AN APPLICATION OF ORTHOGONAL MOMENTS TO PROBLEMS IN STATICALLY INDETERMINATE STRUCTURES*

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1. Numerous methods have been devised for determining moment profiles[1] in statically indeterminate structures. Most of these can be classified as either approximate or exact. The approximate methods are usually simple to apply, but when employing them it is necessary to specify numerical values for the dimensions and stiffnesses (or their ratios) of the structure involved. The exact methods consist in solving systems of linear equations obtained by setting up relations between generalized displacements or by the equivalent means of using Castigliano's Principle. As the degree of indeterminacy of the structure increases, the solution of the equations becomes more laborious. Most of the methods aim at reducing such labor by a suitable choice of unknowns. This paper is an attempt in that direction.

2. Consider a structure whose degree of statical indeterminacy is N. Denote by M the true moment profile in the structure under a given load, and by M_0 the moment profile under the same load when the structure has been made statically determinate by removing N constraints (the so-called basic structure). The effect of the removed constraints may be replaced by the combined effect of N unknown generalized forces (couples and forces) $X_1, X_2, \cdots, X_N$. The generalized displacements of the loaded structure at the points of application and in the directions of $X_1, X_2, \cdots, X_N$, are assumed to be known and will be denoted by $\delta_1, \delta_2, \cdots, \delta_N$. (If X_i is a couple, δ_i is a rotation; if X_i is an ordinary force, δ_i is an ordinary displacement.) Let M_i be the moment profile obtained when the force corresponding to $X_i=1$, $X_j=0$ for $j\neq i$, acts on the (unloaded) basic structure. Then the moment profile X_iM_i represents the effect of the ith constraint. Superposing the moment profiles due to the load and the constraints yields the true moment profile M:

$$M = M_0 + \sum_{i=1}^{N} X_i M_i. \tag{1}$$

The unknown quantities X_i in (1) are to be determined by means of Castigliano's Principle.

Disregarding as usual the contributions of shearing stresses and axial forces, we get for the total strain energy U the following expression:

$$U = \frac{1}{2}\int M^2 dx', \tag{2}$$

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[1] By the moment profile of a structure under a given load we understand the magnitude of the bending moment as a function of position; the graph of this function is commonly called the bending moment diagram.

where $dx' = dx/EI$, E is Young's modulus, I is the moment of inertia of any cross-section about the neutral axis, and the integral is taken over the entire structure. Castigliano's Principle states that

$$\frac{\partial U}{\partial X_i} = \delta_i \qquad (i = 1, 2, \cdots, N),$$

or, by virtue of (1) and (2),

$$M_0M_idx' + \sum_{j=i}^{N} X_j \int M_iM_jdx' = \delta_i \quad (i = 1, 2, \cdots, N). \quad (3)$$

It will be observed that (1) and (3) are still valid if each X_i represents a set of generalized forces (rather than a single force) acting simultaneously at different points. (See for instance Fig. 2d.) In such a case, each δ_i would be made up of the generalized displacements corresponding to the given set of generalized forces.

Previous attacks upon the problem have essentially consisted in choosing the points of application and lines of action of the unknown forces X_i so as to make the system of equations (3) as simple as possible. Thus the moment profiles M_i were completely determined. We propose to reverse this procedure by specifying the moment profiles M_i first. Let us choose these profiles so that

$$\int M_iM_jdx' = 0 \quad (i, j = 1, 2, \cdots, N; i \neq j). \quad (4)$$

Then each of the equations (3) will contain only one unknown, and we get at once

$$X_i = \frac{\delta_i - \int M_0M_idx'}{\int M_i^2dx'} \qquad (i = 1, 2, \cdots, N). \quad (5)$$

The true moment profile M may now be obtained by substituting these values X_i into (1).

It will be recognized that equations (4) require that the moment profiles M_i form an orthogonal system over the structure. Such a system can always be constructed by the standard orthogonalization process from the original set of moment profiles (or any similar set of linearly independent moment profiles).[2] Therefore, the system of orthogonalized moment profiles and the corresponding generalized forces which appear in (5) will consist of linear combinations of the original sets of M_i and X_i, respectively. In most practical applications, the exact form of these relations, as well as the physical interpretation of the generalized forces X_i, is immaterial; only the orthogonalized moment profiles M_i are needed. The simple example that follows will illustrate the notions introduced in the preceding discussion.

