# EFFECT OF A SMALL HOLE ON THE STRESSES IN A UNIFORMLY LOADED PLATE*

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**1. Introduction.** The National Bureau of Standards has recently made tests on steel columns having perforated cover plates.[1] Most of the perforations were of so-called *ovaloid* shape, i.e., that of a square with a semi-circle erected on each of two opposite sides. The tests on the columns included experimental determinations of the distribution of stress in the neighborhood of a perforation, and the results obtained aroused interest in the development of a theory for the distribution of stress in a large, uniformly loaded plate having a single ovaloid hole.

In this paper an *exact* solution to this problem is obtained for a hole having any boundary of which the equation can be expressed in the parametric form

$$x = p \cos \beta + r \cos 3\beta, \qquad y = q \sin \beta - r \sin 3\beta. \tag{1}$$

The plate is supposed in a state of generalized plane stress, the stress[2] at points remote from the hole having the constant normal components $\sigma_x = S_x$, $\sigma_y = S_y$, and the constant shearing component $\tau_{xy} = T_{xy}$.
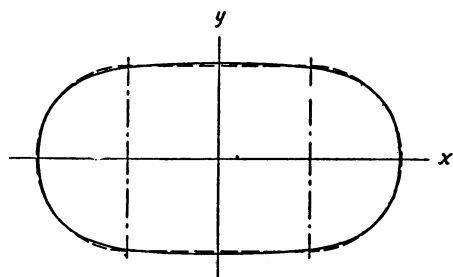


FIG. 1. Actual and approximate ovaloids. The dashed line represents the actual ovaloid and the full line the approximate ovaloid of Eqs. (1) and (2).

Eq. (1) represents a closed curve having symmetry about the $x$-axis and about the $y$-axis. For certain values of $p$, $q$, and $r$ the curve is simple, i.e., it does not cross itself. By adjustment of the values of $p$, $q$, and $r$ a variety of simple closed curves is obtained, including a good approximation to an ovaloid and a good approximation to a square with rounded corners, as well as exact ellipses ($r = 0$) of any eccentricity. The approximate ovaloid obtained by taking

$$p = 2.063, \qquad q = 1.108, \qquad r = -0.079, \tag{2}$$

is shown compared to the actual ovaloid in Fig. 1. The approximate square obtained by taking

$$p = q = 1, \qquad r = -0.14, \tag{3}$$

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[1] Ambrose H. Stang and Martin Greenspan, J. Research NBS **28**, 669, 687; **29**, 279; **30**, 15, 177, 411 (1942–43).

[2] The term *stress* is used throughout to denote the mean value of the stress over the thickness of the plate.

is shown in Fig. 2. The sides of the square are parallel to the axes of coordinates. By taking

$$p = q = 1, \qquad r = 0.14, \tag{4}$$

the same square, but with the diagonals parallel to the axes of coordinates, is obtained. The radius of curvature at the mid-point of the fillet is about 0.086 times the length of the side of the square.

**2. Curvilinear coordinates.** If two sets of curves are defined by

$$f_1(x, y) = \alpha, \qquad f_2(x, y) = \beta, \tag{5}$$

then a pair of values $(\alpha, \beta)$ defines the points at which the corresponding curves (5) intersect, and $(\alpha, \beta)$ are curvilinear coordinates in the



FIG. 2. The approximate square of Eqs. (1) and (3).

$x$, $y$-plane. As a special case, the functions of Eq. (5) may be obtained by equating real and imaginary parts of both sides of

$$w = f(z), \tag{6}$$

where $w = \alpha + i\beta$ and $z = x + iy$. In this case the transformation from the $w$-plane to the $z$-plane is conformal and the two families of Eq. (5) are orthogonal. The expression,

$$\frac{dz}{dw} = \frac{1}{h} e^{i\psi}, \tag{7}$$

defines the *stretch ratio*, $1/h$, of the transformation, and gives $\psi$, the inclination of the curve, $\beta = $ constant, to the $x$-axis.

In the absence of body forces, the condition that the stresses satisfy the conditions of equilibrium is that the normal components, $\sigma_\alpha$ and $\sigma_\beta$, and the shearing component, $\tau_{\alpha\beta}$, can be derived from a stress function, $\phi$, by means of the relations[3]

$$
\begin{aligned}
\sigma_\alpha &= \quad h^2 \frac{\partial^2 \phi}{\partial \beta^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial \beta}\frac{\partial h^2}{\partial \beta} - \frac{\partial \phi}{\partial \alpha}\frac{\partial h^2}{\partial \alpha}\right), \\
\sigma_\beta &= \quad h^2 \frac{\partial^2 \phi}{\partial \alpha^2} - \frac{1}{2}\left(\frac{\partial \phi}{\partial \beta}\frac{\partial h^2}{\partial \beta} - \frac{\partial \phi}{\partial \alpha}\frac{\partial h^2}{\partial \alpha}\right), \\
\tau_{\alpha\beta} &= - h^2 \frac{\partial^2 \phi}{\partial \alpha \partial \beta} - \frac{1}{2}\left(\frac{\partial \phi}{\partial \beta}\frac{\partial h^2}{\partial \alpha} + \frac{\partial \phi}{\partial \alpha}\frac{\partial h^2}{\partial \beta}\right);
\end{aligned}
\tag{8}
$$

and the condition that the expressions (8) satisfy the compatibility conditions is

$$\nabla^4 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \phi = h^2\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right) h^2\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right)\phi = 0. \tag{9}$$

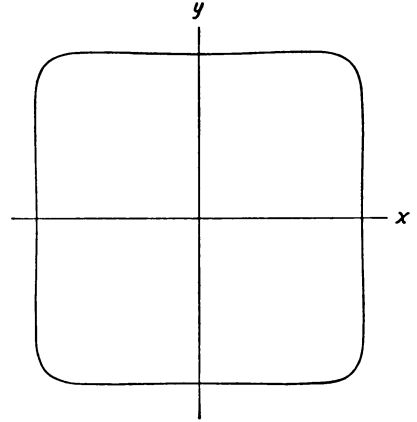If a function, $F$, satisfies Laplace's equation,

$$\nabla^2 F = 0, \tag{10}$$

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[3] A. E. H. Love, *A treatise on the mathematical theory of elasticity*, 4th ed., Cambridge, 1927, p. 91.

then $F$, $xF$, $yF$, and $\rho^2 F = (x^2 + y^2)F$ satisfy Eq. (9). Functions which satisfy Eq. (10) are called harmonic functions, those which satisfy Eq. (9), biharmonic functions.

