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THE ANTENNA PROBLEM*

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1. Introduction. The recent expansion of radio towards ultra short waves has aroused a new interest in theoretical problems of electro-magnetism and especially in the problem of antenna oscillations and radiation properties. The type of approximate discussions used by radio engineers for the case of long wave lengths is of little practical value for ultra short waves, where a more rigorous theory is needed, because the diameter of the antenna wire can no longer be considered as very small when compared to the wave length.

Some older calculations on rather thick antennas have already been found very useful. M. Abraham's[1] discussion of the vibrations of very long ellipsoids has often been referred to. A complete discussion of proper vibrations of ellipsoids of revolution may be found in M. Brillouin's book *Propagation de l'électricité* (Hermann, Paris, 1904, pp. 314–395) with numerical tables for all eccentricities, from the sphere to rather thin ellipsoids. More recently, L. Page and N. I. Adams, and subsequently R. M. Ryder[2] have discussed the free and forced oscillations of all types of prolate ellipsoids of revolution; while Barrow,[3] Schelkunoff,[4] and others have treated the problem of the biconical antenna and its free or forced oscillations. Mie and Debye[5] had formerly discussed the free vibrations of the sphere. In most of these papers, the theory was based on a computation of the whole field distribution around the antenna with the proper boundary conditions on the surface of the antenna. For a perfect metal, for instance, the electric field must be orthogonal to the metal surface.

The aim of the present paper is to emphasize the practical importance of another method based on the use of retarded potentials. The principle of the procedure was indicated a long time ago,[6] and the method was recently applied by Hallen and

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[1] M. Abraham, Ann. d. Physik, **66**, 435 (1898); Math. Ann. **52**, 81 (1899).

[2] L. Page and N. I. Adams, Phys. Rev. **53**, 819 (1938); R. M. Ryder, Appl. Phys. **13**, 327 (1942).

[3] W. L. Barrow, L. J. Chu, J. J. Jansen, Proc. I.R.E., **27**, 769 (1939).

[4] S. A. Schelkunoff, Trans. A.I.E.E., **57**, 744 (1938); Proc. I.R.E., **29**, 493 (1941).

[5] G. Mie, Ann. d. Phys., **25**, 377 (1908); P. Debye, Ann. d. Physik, **30**, 59 (1909).

[6] H. C. Pocklington, Proc. Cambridge Phil. Soc., **9**, 324 (1897); Lord Rayleigh, Proc. Roy. Soc., Ser. A, **87**, 193 (1912); C. W. Oseen, Ark. f. Mat. Astr. Fysik, **9**, No. 12 (1913); L. Brillouin, Radio-électricité, **3**, 147 (1922).

Ronold King[7] to the actual computation of antennas. The finite conductivity of a real metal can be taken into account, but there are still a few basic questions to be discussed, and these will appear more clearly in the problem of a perfect metal with infinite conductivity.

The principle of the method is the following: let us first assume a very thin wire and call s a distance measured along the wire. The problem is to find the current distribution, $I(s, t)$, along the antenna wire. To such a current, I, there corresponds a charge density, $\sigma(s, t)$, by the condition of conservation of electricity

$$-\frac{\partial \sigma}{\partial t} = \frac{\partial I}{\partial s}, \tag{1}$$

or, if we assume the following time dependence $I(s, t) = I(s)e^{i\omega t}$,

$$\sigma(s, t) = \frac{i}{\omega} \frac{\partial I}{\partial s} e^{i\omega t}. \tag{2}$$

Here, real ω means sustained oscillations; while proper oscillations of the antenna array will yield complex proper values ω, the imaginary part corresponding to radiative damping.

An arbitrary current distribution (1), creates an electromagnetic field in the whole space which satisfies Maxwell's equations. This field can be readily computed by the *method of retarded potentials*. In particular, the field on the surface of the metal wire can be obtained in this way; and one may then write the necessary boundary condition, that this electric field shall be orthogonal to the surface. This yields an integro-differential equation which is perfectly rigorous and whose solution is the actual current distribution required.

Using retarded potentials, one is certain to obtain, at a large distance, a field distribution corresponding to a wave spreading out of the antenna. It should be emphasized, however, that the same method can *not always be used* for the computation of oscillations inside a *closed tank resonator*, where the oscillations are of the type of standing waves and have no outside radiation (advanced potentials may sometimes be needed too).

The proper values of this integral equation give the proper frequencies (including damping) of the antenna. The same method can be used to study forced vibrations, if one assumes an outer electric field acting on the antenna (receiving antenna) or a certain electromotive force inserted in the circuit (transmitting antenna). In the second case, one must take into account, for the computation of the retarded potentials, the field radiated from a dipole representing the electromotive force.

Let us discuss the free vibrations of an antenna. The field at a point P is given by the well known formulae:

$$\begin{aligned} h_x &= -\frac{\partial V}{\partial x} - \frac{\partial F_x}{\partial t}, \ldots, \ldots; \quad V = \frac{1}{\epsilon_0}\int \frac{\sigma^* ds}{r}; \\ \mu_0 H_x &= \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \ldots, \ldots; \quad \vec{F} = \mu_0 \int \frac{\vec{i}^* ds}{r}; \end{aligned} \tag{3}$$

[7] E. Hallen, Uppsala Univ. Arsskrift 1930, No. 1; Nova Acta, Uppsala, Ser. 4, **11**, No. 4 (1938); L. V. King, Trans. Roy. Soc. London, **236**, 381 (1937); Ronold King and F. G. Blake, Proc. I.R.E., **30**, 335 (1942).