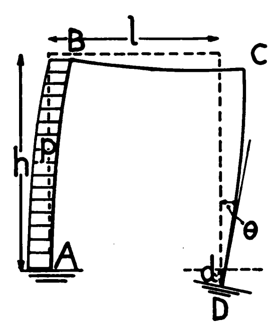

FIG. 1

3. Let us consider the triply indeterminate bent ABCD (Fig. 1), which has undergone a vertical displacement d and a rotation θ at D (as a result of a settlement of

[2] A discussion of orthogonal systems, explaining this process, will be found in R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Berlin, 1931), vol. 1, p. 40, or in E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge, 1927), p. 224.

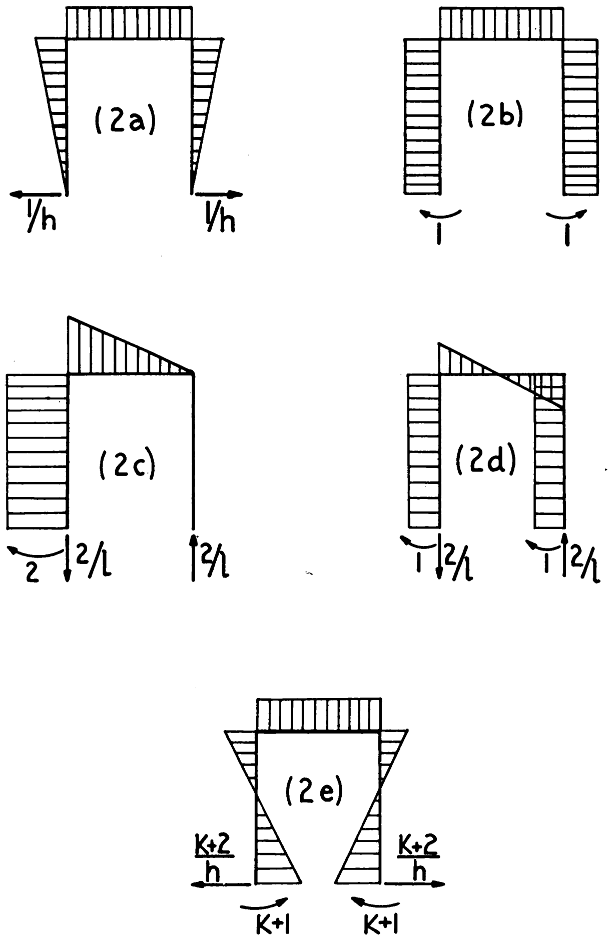

FIG. 2

the foundation). The member AB is subjected to a uniform horizontal pressure p. The stiffnesses EI of the members AB and CD are assumed to be equal. In this and succeeding examples, the positive direction will be taken as downward in vertical members and toward the right in horizontal members.

The solution will be carried out as follows: (1) the construction of an orthogonal system will be shown, and (2) this system will be used to obtain the desired moment profiles.

Figures 2a, 2b, and 2c show three self-equilibrating systems of generalized forces and the corresponding moment profiles; it is obvious that any set of reactions (in equilibrium) can be built up by superposing these three in the right proportions.

We note that the moment profiles M_{2a} and M_{2b} are symmetric with respect to a vertical axis through the midpoint of BC and would be orthogonal to any antisymmetric profile. Therefore we replace M_{2c} by M_{2d} (Fig. 2d). It remains to replace M_{2a} by a linear combination of M_{2a} and M_{2b}, say $\alpha M_{2a}+\beta M_{2b}$, that will be orthogonal to M_{2b}. The orthogonality condition reads

$$0 = \alpha \int M_{2a} M_{2b} dx' + \beta \int M_{2b}^2 dx' = \alpha(l' + h') + \beta(l' + 2h'), \tag{6}$$

where $l' = l/EI_{BC}$, $h' = h/EI_{AB}$. Eq. (6) will be satisfied if $\alpha = \kappa + 2$ and $\beta = -(\kappa+1)$, where $\kappa = l'/h'$. The resulting moment profile M_{2e} is shown in Fig. 2e. The moment profiles M_{2b}, M_{2d}, and M_{2e} form an orthogonal system over the structure. From the manner in which this system was derived, we see that it is independent of the loading and the choice of a basic structure. This is true not only with respect to this particular structure but in general.

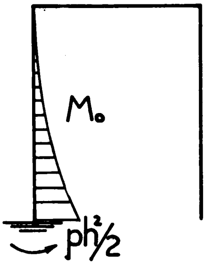

FIG. 3.