**3. The coordinate system.** The solution of the problem is simplified by the use of a coordinate system $(\alpha, \beta)$ such that Eq. (1) of the boundary of the hole reduces to the form $\alpha = \alpha_0$. Such a system is obtained by writing for Eq. (6)

$$z = e^w + abe^{-w} + ac^3 e^{-3w}, \tag{11}$$

or, separating the real and imaginary parts,

$$\left. \begin{array}{l} x = (e^\alpha + abe^{-\alpha}) \cos \beta + ac^3 e^{-3\alpha} \cos 3\beta, \\ y = (e^\alpha - abe^{-\alpha}) \sin \beta - ac^3 e^{-3\alpha} \sin 3\beta. \end{array} \right\} \tag{12}$$

For constant $\alpha$, say $\alpha_0$, Eq. (12) reduces to Eq. (1) for the boundary of the hole, where

$$p = e^{\alpha_0} + abe^{-\alpha_0}, \qquad q = e^{\alpha_0} - abe^{-\alpha_0}, \qquad r = ac^3 e^{-3\alpha_0}. \tag{13}$$

From Eqs. (2) and (13) it is easily calculated that for the approximate ovaloid of Eqs. (1) and (2),

$$e^{\alpha_0} = 1.585, \qquad ab = 0.758, \qquad ac^3 = -0.314.$$

By keeping $ab$ and $ac^3$ fixed and varying $\alpha$ and $\beta$ the appropriate coordinate system
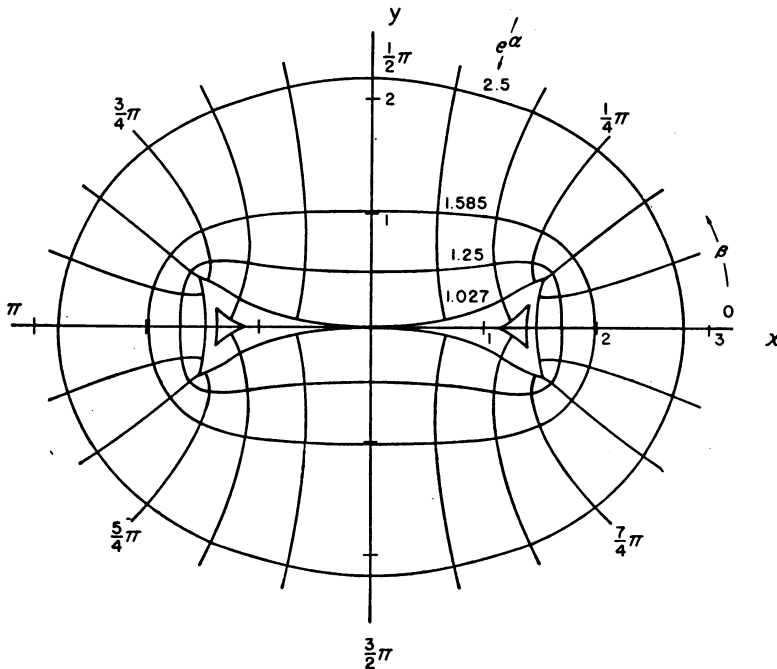


FIG. 3. Coordinate system for problem of ovaloid hole.

for the ovaloid is obtained. This system is shown in Fig. 3. The appropriate systems for the approximate square with rounded corners are similarly obtained. Figure 4 shows the system corresponding to the case of Eq. (3) and Fig. 2, and Fig. 5 shows the system corresponding to the case of Eq. (4).
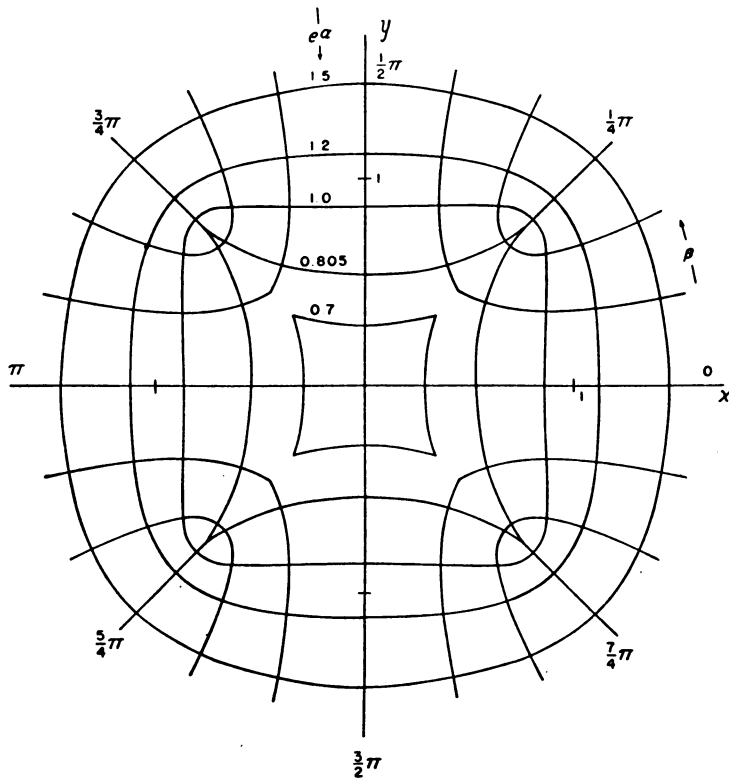
FIG. 4. Coordinate system for problem of square hole with rounded corners, sides of square parallel to Cartesian axes.