$\vec{h}$, electric field; $\vec{H}$, magnetic field; V, scalar potential; $\vec{F}$, vector potential; r distance from the element ds on the circuit to the point P where the field is observed; σ^*, $\vec{i}^*$ charge and the current at the time $t-r/c$; ϵ_0, μ_0 dielectric constant and permeability in vacuum, in non-rationalized units (rationalized units introduce a $1/4\pi$ factor in the formulae for both potentials). Let us assume an antenna consisting of a straight wire along the z axis, extending from $z=0$ to $z=l$. We need the z component, h_z, of the electric field along the wire and must write that this longitudinal component vanishes:

$$h_z = -\frac{\partial V}{\partial z} - \frac{\partial F_z}{\partial t}, \qquad i^* = I(z')e^{i(\omega t - kr)}, \qquad k = \frac{\omega}{c} = \omega\sqrt{\epsilon_0\mu_0}\,,$$
$$V = \frac{ie^{i\omega t}}{\epsilon_0\omega}\int_0^l \frac{\partial I(z')}{\partial z'}\frac{e^{-ikr}}{r}\,dz', \qquad F_z = \mu_0 e^{i\omega t}\int_0^l I(z')\frac{e^{-ikr}}{r}\,dz'. \tag{4}$$

The field at point z is the result of integration over all the points, z', of the antenna. Finally, we obtain the condition

$$h_z = \frac{i}{\epsilon_0\omega}\int_0^l \left[-\frac{\partial I(z')}{\partial z'}\frac{\partial G(r)}{\partial z} - k^2 I(z')G(r)\right]dz' = 0,$$

putting

$$G(r) = \frac{e^{-ikr}}{r}, \qquad r = |z - z'|. \tag{5}$$

This is our fundamental integro-differential equation for the straight antenna.

One difficulty appears immediately: G is infinite for $r=0$, $z=z'$. This means that one must take into account the radius of the wire; but when this radius, a, is explicitly introduced in the calculation, there is an additional condition to be written for both ends of the wire. Here most authors do not attempt to write rigorous conditions; they are satisfied with approximations corresponding to the problem of very thin wires. They neglect a/l but keep terms in Ω^{-1}, Ω^{-2}, $\cdots$ where

$$\Omega = 2\log\frac{l}{a}\cdot \tag{6}$$

Such a procedure is suggested by the similar approximations used by M. Abraham in his discussion of ellipsoids. It should work correctly when $\Omega > 14$, which means $l/a > 1000$, but could certainly not be relied upon for thicker wires.

Furthermore, Oseen and Hallen both use the following assumptions:

$$\begin{aligned} &\text{A)} \quad I(0) = 0,\ I(l) = 0, \text{ current zero at both ends;} \\ &\text{B)} \quad r = [(z - z')^2 + a^2]^{1/2}. \end{aligned} \tag{7}$$

The first condition, A, is not quite correct, since there must be a small current at both ends in order to charge and discharge the terminal capacities. It is only for the case of a hollow cylinder that the current would be exactly zero at both ends; and this hollow pipe is a very special case, as shall be seen later.

The second assumption, B, is explained differently by both writers. Oseen assumes a *current flowing along the axis* of the cylindrical wire and computes the field h_z, Eq. (5), on the surface. Hence his boundary condition (5) is right, but the assumption about the axial current is certainly wrong. Indeed, owing to the skin effect, the actual electric current, in a perfect conductor, flows along the surface. Oseen assumes that the field created by this actual superficial current could be obtained by a fictitious axial current. This may be right for very thin wires, but the assumption is obviously wrong for thick wires or for cylinders of large radius. Moreover, it cannot be proved that the fictitious axial current satisfies the first assumption A. So Oseen hardly justifies the use of both assumptions A and B.

Hallen takes a different point of view. He starts from the well-known property that the current flows along the surface; but instead of computing the field on the surface of the same cylinder, he takes the h_z *field along the axis*. This field must certainly vanish; and from this fact, Eqs. (5) and (7 B) follow. This necessary condition, however, is not sufficient. One may very well have no longitudinal field along the axis and still find a longitudinal field on the surface of the cylinder. These approximations would probably be all right for very thin wires; but they can certainly not be used for thick wires, where B is wrong and A must be replaced by a more elaborate condition, in order to take account of the electric currents and charges on the flat ends of the cylinder.