We choose a convenient basic structure and find the moment profile M_0 due to the load (see Fig. 3). Eq. (1) now takes the form

$$M = M_0 + X_{2b} M_{2b} + X_{2d} M_{2d} + X_{2e} M_{2e}. \tag{7}$$

Clearly, the quantities X may best be interpreted as mere coefficients by which the generalized forces in Figs. 2b, 2d, and 2e must be multiplied in order to yield, upon superposition, the correct reactions at the supports.[3] As for the interpretation of δ_{2b}, δ_{2d}, and δ_{2e}, we recall that these quantities are equal to the partial derivatives, with respect to the corresponding quantities X, of the work done by the reactions. For instance,

$$\delta_{2d} = \frac{\partial}{\partial X_{2d}}\left[\{X_{2d} - X_{2b} + (\kappa+1)X_{2e}\}\theta - \frac{2X_{2d}}{l}d\right] = \theta - \frac{2d}{l}\cdot$$

Similarly $\delta_{2b} = -\theta$ and $\delta_{2e} = (\kappa+1)\theta$.

Next, we evaluate the following important quantities:

$$\int M_{2b}^2 dx' = h'(\kappa+2), \qquad \int M_{2d}^2 dx' = \tfrac{1}{3}h'(\kappa+6),$$
$$\int M_{2e}^2 dx' = \tfrac{1}{3}h'(\kappa+2)(2\kappa+1); \tag{8}$$

$$\int M_0 M_{2b} dx' = \int M_0 M_{2d} dx' = -\tfrac{1}{6}ph^2h', \qquad \int M_0 M_{2e} dx' = \frac{1}{24}ph^2h'(3\kappa+2). \tag{9}$$

Thus the equations (5) specialize to

$$X_{2b} = \frac{-6\theta + ph^2h'}{6h'(\kappa+2)}, \qquad X_{2d} = \frac{6\theta - 12d/l + ph^2h'}{2h'(\kappa+6)},$$
$$X_{2e} = \frac{24(\kappa+1)\theta - (3\kappa+2)ph^2h'}{8h'(\kappa+2)(2\kappa+1)}\cdot \tag{10}$$

To complete the solution we have only to substitute (10) into (7) and tabulate the values of M at A, B, C, and D.

4. In the present method, the problem of solving the system (3) is replaced by that of constructing a set of orthogonal moment profiles. Once such a set is known for a given structure, the solution M corresponding to any loading is obtained merely by evaluating the right-hand member of (5) and substituting into (1).[4] This is an advantage of the present method over others in which the complete system (3) has to be solved anew every time the load is altered. Thus, for important structures, it may be worthwhile to construct an orthogonal set, even in cases where the orthogonalization itself is fairly complicated.

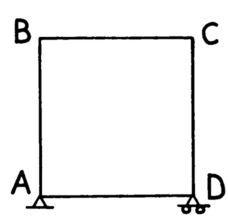

FIG 4.

Fortunately, the rather lengthy standard orthogonalization process (see footnote 2) seldom needs to be used in its entirety, as the preceding example indicates. Shortcuts involving the use of symmetry, antisymmetry, and other characteristics of the

[3] In most problems displacements at the constraints are zero. In such cases it is altogether unnecessary to visualize what particular generalized forces generate the orthogonal moment profiles because that physical notion is needed only for evaluating the δ_i's.

[4] The integrals $\int M_i^2 dx'$ which appear in the denominator of (5) are also independent of the loading and therefore can be computed once for all.

structure are usually available. For instance, in the case of the triply indeterminate frame in Fig. 4 (with $I_{AB}=I_{CD}$ and $I_{BC}=I_{AD}$), we obtain the M_i's immediately from symmetry considerations (Fig. 5).

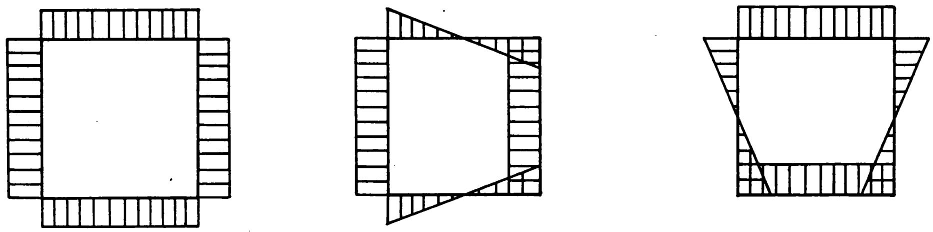

FIG. 5.

The evaluation of integrals $\int M_0 M_i dx'$ in (5) presents few difficulties, because the moments M_i always vary linearly over any member and M_0 can be made to vanish over all but a few members by a suitable choice of the basic structure. Furthermore, there exist tables of such integrals for the common forms of moment profiles (trapezoidal, parabolic, etc.).[5]

Similarly, the remaining operations can be systematized so that even a person with little mathematical training can perform them. This is particularly true when numerical values of the stiffnesses EI are known; then substituting (5) into (1) reduces to taking scalar products on a computing machine.