The coordinates $(\alpha, \beta)$ approach polar coordinates $(\rho, \theta)$ for large $\alpha$ as follows:

$$\lim_{\alpha \to \infty} \alpha = \log \rho, \qquad \lim_{\alpha \to \infty} \beta = \theta. \tag{14}$$

The values of $h^2$ and its derivatives may be computed as follows. From Eq. (11)

$$\frac{dz}{dw} = e^w - abe^{-w} - 3ac^3 e^{-3w}.$$

Hence from Eq. (7),

$$h^{-2} = e^{2\alpha} + a^2b^2 e^{-2\alpha} + 9a^2c^6 e^{-6\alpha} - 2ab \cos 2\beta$$
$$+ 6a^2bc^3 e^{-4\alpha} \cos 2\beta - 6ac^3 e^{-2\alpha} \cos 4\beta,$$

and

$$\frac{1}{2} \frac{\partial h^2}{\partial \alpha} = - h^4(e^{2\alpha} - a^2b^2 e^{-2\alpha} - 27a^2c^6 e^{-6\alpha}$$
$$- 12a^2bc^3 e^{-4\alpha} \cos 2\beta + 6ac^3 e^{-2\alpha} \cos 4\beta),$$

$$\frac{1}{2} \frac{\partial h^2}{\partial \beta} = - h^4(2ab \sin 2\beta - 6a^2bc^3 e^{-4\alpha} \sin 2\beta + 12ac^3 e^{-2\alpha} \sin 4\beta).$$

$$\left. \right\} \tag{15}$$

**4. The boundary conditions.** The statement of the problem may be recapitulated as follows. There is given a large plate containing a small hole of the shape given by Eq. (1). The edge of the hole is free from stress. The plate is in a state of (generalized) plane stress and the components of (mean) stress at points remote from the hole are $\sigma_x = S_x$, $\sigma_y = S_y$, $\tau_{xy} = T_{xy}$; or in polar coordinates,

$$\left.\begin{aligned}
\sigma_\rho &= \frac{S_x + S_y}{2} + \frac{S_x - S_y}{2} \cos 2\theta + T_{xy} \sin 2\theta, \\[2mm]
\sigma_\theta &= \frac{S_x + S_y}{2} - \frac{S_x - S_y}{2} \cos 2\theta - T_{xy} \sin 2\theta, \\[2mm]
\tau_{\rho\theta} &= -\frac{S_x - S_y}{2} \sin 2\theta + T_{xy} \cos 2\theta.
\end{aligned}\right\} \tag{16}$$

The boundary conditions may finally be stated as

$$\left.\begin{aligned}
\sigma_\alpha &= \tau_{\alpha\beta} = 0, \quad (\alpha = \alpha_0), \\[2mm]
\sigma_\alpha &= \frac{S_x + S_y}{2} + \frac{S_x - S_y}{2} \cos 2\beta + T_{xy} \sin 2\beta, \\[2mm]
\sigma_\beta &= \frac{S_x + S_y}{2} - \frac{S_x - S_y}{2} \cos 2\beta - T_{xy} \sin 2\beta, \\[2mm]
\tau_{\alpha\beta} &= -\frac{S_x - S_y}{2} \sin 2\beta + T_{xy} \cos 2\beta, \quad (\alpha = \infty).
\end{aligned}\right\} \tag{17}$$

The last three of Eq. 17 are obtained by substitution of Eq. (14) into Eq. (16).

**5. The stress function.** From the harmonic functions $e^\alpha \sin \beta$ and $e^{-\alpha} \sin \beta$ may be constructed the biharmonic functions $ye^\alpha \sin \beta$ and $ye^{-\alpha} \sin \beta$. From Eq. (12)

$$y = e^\alpha \sin \beta - abe^{-\alpha} \sin \beta - ac^3e^{-3\alpha} \sin 3\beta.$$

Hence

$$ye^\alpha \sin \beta = \tfrac{1}{2}e^{2\alpha} - \tfrac{1}{2}e^{2\alpha} \cos 2\beta - \tfrac{1}{2}ab + \tfrac{1}{2}ab \cos 2\beta + \tfrac{1}{2}ac^3e^{-2\alpha} \cos 4\beta - \tfrac{1}{2}ac^3e^{-2\alpha} \cos 2\beta,$$

$$ye^{-\alpha} \sin \beta = \tfrac{1}{2} - \tfrac{1}{2} \cos 2\beta - \tfrac{1}{2}abe^{-2\alpha} + \tfrac{1}{2}abe^{-2\alpha} \cos 2\beta + \tfrac{1}{2}ac^3e^{-4\alpha} \cos 4\beta - \tfrac{1}{2}ac^3e^{-4\alpha} \cos 2\beta.$$

By dropping the harmonic terms from each of these functions and multiplying by 2 the two biharmonic functions,

$$\phi_a = e^{2\alpha} + ab \cos 2\beta + ac^3e^{-2\alpha} \cos 4\beta,$$

$$\phi_b = -\cos 2\beta - abe^{-2\alpha} - ac^3e^{-4\alpha} \cos 2\beta,$$

are obtained. The biharmonic function

$$\phi_c = ye^{-\alpha} \cos \beta + xe^{-\alpha} \sin \beta = \sin 2\beta - ac^3e^{-4\alpha} \sin 2\beta$$

is obtained in similar fashion.

The biharmonic function $\rho^2$ may be obtained from Eq. (12):

$$\rho^2 = x^2 + y^2 = e^{2\alpha} + a^2b^2e^{-2\alpha} + a^2c^6e^{-6\alpha} + 2ab \cos 2\beta$$
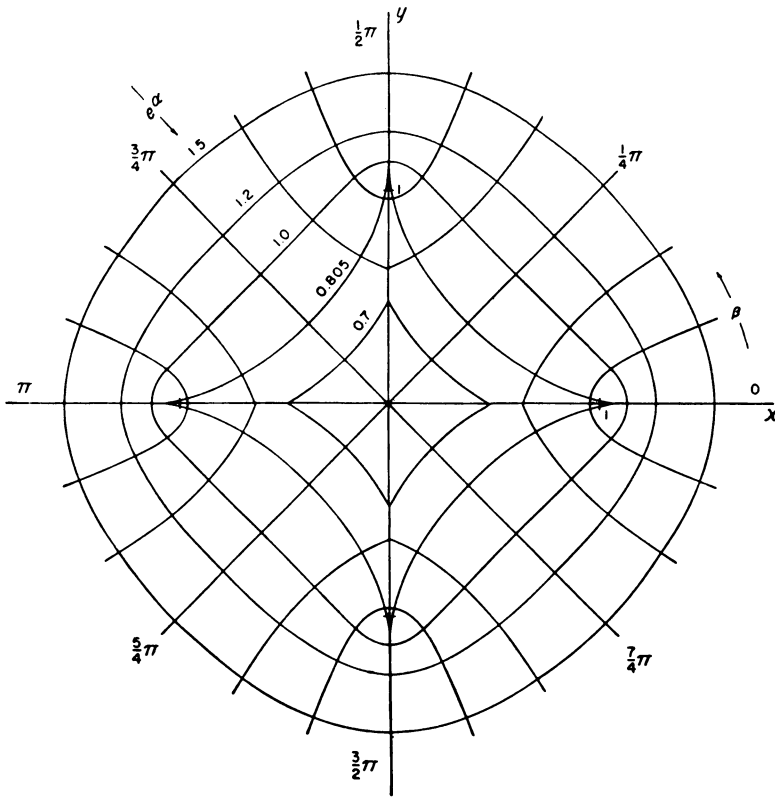$$+ 2a^2bc^3e^{-4\alpha} \cos 2\beta + 2ac^3e^{-2\alpha} \cos 4\beta.$$

FIG. 5. Coordinate system for problem of square hole with rounded corners, diagonals parallel to Cartesian axes.