2. Complete statement for a cylindrical wire of finite radius. The antenna is a solid cylinder of radius a and height l. The oscillations studied are those of cylindrical symmetry where the current is equally distributed around the cylinder and flows along the surface. $I(z', t)$ is the total current at z', and $(1/2\pi)\ I(z', t)d\varphi$ is the current through a small sector $d\varphi$ (Fig. 1); hence, $\sigma(z', t)dz'$, Eqs. (1), (2), is the charge per length dz', all around the cylinder, and $(1/2\pi)\sigma dz' d\varphi$ the charge for a small angle $d\varphi$. For the flat top of the cylinder ($z=l$), we call $I_l(\rho)$ the total radial current crossing a circle of radius ρ; while $\sigma_l(\rho)d\rho$ represents the electric charge between ρ and $\rho+d\rho$:

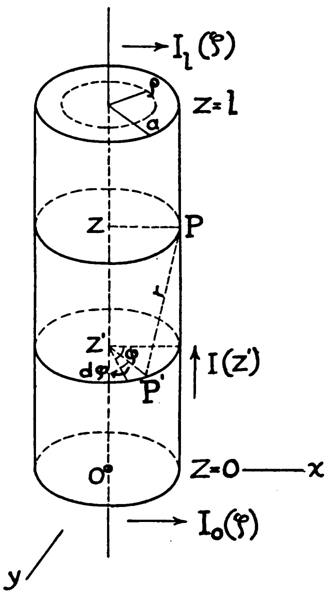

Fig. 1.

$$-\frac{\partial\sigma_l}{\partial t} = \frac{\partial I_l}{\partial\rho}, \qquad \sigma_l = \frac{i}{\omega}\frac{\partial I_l}{\partial\rho}. \tag{8}$$

Similar definitions apply for the bottom of the cylinder ($z=0$) with a current $I_0(\rho)$ and charge $\sigma_0(\rho)d\rho$. The positive signs correspond to the directions indicated by arrows in Fig. 1. The conditions for continuity of the current around the corners, at $z=0$ and $z=l$ read

$$I_l(a) = -I(l), \qquad I_0(a) = I(0). \tag{9}$$

Let us first study the fields and potentials *at a point* $P(z)$ *located on the cylindrical surface.* The potentials due to currents and charges along the cylinder are the following ($e^{i\omega t}$ factors have been dropped):

$$\begin{aligned} V_c(z) &= \frac{i}{\epsilon_0\omega}\int_0^l\int_0^{2\pi}\frac{\partial I(z')}{\partial z'}\frac{e^{-ikr}}{r}\frac{d\varphi}{2\pi}dz', \\ F_{cz}(z) &= \mu_0\int_0^l\int_0^{2\pi} I(z')\frac{e^{-ikr}}{r}\frac{d\varphi}{2\pi}dz'. \end{aligned} \tag{10}$$

The currents along the cylinder flow vertically; hence, there are no horizontal components F_{cx}, F_{cy} of the vector potential. The distance r is shown in Fig. 1.

On the flat top of the cylinder, the current flows radially in the horizontal plane; hence, the F_{lz} component is zero, but we find horizontal components, F_{lx} and F_{ly}, of the vector potential:

$$F_{lx}(z) = \mu_0 \int_0^a \int_0^{2\pi} I_l(\rho) \frac{e^{-ikr}}{r} \cos\varphi \frac{d\varphi}{2\pi} d\rho,$$
$$F_{ly}(z) = \mu_0 \int_0^a \int_0^{2\pi} I_l(\rho) \frac{e^{-ikr}}{r} \sin\varphi \frac{d\varphi}{2\pi} d\rho = 0. \tag{11}$$

The transverse components F_{ly}, for a point P in the x-z-plane, is obviously zero by symmetry:

$$V_l(z) = \frac{i}{\epsilon_0\omega} \int_0^a \int_0^{2\pi} \frac{\partial I_l}{\partial\rho} \frac{e^{-ikr}}{r} \frac{d\varphi}{2\pi} d\rho \tag{12}$$

and similar formulae for the potentials F_{0x} and V_0 due to currents and charges on the bottom of the cylinder.

The φ integrals are of two fundamental types which will now be explained in connection with Fig. 2.

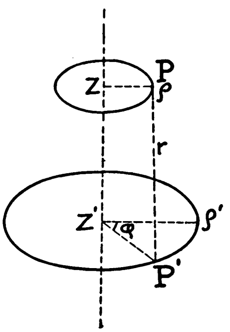

FIG. 2.

$$r^2 = (z - z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos\varphi, \tag{13}$$

$$G_k(\rho, \rho', z - z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ikr}}{r} d\varphi, \tag{14}$$

$$C_k(\rho, \rho', z - z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ikr}}{r} \cos\varphi d\varphi. \tag{15}$$

G_k and C_k are two functions which will be discussed more fully in section 5. They are symmetrical in ρ, ρ' and even functions of $z-z'$. With these functions, our formulae (10, 11, 12) read

$$V_c(z) = \frac{i}{\epsilon_0\omega} \int_0^l \frac{\partial I}{\partial z'} G_k(a, \rho, z - z')dz',$$
$$F_{cz}(z) = \mu_0 \int_0^l I(z')G_k(a, \rho, z - z')dz',$$
$$V_l(z) = \frac{i}{\epsilon_0\omega} \int_0^a \frac{\partial I_l}{\partial\rho'} G_k(\rho', \rho, z - l)d\rho', \tag{16}$$
$$F_{lx}(z) = \mu_0 \int_0^a I_l C_k(\rho', \rho, z - l)d\rho',$$

where $\rho = a$ for the point $P(z)$ on the cylinder.