5. We conclude with an example illustrating an efficient arrangement of the work. Consider the four-legged bent shown in Fig. 6; assume that $I_{AB}=I_{CD}=I_{EF}=I_{GH}$ and $I_{BC}=I_{CF}=I_{FG}$, and denote by κ the ratio $l'/h'=lI_{AB}/hI_{BC}$, as before.

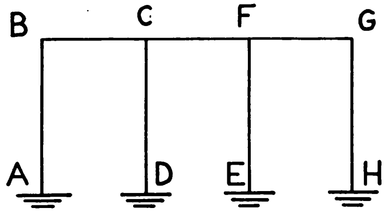

FIG. 6.

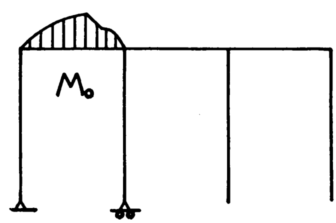

FIG. 7.

Since the structure is statically indeterminate of the ninth degree, we obtain nine orthogonal moment profiles $M_1, M_2, \cdots, M_9$, whose values at different points of the structure are given in the first ten columns of Table I. (We understand by $M_i(CB)$ the value of M_i at the end of C of the member BC, and give a similar meaning to $M_i(AB)$, $M_i(BC)$, etc.) It will be observed that M_1, M_2, M_3, M_7, M_8, and M_9 are essentially reproductions of the moment profiles M_{2b}, M_{2d}, and M_{2e} found for the two-legged bent in section 3. Considerations of symmetry are helpful in constructing M_4, M_5, and M_6. The integrals $\int M_i^2 dx'$ are now computed by means of the tables referred to in footnote 5, and are given in the eleventh column of our table.

The parts of the table so far discussed can be used to find the moment profile due to any loading of the bent. Suppose the member BC is subjected to a set of vertical

[5] See, e.g., H. F. B. Müller-Breslau, *Die Graphische Statik der Baukonstruktionen* (Leipzig, 1925), vol. 2, part 2, p. 56.

loads and moments. By selecting the basic structure as indicated in Fig. 7, we confine the moment profile M_0 to BC.

Defining

$$P_l = \frac{1}{l^2}\int_{BC} M_0 x dx, \qquad P_r = \frac{1}{l^2}\int_{BC} M_0(l - x)dx,$$

we see at once that

$$\int M_0 M_i dx' = l'[P_r M_i(BC) + P_l M_i(CB)] \qquad (i = 1, 2, \cdots, 9). \qquad (11)$$

The values X_i (twelfth column) are now obtained by dividing the negatives of the right members of (11) by the corresponding values in the eleventh column.

To find the value of the true moment profile at (say) A, we multiply each term M_i(AB) (first column) by the corresponding X_i and add the results, and similarly for other points. It would ordinarily not be necessary to carry out the algebraic simplification of these sums, but this has been done for the sake of compactness:

$$M(AB) = \frac{\kappa}{ST}\{(4\kappa^3 - 170\kappa^2 - 414\kappa - 225)P_r - 2(\kappa + 1)(10\kappa^2 - 6\kappa - 3)P_l\},$$

$$M(BC) = \frac{\kappa}{ST}\{-(64\kappa^3 + 628\kappa^2 + 1188\kappa + 585)P_r + 2(\kappa + 1)(16\kappa^2 + 102\kappa + 51)P_l\},$$

$$M(CB) = \frac{1}{ST}\{2\kappa(16\kappa^3 + 118\kappa^2 + 153\kappa + 51)P_r - 4(\kappa + 1)(16\kappa^3 + 141\kappa^2 + 195\kappa + 72)P_l\},$$

$$M(DC) = \frac{\kappa}{ST}\{(20\kappa^3 + 96\kappa^2 + 233\kappa + 135)P_r + 2(-2\kappa^3 + 51\kappa^2 + 91\kappa + 39)P_l\},$$

$$M(CF) = \frac{1}{ST}\{\kappa(140\kappa^2 + 412\kappa + 237)P_r - 2(86\kappa^3 + 355\kappa^2 + 411\kappa + 144)P_l\},$$

$$M(FC) = \frac{1}{ST}\{-\kappa(124\kappa^2 + 248\kappa + 93)P_r + 2(70\kappa^3 + 179\kappa^2 + 144\kappa + 36)P_l\},$$