The non-harmonic stress functions required by this problem are

$$\phi_1 = 2\phi_a - 2ab\phi_b - \rho^2,$$

$$\phi_2 = -\phi_b,$$

and

$$\phi_6 = \phi_c,$$

or

$$\phi_1 = e^{2\alpha} + a^2b^2e^{-2\alpha} - a^2c^6e^{-6\alpha} + 2ab\cos 2\beta,$$

$$\phi_2 = abe^{-2\alpha} + \cos 2\beta + ac^3e^{-4\alpha}\cos 2\beta,$$

and

$$\phi_6 = \sin 2\beta - ac^3e^{-4\alpha}\sin 2\beta.$$

In addition, the harmonic stress functions,

$$\phi_3 = e^{2\alpha}\cos 2\beta, \quad \phi_4 = e^{-2\alpha}\cos 2\beta, \quad \phi_5 = \alpha, \quad \phi_7 = e^{2\alpha}\sin 2\beta, \quad \text{and} \quad \phi_8 = e^{-2\alpha}\sin 2\beta$$

will be required.

The complete stress function may be written

$$\phi = C_1\phi_1 + C_2\phi_2 + C_3\phi_3 + C_4\phi_4 + C_5\phi_5 + C_6\phi_6 + C_7\phi_7 + C_8\phi_8, \tag{18}$$

where the $C$'s are to be adjusted so that the stresses derived from $\phi$ meet boundary conditions (17). Also

$$\frac{\partial \phi}{\partial \alpha} = 2C_1(e^{2\alpha} - a^2b^2e^{-2\alpha} + 3a^2c^6e^{-6\alpha}) - 2C_2(abe^{-2\alpha} + 2ac^3e^{-4\alpha} \cos 2\beta)$$
$$+ 2C_3e^{2\alpha} \cos 2\beta - 2C_4e^{-2\alpha} \cos 2\beta + C_5 + 4C_6ac^3e^{-4\alpha} \sin 2\beta$$
$$+ 2C_7e^{2\alpha} \sin 2\beta - 2C_8e^{-2\alpha} \sin 2\beta, \tag{19a}$$

$$\frac{\partial \phi}{\partial \beta} = - 4C_1ab \sin 2\beta - 2C_2(\sin 2\beta + ac^3e^{-4\alpha} \sin 2\beta)$$
$$- 2C_3e^{2\alpha} \sin 2\beta - 2C_4e^{-2\alpha} \sin 2\beta + 2C_6(\cos 2\beta - ac^3e^{-4\alpha} \cos 2\beta)$$
$$+ 2C_7e^{2\alpha} \cos 2\beta + 2C_8e^{-2\alpha} \cos 2\beta, \tag{19b}$$

$$\frac{\partial^2 \phi}{\partial \alpha^2} = 4C_1(e^{2\alpha} + a^2b^2e^{-2\alpha} - 9a^2c^6e^{-6\alpha}) + 4C_2(abe^{-2\alpha} + 4ac^3e^{-4\alpha} \cos 2\beta)$$
$$+ 4C_3e^{2\alpha} \cos 2\beta + 4C_4e^{-2\alpha} \cos 2\beta - 16C_6ac^3e^{-4\alpha} \sin 2\beta$$
$$+ 4C_7e^{2\alpha} \sin 2\beta + 4C_8e^{-2\alpha} \sin 2\beta, \tag{19c}$$

$$\frac{\partial^2 \phi}{\partial \beta^2} = - 8C_1ab \cos 2\beta - 4C_2(\cos 2\beta + ac^3e^{-4\alpha} \cos 2\beta)$$
$$- 4C_3e^{2\alpha} \cos 2\beta - 4C_4e^{-2\alpha} \cos 2\beta - 4C_6(\sin 2\beta - ac^3e^{-4\alpha} \sin 2\beta)$$
$$- 4C_7e^{2\alpha} \sin 2\beta - 4C_8e^{-2\alpha} \sin 2\beta, \tag{19d}$$

$$\frac{\partial^2 \phi}{\partial \alpha \partial \beta} = 8C_2ac^3e^{-4\alpha} \sin 2\beta - 4C_3e^{2\alpha} \sin 2\beta + 4C_4e^{-2\alpha} \sin 2\beta$$
$$+ 8C_6ac^3e^{-4\alpha} \cos 2\beta + 4C_7e^{2\alpha} \cos 2\beta - 4C_8e^{-2\alpha} \cos 2\beta. \tag{19e}$$

**6. The stresses.** Substitution of Eqs. (15) and (19) into Eq. (8) gives the stresses in the form

$$\frac{\sigma_\alpha}{h^4} = 2C_1(A_{10} + A_{12} \cos 2\beta + A_{14} \cos 4\beta)$$
$$+ 2C_2(A_{20} + A_{22} \cos 2\beta + A_{24} \cos 4\beta + A_{26} \cos 6\beta)$$
$$+ 2C_3(A_{30} + A_{32} \cos 2\beta + A_{34} \cos 4\beta + A_{36} \cos 6\beta)$$
$$+ 2C_4(A_{40} + A_{42} \cos 2\beta + A_{44} \cos 4\beta + A_{46} \cos 6\beta)$$
$$+ C_5(A_{50} + A_{52} \cos 2\beta + A_{54} \cos 4\beta)$$
$$- 2C_6(A_{62} \sin 2\beta + A_{64} \sin 4\beta + A_{66} \sin 6\beta)$$
$$- 2C_7(A_{72} \sin 2\beta + A_{74} \sin 4\beta + A_{76} \sin 6\beta)$$
$$- 2C_8(A_{82} \sin 2\beta + A_{84} \sin 4\beta + A_{86} \sin 6\beta), \tag{20}$$