We now are in a position to compute the longitudinal field $h_z(z)$ at the point P, according to Eqs. (4) and (16).

$$h_z = -\frac{\partial V_c}{\partial z} - \frac{\partial F_{cz}}{\partial t} - \frac{\partial V_0}{\partial z} - \frac{\partial V_l}{\partial z},$$

$$\begin{aligned} -\frac{\epsilon_0 \omega}{i} h_z(z) &= \int_0^l \left[\frac{\partial I}{\partial z'}\frac{\partial}{\partial z}G_k(a, a, z - z') + k^2 I(z')G_k(a, a, z - z')\right]dz' \\ &+ \int_0^a \frac{\partial I_l}{\partial \rho'}\frac{\partial}{\partial z}G_k(\rho', a, z - l)d\rho' + \int_0^a \frac{\partial I_0}{\partial \rho'}\frac{\partial}{\partial z}G_k(\rho', a, z)d\rho' = 0. \end{aligned} \tag{17}$$

This is the first integral equation of the problem which corresponds to Eq. (5) for the simplified example of a thin wire. It should be noticed immediately that in the first integral

$$\frac{\partial}{\partial z}G_k(a, a, z - z') = -\frac{\partial}{\partial z'}G_k(a, a, z - z'). \tag{17a}$$

This transformation will be very useful, afterwards, in applying integration by parts.

Another integral equation is obtained by writing the fact that the horizontal field component is zero at a point $P(\rho)$ on the top of the cylinder:

$$h_x(\rho) = -\frac{\partial V_c}{\partial x} - \frac{\partial V_l}{\partial x} - \frac{\partial V_0}{\partial x} - \frac{\partial F_{lx}}{\partial t} - \frac{\partial F_{0x}}{\partial t}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \rho},$$

$$\begin{aligned} -\frac{\epsilon_0 \omega}{i} h_x(\rho) &= \int_0^l \frac{\partial I}{\partial z'}\frac{\partial}{\partial \rho}G_k(a, \rho, l - z')dz' \\ &+ \int_0^a \frac{\partial I_l}{\partial \rho'}\frac{\partial}{\partial \rho}G_k(\rho', \rho, 0)d\rho' + \int_0^a \frac{\partial I_0}{\partial \rho'}\frac{\partial}{\partial \rho}G_k(\rho', \rho, l)d\rho' \\ &+ k^2 \int_0^a [I_l(\rho')C_k(\rho', \rho, 0) + I_0(\rho')C_k(\rho', \rho, l)]d\rho' = 0. \end{aligned} \tag{18}$$

A similar equation could be written for the bottom of the cylinder; but this is actually not needed, since it reduces to (18) by reason of symmetry.

The proper oscillations of the cylinder can be divided into two groups:

$$\begin{aligned} &\text{symmetrical oscillations } I_l(\rho') = I_0(\rho'),\ I(l - z) = -I(z), \\ &\qquad \frac{\partial I(l - z)}{\partial z} = \frac{\partial I(z)}{\partial z}; \\ &\text{antisymmetrical oscillations } I_l(\rho') = -I_0(\rho'),\ I(l - z) = I(z), \\ &\qquad \frac{\partial I(l - z)}{\partial z} = -\frac{\partial I(z)}{\partial z}. \end{aligned} \tag{19}$$

These two types will be discussed together in the following formulae. The upper sign corresponds to symmetrical and the lower sign to antisymmetrical vibrations.

3. Discussion of the first integral equation (17). Wave propagation along the cylinder. Equation (17) can now be written in the following way:

$$\int_0^l \left[-\frac{\partial I}{\partial z'} \frac{\partial}{\partial z'} G_k(a, a, z - z') + k^2 I(z') G_k(a, a, z - z') \right] dz'$$
$$= -\int_0^a \frac{\partial I_l}{\partial \rho'} \frac{\partial}{\partial z} [G_k(\rho', a, z - l) \pm G_k(\rho', a, z)] d\rho' = R(I_l, z). \tag{20}$$

The left hand integral contains only vertical currents, $I(z')$, along the cylindrical boundary; while the right hand terms, R, show the coupling between these vertical currents and the currents or charges on both flat ends of the cylinder.

Let us integrate the left hand integral by parts, starting from $\partial G_k/\partial z'$:

$$\int_0^l \left[\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') \right] G_k(a, a, z - z') dz' = \frac{\partial I}{\partial z'} G_k(a, a, z - z') \Big|_{z'=0}^{z'=l} + R(I_l, z)$$
$$= \left(\frac{\partial I}{\partial z'} \right)_{z'=l} [G_k(a, a, z - l) \mp G_k(a, a, z)] + R(I_l, z). \tag{21}$$

This new formula has been obtained without any approximations. Let us now make a few simplifying assumptions, in order to get a better understanding of the meaning of this equation.

