1}[\theta_1(n^2 - p^2/r^2)^{1/2} + \theta_2(n'^2 - p^2/r^2)^{1/2}], \tag{4.12}$$

where θ_1 and θ_2 are each ± 1, for the moment undetermined. We remove the ambiguity of sign by considering the case $\xi = \eta = 0$, so that by (2.8) $\xi' = Cx$, $\eta' = Cy$. It is evident from elementary considerations that C has the same sign as $(n - kn')/r$. Therefore $\theta_1 = 1$, $\theta_2 = -k$, and so in general

$$C = r^{-1}[(n^2 - p^2/r^2)^{1/2} - k(n'^2 - p^2/r^2)^{1/2}]. \tag{4.13}$$

By (2.11) and (4.9) we have

$$D = n^{-2}[p^2/r^2 + k(n^2 - p^2/r^2)^{1/2}(n'^2 - p^2/r^2)^{1/2} - u_2C]. \tag{4.14}$$

We verify that if $x = y = 0$, then $D = kn'/n$, as it must be by (2.8) from elementary considerations. It is easy to check that (3.13) is satisfied by (4.13), (4.14).

For the case of refraction ($k = 1$) the formula (4.13) agrees with Herzberger's equation (36)[1], except for a reversal of sign.