$$\frac{\sigma_\beta}{h^4} = 2C_1(B_{10} + B_{12} \cos 2\beta + B_{14} \cos 4\beta + B_{16} \cos 6\beta)$$
$$+ 2C_2(B_{20} + B_{22} \cos 2\beta + B_{24} \cos 4\beta + B_{26} \cos 6\beta)$$
$$- 2C_3(B_{30} + B_{32} \cos 2\beta + B_{34} \cos 4\beta + B_{36} \cos 6\beta)$$
$$- 2C_4(B_{40} + B_{42} \cos 2\beta + B_{44} \cos 4\beta + B_{46} \cos 6\beta)$$
$$- C_5(B_{50} + B_{52} \cos 2\beta + B_{54} \cos 4\beta)$$
$$- 2C_6(B_{62} \sin 2\beta + B_{64} \sin 4\beta + B_{66} \sin 6\beta)$$
$$+ 2C_7(B_{72} \sin 2\beta + B_{74} \sin 4\beta + B_{76} \sin 6\beta)$$
$$+ 2C_8(B_{82} \sin 2\beta + B_{84} \sin 4\beta + B_{86} \sin 6\beta), \tag{21}$$

$$\frac{\tau_{\alpha\beta}}{h^4} = 12C_1(D_{12}\sin 2\beta + D_{14}\sin 4\beta + D_{16}\sin 6\beta)$$

$$- 2C_2(D_{22}\sin 2\beta + D_{24}\sin 4\beta + D_{26}\sin 6\beta)$$

$$+ 2C_3(D_{32}\sin 2\beta + D_{34}\sin 4\beta + D_{36}\sin 6\beta)$$

$$- 2C_4(D_{42}\sin 2\beta + D_{44}\sin 4\beta + D_{46}\sin 6\beta)$$

$$+ 2C_5(D_{52}\sin 2\beta + D_{54}\sin 4\beta)$$

$$- 2C_6(D_{60} + D_{62}\cos 2\beta + D_{64}\cos 4\beta + D_{66}\cos 6\beta)$$

$$+ 2C_7(D_{70} + D_{72}\cos 2\beta + D_{74}\cos 4\beta + D_{76}\cos 6\beta)$$

$$- 2Cs(D_{80} + D_{82}\cos 2\beta + D_{84}\cos 4\beta + D_{86}\cos 6\beta), \tag{22}$$

in which

$$A_{10} = e^{4\alpha} + 4a^2b^2 - (24a^2c^6 + 18a^3b^2c^3 - a^4b^4)e^{-4\alpha} + 24a^4b^2c^6e^{-8\alpha} - 81a^4c^{12}e^{-12\alpha},$$

$$A_{12} = -4\big[abe^{2\alpha} - (3a^2bc^3 - a^3b^3)e^{-2\alpha} + (9a^3bc^6 - 3a^4b^3c^3)e^{-6\alpha} + 9a^4bc^9e^{-10\alpha}\big],$$

$$A_{14} = 2(3ac^3 + a^2b^2 - 6a^3b^2c^3e^{-4\alpha} + 9a^3c^9e^{-8\alpha}),$$

$$A_{20} = 2ab - (6a^2bc^3 - a^3b^3)e^{-4\alpha} + 30a^3bc^6e^{-8\alpha},$$

$$A_{22} = -2\big[e^{2\alpha} - (4ac^3 - a^2b^2)e^{-2\alpha} + 6(a^2c^6 - a^3b^2c^3)e^{-6\alpha} - 18a^3c^9e^{-10\alpha}\big],$$

$$A_{24} = ab - 8a^2bc^3e^{-4\alpha} + 9a^3bc^6e^{-8\alpha}, \qquad A_{26} = -6a^2c^6e^{-6\alpha},$$

$$A_{30} = 3abe^{2\alpha} - 15a^2bc^3e^{-2\alpha}, \qquad A_{32} = -(e^{4\alpha} - 15ac^3 + 3a^2b^2 + 45a^2c^6e^{-4\alpha}),$$

$$A_{34} = abe^{2\alpha} - 9a^2bc^3e^{-2\alpha}, \qquad A_{36} = 3ac^3,$$

$$A_{40} = 3abe^{-2\alpha} - 3a^2bc^3e^{-6\alpha}, \qquad A_{42} = -\big[3 - (9ac^3 - a^2b^2)e^{-4\alpha} - 9a^2c^6e^{-8\alpha}\big],$$

$$A_{44} = abe^{-2\alpha} + 3a^2bc^3e^{-6\alpha}, \qquad A_{46} = -3ac^3e^{-4\alpha},$$

$$A_{50} = e^{2\alpha} - a^2b^2e^{-2\alpha} - 27a^2c^6e^{-6\alpha}, \qquad A_{52} = -2a^2bc^3e^{-4\alpha}, \qquad A_{54} = 6ac^3e^{-2\alpha},$$

$$A_{62} = 2\big[e^{2\alpha} + (a^2b^2 + 4ac^3)e^{-2\alpha} + 6a^2c^6e^{-6\alpha} + 18a^3c^9e^{-10\alpha}\big],$$

$$A_{64} = -(ab - 2a^2bc^3e^{-4\alpha} - 9a^3bc^6e^{-8\alpha}), \qquad A_{66} = -6a^2c^6e^{-6\alpha},$$

$$A_{72} = e^{4\alpha} + 3(a^2b^2 + 5ac^3) + 45a^2c^6e^{-4\alpha}, \quad A_{74} = -(abe^{2\alpha} - 9a^2bc^3e^{-2\alpha}), \quad A_{76} = -3ac^3,$$

$$A_{82} = 3 + (a^2b^2 + 9ac^3)e^{-4\alpha} - 9a^2c^6e^{-8\alpha}, \quad A_{84} = -(abe^{-2\alpha} + 3a^2bc^3e^{-6\alpha}), \quad A_{86} = 3ac^3e^{-4\alpha},$$

$$B_{10} = e^{4\alpha} + 4a^2b^2 + (24a^2c^6 + 6a^3b^2c^3 + a^4b^4)e^{-4\alpha} - 24a^4b^2c^6e^{-8\alpha} - 81a^4c^{12}e^{-12\alpha},$$

$$B_{12} = -4\big[abe^{2\alpha} - (3a^2bc^3 - a^3b^3)e^{-2\alpha} - 9a^3bc^6e^{-6\alpha} + 18a^4bc^9e^{-10\alpha}\big],$$

$$B_{14} = -2(9ac^3 - a^2b^2 + 6a^3b^2c^3e^{-4\alpha} - 45a^3c^9e^{-8\alpha}), \qquad B_{16} = 12a^2bc^3e^{-2\alpha},$$