For a very thin and long wire, $l \gg a$, we may neglect the $R(I_l, z)$ term, as both charges and currents on the flat terminals become very small. Furthermore, at a certain distance from the terminals, $G_k(a, a, z-l)$ and $G_k(a, a, z)$ are also very small, since G_k decreases approximately like $1/r$ for large distances. The only important term is the one on the left, which has the obvious solution

$$\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') = 0, \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}. \tag{22}$$

This shows wave propagation with the velocity of light along the major part of the wire. This result is obtained under the assumption $l \gg a$ and without any restriction about the wave length λ, which can be of the order a or even smaller; but it holds only for the medium part of the wire, far away from both ends.[8]

This shows the connection with the usual elementary theory of antennas. The classical discussion[9] starts from the assumption of sinusoidal standing waves along the wire, which cancels out completely the left hand integral in equation (21). Then, using this current distribution, the longitudinal field along the wire may be computed; and according to (17) and (21) it comes out as

$$h_z(z) = \frac{i}{\epsilon_0 \omega} \left\{ \left(\frac{\partial I}{\partial z'} \right)_{z'=l} [G_k(a, a, z - l) \mp G_k(a, a, z)] + R(I_l, z) \right\}. \tag{23}$$

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[8] It should be emphasized, here, that our discussion is limited to the case of oscillations with cylindrical symmetry (see beginning of Section 2). Vibrations with nodal lines parallel to the axis are not included.

[9] L. Brillouin, Radio-électricité, loc. cit.

J. A. Stratton, *Electromagnetic theory*, McGraw-Hill, New York, 1941, pp. 455–460. Stratton uses rational units, hence a $1/4\pi$ factor before the integrals, and he uses the opposite sign before i.

This plays the role of a small additional average impedance Z along the antenna, which can be defined by

$$Z\,\overline{I^2} = \int_0^l h_z(z) I(z) dz. \tag{23a}$$

The real part of Z is called the *radiation resistance*, Z_r, and the expression $Z_r\overline{I^2}$ represents the energy, W, radiated at large distance (see Stratton, p. 458), from which the damping of the antenna oscillations may be computed. For a very thin wire, one may neglect the term $R(I_l, z)$, which represents the role played by the currents and charges on both flat terminals of the wire; and one may take for G_k the expression $(1/r)e^{-ikr}$ as in Eq. (5). With these approximations, our equation (23) becomes identical with Stratton's Eq. (76a), p. 457.

It should be noticed that Eq. (23) is physically wrong, as we know in advance that the longitudinal electric field along the wire is zero. These equations (23) and (23a) merely represent a second approximation in a system of successive approximations starting from (22). An attempt will be made, in the next section, to build up a consistent system of approximations of similar structure.

Returning now to Eq. (20), we may try another integration by parts, starting from $\partial I/\partial z'$, which yields

$$\int_0^l I(z') \left[\frac{\partial^2 G_k}{\partial z'^2} + k^2 G_k(a, a, z - z') \right] dz'$$
$$= I(z') \frac{\partial}{\partial z'} G_k(a, a, z - z') \Big|_{z'=0}^{z'=l} + R(I_l, z). \tag{24}$$

Let us again discuss this equation for a very thin wire. The term $R(I_l, z)$ represents the role of both terminals and may be neglected, $I(z')$ is zero at both ends ($z'=0$, $z'=l$), and consequently all the right hand terms are zero. This transformation is very closely connected with the one used by Schelkunoff and Feldman[10] in a recent paper. These authors discuss the problem of forced vibrations in a transmission antenna, instead of the free vibrations which we have in mind. They use both approximations (7A) and (7B) of Oseen and Hallen and take for G the simplified expression $(1/r)e^{-ikr}$, Eq. (5). These approximations may apply for a very thin wire. Furthermore, they split the $(1/r)e^{-ikr}$ function into its real and imaginary parts before performing the integration by parts. Their final result is actually identical with the one derived from the elementary theory and Eq. (23). This is not surprising, as both methods are very closely connected.

4. Principle of a method of successive approximations. As stated in the preceding section, it seems possible to build up a method of successive approximations in order to solve Eq. (21) along a way rather similar to the one followed in the classical elementary discussion.

First of all, we may split the integro-differential equation (21) into an integral equation and a differential equation, by writing:

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[10] S. A. Schelkunoff and C. B. Feldman, Proc. I.R.E., **30**, 511 (1942).

$$\int_0^l F(z')G_k(a, a, z - z')dz' = R'(z), \tag{25}$$

where $R'(z) = (\partial I/\partial z')_{z'=l}[G_k(a, a, z-l) \mp G_k(a, a, z)] + R(I_l, z)$,

$$\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') = F(z'). \tag{26}$$

The first equation is an integral equation of the first kind, with the kernel $G_k(z-z')$. Its solution can be written with the help of the resolving kernel $H_k(z'-z'')$, which satisfies the following conditions

$$\int_0^l G_k(z - z')H_k(z' - z'')dz' = \delta(z - z''), \tag{27}$$

$$F(z') = \int_0^l R'(z'')H_k(z' - z'')dz'', \tag{28}$$

where δ means a delta function. Hence, the first question is to build up the resolving kernel H_k, a problem for which some general methods have been developed. This being done, we are left with Eq. (26) to which we apply the usual Rayleigh-Schrödinger method of successive approximations. Let us first notice that the G_k function becomes very large for $z = z'$ which, according to (27), means that H_k is small. Thus we may rewrite (26) and state explicitly by an ϵ coefficient the smallness of the right hand term:

$$\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') = \epsilon\varphi(z'), \qquad F = \epsilon\varphi. \tag{26a}$$