$$M(EF) = \frac{1}{ST}\{\kappa(36\kappa^3 + 260\kappa^2 + 377\kappa + 135)P_r - 2\kappa(18\kappa^3 + 125\kappa^2 + 176\kappa + 69)P_l\},$$

$$M(GF) = \frac{1}{ST}\{-\kappa(108\kappa^2 + 208\kappa + 135)P_r + 2\kappa(54\kappa^2 + 73\kappa + 21)P_l\},$$

$$M(FG) = \frac{1}{ST}\{2\kappa(54\kappa^2 + 73\kappa + 21)P_r + 2(-54\kappa^3 - 38\kappa^2 + 51\kappa + 36)P_l\},$$

$$M(HG) = \frac{1}{ST}\{\kappa(36\kappa^3 + 198\kappa^2 + 284\kappa + 135)P_r - 2\kappa(18\kappa^3 + 90\kappa^2 + 104\kappa + 33)P_l\}.$$

The arrangement given here is especially adapted to the use of computing machines in case numerical values of κ and the other quantities appearing are known.[6]

[6] After the manuscript of this paper had been completed, our attention was drawn to a work along similar lines by S. Müller (S. Müller, *Zur Berechnung mehrfach statisch unbestimmter Tragwerke*, Zentralblatt der Bauverw. **1907**, p. 23). Müller introduces a "system of forces X_i" instead of single forces, and reduces the system of equations to the diagonal form. However, his point of view and emphasis are quite different from ours, and his procedure is more lengthy than that presented here.

TABLE I

	$M_i(AB)$	$M_i(BC)$	$M_i(CB)$	$M_i(DC)$	$M_i(CF)$	$M_i(FC)$	$M_i(EF)$	$M_i(FG)$	$M_i(GF)$	$M_i(HG)$	$\int M_i^2 dx'$	$-X_i$
M_1	$\kappa+1$	1	1	$-(\kappa+1)$	—	—	—	—	—	—	$h'(\kappa+2)(2\kappa+1)/3$	$3\kappa(P_r+P_l)/(\kappa+2)(2\kappa+1)$
M_2	-1	1	1	1	—	—	—	—	—	—	$h'(\kappa+2)$	$\kappa(P_r+P_l)/(\kappa+2)$
M_3	-1	1	-1	-1	—	—	—	—	—	—	$h'(\kappa+6)/3$	$3\kappa(P_r-P_l)/(\kappa+6)$
M_4	$-\kappa(3\kappa+7)$	-11κ	$13\kappa+12$	$-\kappa(\kappa-5)$	λ	λ	$\kappa(\kappa-5)$	$13\kappa+12$	-11κ	$\kappa(3\kappa+7)$	$h'\kappa\lambda R$	$[-11\kappa P_r+(13\kappa+12)P_l]/\lambda R$
M_5	$-\kappa(3\kappa+7)$	-11κ	$13\kappa+12$	$-\kappa(\kappa-5)$	λ	$-\lambda$	$-\kappa(\kappa-5)$	$-(13\kappa+12)$	$+11\kappa$	$-\kappa(3\kappa+7)$	$h'\kappa\lambda S/3$	$3[-11\kappa P_r+(13\kappa+12)P_l]/\lambda S$
M_6	$2\kappa^2+8\kappa+5$	$6\kappa+5$	$-2(3\kappa+2)$	$2\kappa^2+9\kappa+5$	$2\kappa+1$	$2\kappa+1$	$-(2\kappa^2+9\kappa+5)$	$-2(3\kappa+2)$	$6\kappa+5$	$-(2\kappa^2+8\kappa+5)$	$h'RT/3$	$3\kappa[(6\kappa+5)P_r-2(3\kappa+2)P_l]/RT$
M_7	—	—	—	—	—	—	$\kappa+1$	1	1	$-(\kappa+1)$	$h'(\kappa+2)(2\kappa+1)/3$	0
M_8	—	—	—	—	—	—	-1	1	1	1	$h'(\kappa+2)$	0
M_9	—	—	—	—	—	—	1	-1	1	1	$h'(\kappa+6)/3$	0

$R=4\kappa^2+31\kappa+20,$ $T=4\kappa^2+10\kappa+5,$ $M_i(FE)=M_i(FC)-M_i(FG),$ $M_i(BA)=-M_i(BC)$

$S=4\kappa^2+41\kappa+36,$ $\lambda=2(2\kappa+1)(\kappa+6)$ $M_i(CD)=M_i(CB)-M_i(CF),$ $M_i(GH)=M_i(GF)$