$$B_{20} = 2ab - (6a^2bc^3 - a^3b^3)e^{-4\alpha} + 6a^3bc^6e^{-8\alpha},$$

$$B_{22} = 2\big[2(ac^3 - a^2b^2)e^{-2\alpha} - 3(4a^2c^6 - a^3b^2c^3)e^{-6\alpha} + 9a^3c^9e^{-10\alpha}\big],$$

$$B_{24} = ab - 16a^2bc^3e^{-4\alpha} + 9a^3bc^6e^{-8\alpha}, \qquad B_{26} = 6(ac^3e^{-2\alpha} - 2a^2c^6e^{-6\alpha}),$$

$$B_{30} = 3abe^{2\alpha} - 15a^2bc^3e^{-2\alpha}, \qquad B_{32} = -(e^{4\alpha} - 15ac^3 + 3a^2b^2 + 45a^2c^6e^{-4\alpha}),$$

$$B_{34} = abe^{2\alpha} - 9a^2bc^3e^{-2\alpha}, \qquad B_{36} = 3ac^3,$$

$$B_{40} = 3abe^{-2\alpha} - 3a^2bc^3e^{-6\alpha}, \qquad B_{42} = -\big[3 - (9ac^3 - a^2b^2)e^{-4\alpha} - 9a^2c^6e^{-8\alpha}\big],$$

$$B_{44} = abe^{-2\alpha} + 3a^2bc^3e^{-6\alpha}, \qquad B_{46} = -3ac^3e^{-4\alpha},$$

$$B_{50} = e^{2\alpha} - a^2b^2e^{-2\alpha} - 27a^2c^6e^{-6\alpha}, \qquad B_{52} = -12a^2bc^3e^{-4\alpha}, \qquad B_{54} = 6ac^3e^{-2\alpha},$$

$$B_{62} = 2[2ac^3e^{-2\alpha} + 3(a^3b^2c^3 + 4a^2c^6)e^{-6\alpha} + 9a^3c^9e^{-10\alpha}],$$

$$B_{64} = -(ab + 4a^2bc^3e^{-4\alpha} - 9a^3bc^6e^{-8\alpha}), \qquad B_{66} = -6(ac^3e^{-2\alpha} + 2a^2c^6e^{-6\alpha}),$$

$$B_{72} = e^{4\alpha} + 3(5ac^3 + a^2b^2) + 45a^2c^6e^{-4\alpha}, \quad B_{74} = -(abe^{2\alpha} - 9a^2bc^3e^{-2\alpha}), \quad B_{76} = -3ac^3,$$

$$B_{82} = 3 + (9ac^3 + a^2b^2)e^{-4\alpha} - 9a^2c^6e^{-8\alpha}, \, B_{84} = -(abe^{-2\alpha} + 3a^2bc^3e^{-6\alpha}), \, B_{86} = 3ac^3e^{-4\alpha},$$

$$D_{12} = (10a^3bc^6 + a^4b^3c^3)e^{-6\alpha} - 3a^4bc^9e^{-10\alpha}, \, D_{14} = 2(ac^3 + 3a^3c^9e^{-8\alpha}), \, D_{16} = -a^2bc^3e^{-2\alpha},$$

$$D_{22} = e^{2\alpha} + (2ac^3 + a^2b^2)e^{-2\alpha} - 3(2a^2c^6 + a^3b^2c^3)e^{-6\alpha} + 9a^3c^9e^{-10\alpha},$$

$$D_{24} = 4a^2bc^3e^{-4\alpha}, \qquad D_{26} = 3(ac^3e^{-2\alpha} + a^2c^6e^{-6\alpha}),$$

$$D_{32} = e^{4\alpha} + 15ac^3 + 3a^2b^2 + 45a^2c^6e^{-4\alpha}, \quad D_{34} = -(abe^{2\alpha} - 9a^2bc^3e^{-2\alpha}), \quad D_{36} = -3ac^3,$$

$$D_{42} = 3 + (9ac^3 + a^2b^2)e^{-4\alpha} - 9a^2c^6e^{-8\alpha}, \, D_{44} = -(abe^{-2\alpha} + 3a^2bc^3e^{-6\alpha}), \, D_{46} = 3ac^3e^{-4\alpha},$$

$$D_{52} = ab - 3a^2bc^3e^{-4\alpha}, \qquad D_{54} = 6ac^3e^{-2\alpha}, \qquad D_{60} = 12a^3bc^6e^{-8\alpha},$$

$$D_{62} = -[e^{2\alpha} - (2ac^3 + a^2b^2)e^{-2\alpha} - 3(2a^2c^6 + a^3b^2c^3)e^{-6\alpha} - 9a^3c^9e^{-10\alpha}],$$

$$D_{64} = 4a^2bc^3e^{-4\alpha}, \qquad\qquad D_{66} = -3(ac^3e^{-2\alpha} - a^2c^6e^{-6\alpha}),$$

$$D_{70} = 3(abe^{2\alpha} - 5a^2bc^3e^{-2\alpha}), \qquad D_{72} = -[e^{4\alpha} - 3(5ac^3 - a^2b^2) + 45a^2c^6e^{-4\alpha}],$$

$$D_{74} = abe^{2\alpha} - 9a^2bc^3e^{-2\alpha}, \qquad D_{76} = 3ac^3,$$

$$D_{80} = 3(abe^{-2\alpha} - a^2bc^3e^{-6\alpha}), \qquad D_{82} = -[3 - (9ac^3 - a^2b^2)e^{-4\alpha} - 9a^2c^6e^{-8\alpha}],$$

$$D_{84} = abe^{-2\alpha} + 3a^2bc^3e^{-6\alpha}, \qquad D_{86} = -3ac^3e^{-4\alpha}.$$

Boundary conditions (17) are satisfied by substitution for the $C$'s in Eqs. (20), (21), and (22) of

$$\left.\begin{aligned}
C_1 &= \tfrac{1}{4}(S_x + S_y), \qquad C_3 = -\tfrac{1}{4}(S_x - S_y), \qquad C_7 = -\tfrac{1}{2}T_{xy}, \\
-2(1 - ac^3e^{-4\alpha_0})C_2 &= ab(S_x + S_y) - e^{2\alpha_0}(S_x - S_y), \\
4(1 - ac^3e^{-4\alpha_0})C_4 &= 4a^2bc^3e^{-2\alpha_0}(S_x + S_y) - (e^{4\alpha_0} + 3ac^3)(S_x - S_y), \\
-2(1 - ac^3e^{-4\alpha_0})C_5 &= [e^{2\alpha_0} - (ac^3 - a^2b^2)e^{-2\alpha_0} + (3a^2c^6 + a^3b^2c^3)e^{-6\alpha_0} \\
&\qquad - 3a^3c^9e^{-10\alpha_0}](S_x + S_y) - 2ab(S_x - S_y), \\
(1 + ac^3e^{-4\alpha_0})C_6 &= e^{2\alpha_0}T_{xy}, \qquad -2(1 + ac^3e^{-4\alpha_0})C_8 = (e^{4\alpha_0} - 3ac^3)T_{xy}.
\end{aligned}\right\} \quad (23)$$

The case $ac^3e^{-4\alpha_0} = \pm 1$, for which some of the $C$'s in Eq. (23) are infinite, does not correspond to a simple curve for $\alpha = \alpha_0$ and hence is excluded.