Then we use the following expansions:

$$\begin{aligned} I(z') &= I_0(z') + \epsilon I_1(z') + \epsilon^2 I_2(z') \cdots, \\ k^2 &= k_0^2 + \epsilon\chi_1 + \epsilon^2\chi_2 \cdots \end{aligned} \tag{29}$$

and obtain the successive approximations:

$$\begin{aligned} \frac{\partial^2 I_0}{\partial z'^2} + k_0^2 I_0 &= 0, \\ \frac{\partial^2 I_1}{\partial z'^2} + k_0^2 I_1 &= -\chi_1 I_0 + \varphi, \\ \frac{\partial^2 I_2}{\partial z'^2} + k_0^2 I_2 &= -\chi_2 I_0 - \chi_1 I_1 \cdots \end{aligned} \tag{30}$$

I_0 is a sinusoidal function, as in the elementary treatment,

$$I_0 = A \sin k_0(z' + \zeta)$$

where the ζ constant is necessary in order to give a small but finite value for the current I_0 at the bottom of the cylinder ($z' = 0$). This is needed for the junction with the currents on the lower flat end of the cylinder. By symmetry, the correction at the upper end must also be ζ; hence,

$$k_0(l + 2\zeta) = n\pi, \qquad l + 2\zeta = n\,\frac{\lambda_0}{2}, \qquad k_0 = \frac{2\pi}{\lambda_0}\cdot \tag{31}$$

The constant ζ will be determined by means of the second integral equation (18) for the flat terminals. Now let us turn to the second equation (30). As is well known, it is necessary for the right hand term to be orthogonal to the solution of the homogeneous equation, which means

$$\int_0^l \sin k_0(z' + \zeta)[-\chi_1 I_0 + \varphi]dz' = 0$$

or

$$A\chi_1 \int_0^l \sin^2 k_0(z' + \zeta)dz' = \int_0^l \varphi \sin k_0(z' + \zeta)dz'. \tag{32}$$

This yields the correction χ_1 to the proper value k_0^2. It is readily seen that equation (32) is very similar to the relation (23a) used in the elementary theory to obtain the average "radiation resistance" of the antenna and thence the damping coefficient in the proper oscillations. The important point, however, is that equation (32) contains φ, which is not R' but is computed from R' by means of (28)–(26a).

Once χ_1 is obtained, the second equation (30) can be solved; then χ_2 is first computed by a similar orthogonality condition, and so on. Hence, the whole procedure should yield a solution along lines parallel to the elementary treatment and show how far the usual formulae can be trusted.

We may already go one step further and write the general expression of the function $F(z')$ on the basis of Eqs. (25) and (27):

$$\begin{aligned}\epsilon\varphi(z') = F(z') &= \int_0^l R'(z'')H_k(z' - z'')dz'' \\ &= \left(\frac{\partial I_0}{\partial z'}\right)_{z'=l}[\delta(z' - l) \mp \delta(z')] + \int_0^l R(I_l, z'')H_k(z' - z'')dz''.\end{aligned} \tag{33}$$

The δ functions appear here automatically, because G_k is an even function of $(z-z')$, and so is H_k for $z'-z''$; hence, the integral in (28) comes out as

$$\int G_k(z'' - l)H_k(z' - z'')dz'' = \int G_k(l - z'')H_k(z'' - z')dz'' = \delta(z' - l)$$

according to (27).

We can use the new expression (28) for the discussion of some simplified examples. Let us start with the *wire of vanishing radius*. The whole $R(I_l, z'')$ term, which represents the terminal effect, drops out; and we are left with an equation

$$\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') = F(z') = \left(\frac{\partial I}{\partial z'}\right)_{z'=l}[\delta(z' - l) \mp \delta(z')]. \tag{34}$$

from (26) and (28). The condition on both terminals is obviously $I(0) = I(l) = 0$; hence $\zeta = 0$ in (31), which results in the following equation:

$$I_0 = A \sin k_0 z', \qquad k_0 = n\pi/l. \tag{35}$$

$n=2m+1$: symmetrical oscillation, sign $- = (-1)^n$ in bracket,
$n=2m$: antisymmetrical oscillation, sign $+ = (-1)^n$ in bracket.

$\epsilon\varphi(z') = F(z') = Ak_0 \left[(-1)^n \delta(z'-l) + \delta(z')\right]$ as $\cos k_0 l = (-1)^n$ and Eq. (32) reduces to

$$\epsilon\chi_1 \frac{l}{2} = k_0 \int_0^l \left[(-1)^n \delta(z'-l) + \delta(z')\right] \sin k_0 z' dz'$$
$$= k_0\left[(-1)^n \sin(k_0 l) + \sin(k_0 0)\right] = 0 \tag{36}$$

which gives no damping at all. The physical explanation is the following: a finite amount of energy is radiated per second; but this does not mean any damping of the oscillations, because the energy accumulated in the field around the wire is infinite. As a matter of fact, both electric and magnetic fields are infinite as $1/r$ near the wire of infinitely small radius. The square of the field is of the order $1/r^2$; and the energy is obtained by multiplying by $2\pi r\, dr$ and integrating with respect to r, which gives logarithmic infinite terms. The situation is similar to the one obtained in a circuit with infinite L, zero capacity, and finite resistance R, which yields a negligible damping coefficient $R/2L$.