**7. Stresses along the inner boundary.** The tangential stress in the boundary $\alpha = \alpha_0$ is

$$\sigma_t = (\sigma_\beta)_{\alpha=\alpha_0}.$$

However, it is simpler to compute it as follows. From Eq. (8),

$$\frac{\sigma_\alpha + \sigma_\beta}{h^2} = \frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial\beta^2} \cdot$$

Hence from Eq. (19),

$$\frac{\sigma_\alpha + \sigma_\beta}{h^2} = 4C_1[e^{2\alpha} + a^2b^2e^{-2\alpha} - 9a^2c^6e^{-6\alpha} - 2ab\cos 2\beta]$$
$$+ 4C_2[abe^{-2\alpha} - (1 - 3ac^3e^{-4\alpha})\cos 2\beta] - 4C_6[1 + 3ac^3e^{-4\alpha}]\sin 2\beta. \quad (24)$$
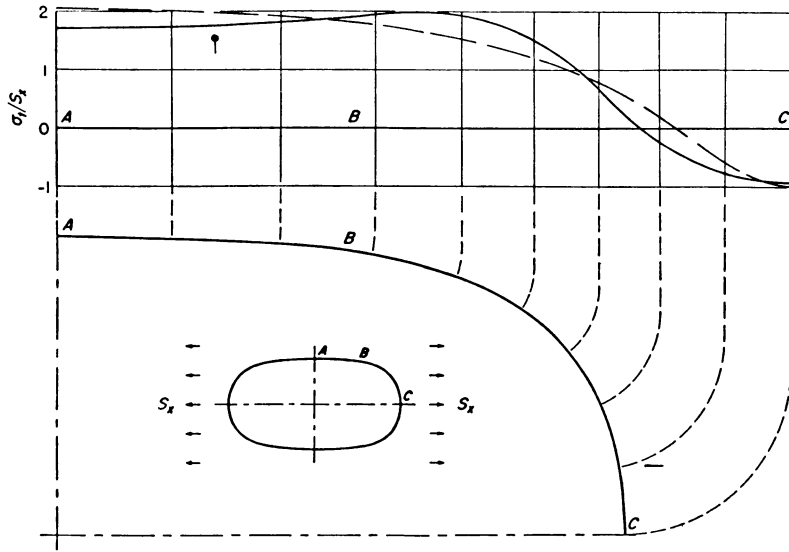
FIG. 6. Ovaloid hole, tension parallel to long axis. Distribution of stress along ovaloid boundary. The dashed curve shows the distribution of stress for the case of an elliptical boundary having the same ratio of major to minor axis and the same rectified length as the ovaloid boundary.
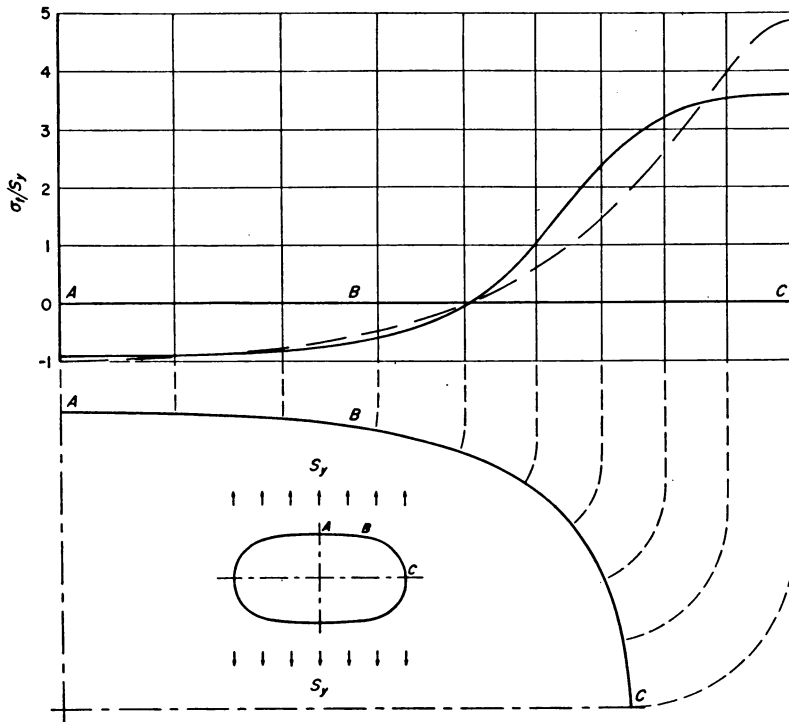


FIG. 7. Ovaloid hole, tension parallel to short axis. Distribution of stress along the ovaloid boundary. The dashed curve shows the distribution of stress for the case of an elliptical boundary having the same ratio of major to minor axis and the same rectified length as the ovaloid boundary.
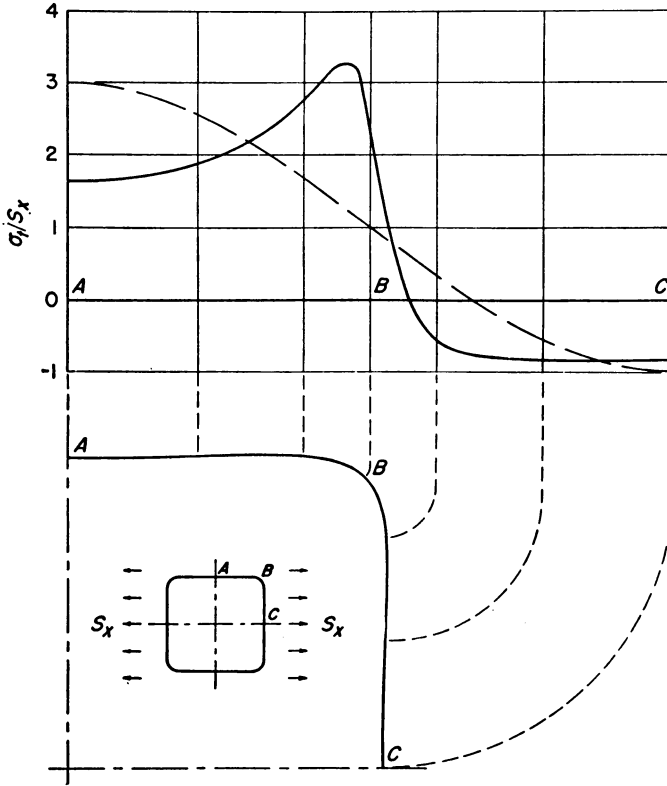
FIG. 8. "Square" hole, tension parallel to side.
The dashed curve shows the distribution of stress for the case of a circular boundary having the same rectified length as the "square" boundary.

For $\alpha = \alpha_0$, $\sigma_\alpha = 0$; hence

$$\sigma_t = (\sigma_\beta)_{\alpha=\alpha_0} = (\sigma_\alpha + \sigma_\beta)_{\alpha=\alpha_0},$$

or from Eq. (24),

$$\frac{\sigma_t}{h_0^2} = 4C_1[e^{2\alpha_0} + a^2b^2e^{-2\alpha_0} - 9a^2c^6e^{-6\alpha_0} - 2ab\cos 2\beta]$$
$$+ 4C_2[abe^{-2\alpha_0} - (1 - 3ac^3e^{-4\alpha_0})\cos 2\beta] - 4C_6[1 + 3ac^3e^{-4\alpha_0}]\sin 2\beta, \quad (25)$$

in which $h_0$ denotes the value of $h$ for $\alpha = \alpha_0$.

Substitution into Eq. (25) of $h_0$ from Eq. (15), of $C_1$, $C_2$, and $C_6$ from Eq. (23), and replacement of the constants $a$, $b$, $c$, and $\alpha_0$ by their values obtained from Eq. (13) gives, finally

$$[(p^2 + 6rq)\sin^2\beta + (q^2 + 6rp)\cos^2\beta - 6r(p + q)\cos^2 2\beta + 9r^2]\sigma_t$$

$$= (S_x + S_y)(p^2\sin^2\beta + q^2\cos^2\beta - 9r^2) - T_{xy}(p + q)^2\frac{p + q + 6r}{p + q + 2r}\sin 2\beta$$

$$- \frac{(p^2 - q^2)(S_x + S_y) - (p + q)^2(S_x - S_y)}{p + q - 2r}[(p - 3r)\sin^2\beta - (q - 3r)\cos^2\beta]. \quad (26)$$

**8. Some special cases.** The components of stress at any point in the plate may be computed from Eqs. (20), (21), (22), and (23). Of especial interest, however, are the values of $\sigma_t$, the tangential stress along the inner boundary, $\alpha = \alpha_0$, at points of which the numerically greatest normal and shearing stresses may be expected to occur.

In this section the values of $\sigma_t$ are computed and shown for several simple cases.

*Case* 1 (*Fig.* 6). Ovaloid hole, tension parallel to long axis. In this case $\sigma_t$ is obtained from Eq. (26) with $S_y = T_{xy} = 0$ and $p$, $q$, and $r$ as given by Eq. (2). Then

$$\frac{\sigma_t}{S_x} = \frac{4.915 - 7.133 \cos 2\beta}{3.723 - 2.316 \cos 2\beta + \cos 4\beta}.$$

*Case* 2 (*Fig.* 7). Ovaloid hole, tension parallel to short axis. Here $S_x = T_{xy} = 0$. Then

$$\frac{\sigma_t}{S_y} = \frac{1.079 + 7.517 \cos 2\beta}{3.723 - 2.316 \cos 2\beta + \cos 4\beta}.$$



*Case* 3 (*Fig.* 8). "Square" hole, tension parallel to side. Here $S_y = T_{xy} = 0$ and $p$, $q$, and $r$ are given by Eq. (3). Then

$$\frac{\sigma_t}{S_x} = \frac{.981 - 2.967 \cos 2\beta}{1.401 + \cos 4\beta}.$$

*Case* 4 (*Fig.* 9). "Square" hole, tension parallel to diagonal. Here $S_y = T_{xy} = 0$ and $p$, $q$, and $r$ are as given by Eq. (4). Then

$$\frac{\sigma_t}{S_x} = \frac{.981 - 1.606 \cos 2\beta}{1.401 - \cos 4\beta}.$$

In each of Figs. 6, 7, 8, and 9 the values of $\sigma_t/S_x$ or $\sigma_t/S_y$ are plotted along the development of one quadrant of the inner boundary of the plate. For comparison, there is shown by means of the dashed curve in each figure the distribution of $\sigma_t/S_x$ or $\sigma_t/S_y$ for the case of an elliptical boundary having the same ratio of major axis to minor axis and the same rectified length as the actual boundary.
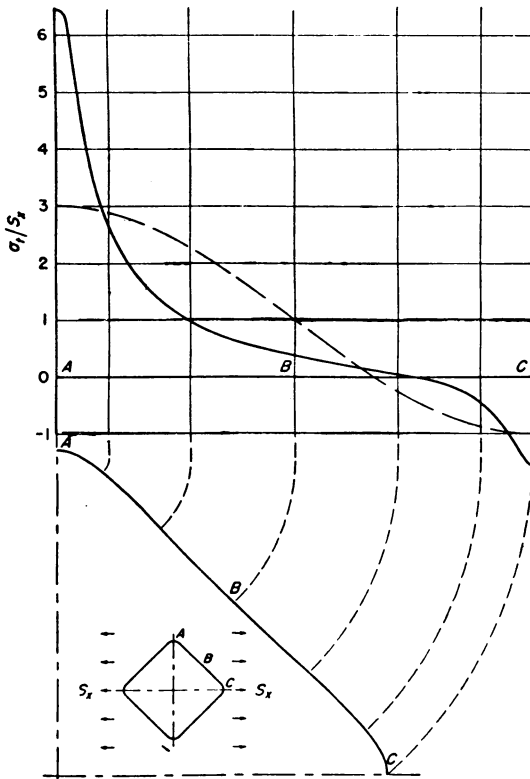
FIG. 9. "Square" hole, tension parallel to diagonal. The dashed curve shows the distribution of stress for the case of a circular boundary having the same rectified length as the "square" boundary.