This shows the difficulties involved in the assumption (7A), as put forth by Oseen and Hallen. When such a condition is used in the rigorous Eqs. (25), (26), it leads directly to (36) and yields practically no damping.

Such is also the case for a *hollow cylinder*. Here again, there is no end effect, no terminals, no R term, and condition (7A) holds good. The whole procedure from (34) to (36) repeats itself and shows again no damping. Of course, the G_k and H_k functions would differ materially in both cases; but these functions have been eliminated from Eq. (34) and finally drop out.

The explanation is similar to the one given for the thin wire, but not quite so obvious. The problem of a hollow cylinder of indefinitely small thickness must be considered as the limit of a cylinder of finite wall thickness, as represented in Fig. 3. On such a cylinder, one should take into account, separately, a current I_l flowing along the external surface of the cylinder and another current I_i along the internal surface. At the limit, these two currents merge into a single one, for which the theory indicates a sinusoidal distribution. Hence, for a cylinder of finite thickness, there certainly is a current flowing around the edge of the cylinder, as shown in Fig. 3. On this edge, one must also consider the electric charge; and this results in an accumulation of electric fields and of electro-magnetic energy near the cylinder, while the energy radiated per second at large distance remains finite. Hence the damping becomes negligible.

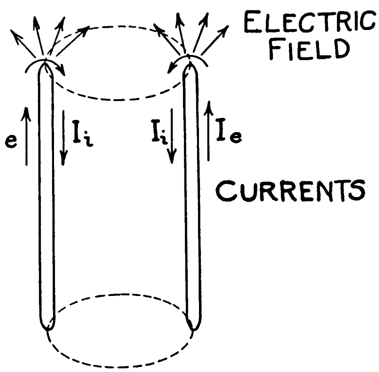

FIG. 3.

The result is general and applies for any hollow cylinder of indefinitely small thickness, whatever the shape of the cross-section might be. The field distribution inside the cylinder should correspond to a superposition of E_0 waves (transverse magnetic) and should show a strong decay from both ends down to the middle part of the cylinder, especially when the diameter of the cylinder is small compared to the wave length.

These two simple examples show the importance of the role played by the shape of both terminals and the danger of using assumptions like (7A) or (7B).

5. Some important formulae. We have introduced in (14), (15) two fundamental functions:

$$G_k(\rho, \rho', \zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ikr}}{r}\, d\varphi, \qquad \zeta = z - z',$$

$$C_k(\rho, \rho', \zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ikr}}{r} \cos \varphi d\varphi, \tag{37}$$

$$r^2 = \zeta^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos \varphi = q - 2p \cos \varphi, \qquad q \geqq 2p,$$

$$q = \zeta^2 + \rho^2 + \rho'^2, \qquad p = \rho\rho'.$$

From these relations, we see that G_k and C_k depend upon ρ, ρ', ζ only through the two combinations p and q. Furthermore, it is easily proved that

$$\frac{\partial G_k}{\partial p} = -2\frac{\partial C_k}{\partial q} = -\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^{-ikr}}{r}\right) \cos \varphi d\varphi. \tag{38}$$

C_k and G_k being both zero at infinity, this can be written as

$$C_k = \frac{1}{2}\int_q^{\infty} \frac{\partial G_k}{\partial p}\, dq. \tag{39}$$

These integrals are closely connected with the complete elliptic integrals K and D,[11] as is seen for a thin wire when the radius a is small compared with the wave length (ka small). The following expansions can be used:

$$\begin{aligned} r &= \sqrt{q - 2p \cos \varphi} = \sqrt{q} + [\sqrt{q - 2p \cos \varphi} - \sqrt{q}] \\ e^{-ikr} &= e^{-ik\sqrt{q}}\{1 - ik[\sqrt{q - 2p \cos \varphi} - \sqrt{q}] \cdots \}. \end{aligned} \tag{40}$$

The bracket [ ] is of the order of magnitude of a, and its product when multiplied by k is small:

$$G_k = e^{-ik\sqrt{q}}\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ik\sqrt{q}}{[q - 2p \cos \varphi]^{1/2}}\, d\varphi - ik \cdots \right\}$$

$$C_k = e^{-ik\sqrt{q}}\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ik\sqrt{q}}{[q - 2p \cos \varphi]^{1/2}} \cos \varphi d\varphi - 0 + \cdots \right\}. \tag{41}$$

We may write

$$q - 2p \cos \varphi = (q + 2p)(1 - \kappa^2 \sin^2 \psi)$$

$$\kappa^2 = \frac{4p}{q + 2p}, \qquad \psi = \frac{\varphi - \pi}{2}. \tag{42}$$

Hence

[11] E. Jahnke and F. Emde, *Tables of functions*, 2nd ed., Springer, Berlin, 1933, pp. 127–145.

$$\int_0^{2\pi} \frac{d\varphi}{[q - 2p \cos \varphi]^{1/2}} = \frac{2}{[q + 2p]^{1/2}} \int_{-\pi/2}^{\pi/2} \frac{d\psi}{[1 - \kappa^2 \sin^2 \psi]^{1/2}} = \frac{4}{[q + 2p]^{1/2}} K(\kappa)$$

and

$$G_k = e^{-ik\sqrt{q}} \left[\frac{2}{\pi} \cdot \frac{1 + ik\sqrt{q}}{[q + 2p]^{1/2}} K(\kappa) - ik \cdots \right]. \tag{43}$$

When $\zeta \to 0$, the variables q and p retain finite values; but when at the same time $\rho = \rho'$, $q = 2p$, then κ is 1 and K is logarithmically infinite. This could easily be foreseen and does not make any special trouble in the integrations. The second integral C_k is transformed in a similar way:

$$\int_0^{2\pi} \frac{\cos \varphi d\varphi}{[q - 2p \cos \varphi]^{1/2}} = \frac{2}{[q + 2p]^{1/2}} \int_{-\pi/2}^{\pi/2} \frac{2 \sin^2 \psi - 1}{[1 - \kappa^2 \sin^2 \psi]^{1/2}} d\psi$$

$$= \frac{4}{[q + 2p]^{1/2}} [2D(\kappa) - K(\kappa)], \tag{44}$$

$$C_k = e^{-ik\sqrt{q}} \left\{ \frac{2}{\pi} \cdot \frac{1 + ik\sqrt{q}}{[q + 2p]^{1/2}} [2D(\kappa) - K(\kappa)] \cdots \right\}.$$

These approximate formulae should be used for a thin wire and represent the first two terms in an expansion when a/λ is small but not negligible. For the fundamental vibration, λ is of the order of $2l$ (twice the length of the antenna). Hence using the expansions (43), (44), one should be able to go one step further than Oseen or Hallen, who completely neglected a/l and were satisfied with keeping terms in Ω^{-1}, Ω^{-2}, where

$$\Omega = 2 \log \frac{l}{a} \cdot \tag{6a}$$

This parameter comes in, when integrations are performed on D and K for κ near 1,

$$\kappa'^2 = 1 - \kappa^2 = \frac{q - 2p}{q + 2p} = \frac{\zeta^2 + (\rho - \rho')^2}{\zeta^2 + (\rho + \rho')^2}, \quad \text{small;} \quad \zeta = z - z'.$$

This happens when z and z' are nearly equal for two points on the cylindrical surface $\rho = \rho' = a$. It happens again for two points on one of the flat terminals, when $z = z' = 0$ or l, and ρ is nearly ρ'. In such cases, K and D are represented by the following expansions (Jahnke-Emde, p. 145)

$$K = \Lambda + \frac{\Lambda - 1}{4} \kappa'^2 \cdots, \qquad D = \Lambda - 1 + \tfrac{3}{4}(\Lambda - \tfrac{4}{3})\kappa'^2 \cdots,$$

$$\Lambda = \log \frac{4}{\kappa'} = \log 4 - \tfrac{1}{2} \log \kappa'^2 = \log 4 - \tfrac{1}{2} \log \frac{(z - z')^2 + (\rho - \rho')^2}{(z - z')^2 + (\rho + \rho')^2} \cdot \tag{45}$$

Integration and averaging process carried out on Λ will introduce the parameter Ω.

Finally, let us discuss the dependence on k of both functions G_k and C_k. From the definition itself (37), it is seen that both functions can be expressed in terms of G_1, C_1 corresponding to $k = 1$,

$$G_k = \frac{k}{2\pi}\int \frac{e^{-ikr}}{kr}\,d\varphi = kG_1(kr) \text{ hence:}$$

$$\begin{aligned} G_k(\rho, \rho', \zeta) &= kG_1(k\rho, k\rho', k\zeta) = kG_1(k^2q, k^2p), \\ C_k(\rho, \rho', \zeta) &= kC_1(k\rho, k\rho', k\zeta) = kC_1(k^2q, k^2p). \end{aligned} \tag{46}$$

The same decomposition can be seen from the expansions (41).

6. Conclusions. The preceding sections show clearly the importance of the role played by both end-surfaces, whose exact shape should be taken into consideration very carefully. We have shown, on the example of plane terminations, that the problem consists in finding *two* unknown current distributions, one for the cylindrical surface and one for the (symmetrical) terminal surfaces, and this requires solving *two* integral equations. This is the essential difference from the problems of the proper oscillations of *one* closed algebraic surface, such as an ellipsoid. For plane terminations, a complete study of equations (17) and (18) should be affected, and the successive approximations should be worked out simultaneously on both equations. Other shapes of end-surfaces, like half spherical or half ellipsoidal terminals, would certainly yield quite different results. A discussion of this problem is not attempted in the present paper, the aim of which was merely to offer a precise statement of the mathematical theory of antennas and to emphasize some difficulties which seemed to have been overlooked by previous authors.