THE THERMAL-STRESS AND BODY-FORCE PROBLEMS OF
THE INFINITE ORTHOTROPIC SOLID*

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1. Introduction. Elastic problems dealing with orthotropic materials have had
considerable investigation in recent years,1 but up to the present time, such investiga-
tion has been largely limited to a consideration of the problems involving thin plates
of this material.

In the present paper, two problems dealing with the stresses and displacements
in an infinite elastic orthotropic solid are solved, and in each case the results are ob-
tained in terms of three independent displacement potentials. The two solutions are:
1) the displacement potentials arising from an arbitrary distribution of temperature
within a finite region of the solid (the temperature being measured from an arbitrary
datum) and 2) the potentials arising from an arbitrary distribution of body force
within a finite region. Each of these problems reduces to the solution of three simulta-
neous partial differential equations, which are transformed, through the use of
Fourier integrals, into individual solutions for each potential. The expressions for
these potentials are reduced to the form of Newtonian potential integrals for those
cases where sufficient symmetry of the material properties exists to allow such a re-
duction. In the more complicated cases, the results are still expressed in closed form
in terms of definite integrals.

2. The thermo-elastic problem. The conditions under which the thermo-elastic
problem will be formulated and solved are the following. The material is to be homo-
geneous, orthotropic, and elastic, throughout the infinite region, and is to be within
that class of orthotropic materials which has three coefficients of temperature ex-
pansion, αj, associated with the three principal directions of the material. The body
forces will be taken as vanishing, since any problem involving both thermal and body
force effects has a solution which is merely the superposition of the two individual
solutions. The temperature distribution is to be an arbitrary function of position with
the restrictions that this function must vanish everywhere outside some finite region,
be continuous everywhere and be differentiable everywhere except on a finite number
of surfaces.

The fundamental relations needed to formulate the problem mathematically are:
the equations of equilibrium of an element of the material; the thermo-elastic equa-
tions, that is, the relations between strains, stresses and temperature; and the rela-
tions between strains and displacements.

The equations of equilibrium are found by a consideration of the equilibrium of
a rectangular parallelepiped of the material under general loading. Since these equa-
tions are independent of the type of material under consideration, they are given, as

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1 See, for example, A. E. Green, and G. I. Taylor, Stress distributions in aeolotropic plates, Proc. Roy.
Soc. A 173, 163 (1939).
in the isotropic case for zero body force,\(^2\) by three equations of the type,

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad (1)
\]

where the notation is the conventional one.

The orthotropic material has been defined as one whose Hooke’s law has the form indicated by equations (2), when \(T\) is identically zero. The effect of temperatures, different from datum, is to produce normal strains in the three principal directions of the material, as specified under the conditions of the problem. Hence, when the coordinate axes are taken parallel to the principal directions, the general formulas for the strains have the form,

\[
e_s = a_{11} \sigma_x + a_{12} \sigma_y + a_{13} \sigma_z + \alpha_1 T, \ldots; \quad \gamma_{yz} = a_{44} \tau_{yz}, \ldots \quad (2)
\]

If we now define three displacement potentials, \(\phi_j\), such that

\[
u = \frac{\partial \phi_1}{\partial x}, \quad \phi_1 = \frac{\partial \phi_2}{\partial y}, \quad w = \frac{\partial \phi_3}{\partial z},
\]

and such that \(\phi_j\) and its derivatives vanish at infinity, the conventional definitions of the strains become,

\[
e_s = \frac{\partial \phi_1}{\partial x} = \frac{\partial^2 \phi_1}{\partial x^2}, \ldots; \quad \gamma_{yz} = \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_3}{\partial z} = \frac{\partial^2}{\partial y \partial z} (\phi_2 + \phi_3), \ldots \quad (3)
\]

Combining now, equations (1), (2), and (3), we obtain three equations of which the following is the first:

\[
\frac{\partial}{\partial x} \left[ \left( b_{11} \frac{\partial^2}{\partial x^2} + b_{12} \frac{\partial^2}{\partial y^2} + b_{13} \frac{\partial^2}{\partial z^2} \right) \phi_1 + c_{12} \frac{\partial^2}{\partial y^2} \phi_2 + c_{13} \frac{\partial^2}{\partial z^2} \phi_3 \right] = -\beta_1 \frac{\partial T}{\partial x}. \quad (4)
\]

Each of these may be integrated once to give,\(^3\)

\[
\left( b_{11} \frac{\partial^2}{\partial x^2} + b_{12} \frac{\partial^2}{\partial y^2} + b_{13} \frac{\partial^2}{\partial z^2} \right) \phi_1 + c_{12} \frac{\partial^2}{\partial y^2} \phi_2 + c_{13} \frac{\partial^2}{\partial z^2} \phi_3 = -\beta_1 T, \ldots \quad (4a)
\]

The arbitrary functions which appear in each of the foregoing integrations must each vanish, since, for example, in the first equation, all terms vanish when \(x\) is infinite and \(y, z\) are finite, implying that all functions independent of \(x\) must vanish identically.

Due to the convenient form of the boundary conditions, these equations are easily integrated by the following procedure. Multiply each equation through by \(e^{-i\left(e^{2}+v^2+i^2\right)}\) and integrate over the whole region, integrating by parts those terms containing derivatives of \(\phi_j\). This operation produces the following three equations, using the abbreviated forms defined below in equations (6).


\(^3\) The \(b_{ij}, c_{ij}\) and \(\beta_j\) are combinations of elastic and thermal constants arising from the above operation. The manner in which these constants appear in the second and third of these is easily deduced from equations (5).
\[(b_{11} \xi^2 + b_{12} \eta^2 + b_{15} \zeta^2) E_1 + c_{12} \eta^2 E_2 + c_{13} \zeta^2 E_3 = \beta_1 S,\]
\[c_{12} \xi^2 E_1 + (b_{66} \xi^2 + b_{22} \eta^2 + b_{44} \zeta^2) E_2 + c_{23} E_3 = \beta_2 S,\]
\[c_{13} \xi^2 E_1 + c_{27} \eta^2 E_2 + (b_{66} \xi^2 + b_{44} \eta^2 + b_{28} \zeta^2) E_3 = \beta_3 S,\]

where,
\[E_j = \int \int \int_{-\infty}^{\infty} \phi_j e^{-i(\xi \xi + \eta \eta + \zeta \zeta)} d\xi d\eta d\zeta,\]
\[S = \int \int \int T e^{-i(\xi \xi + \eta \eta + \zeta \zeta)} d\xi d\eta d\zeta.\]

Equations (5) are easily solved for the \(E_j\), and yield the expressions,
\[E_i = F_i(\xi, \eta, \zeta) S,\]

where the \(F_i\) become ratios of homogeneous polynomials in \(\xi^2, \eta^2\) and \(\zeta^2\).

Noting now, that by their definitions, the \(E_j\) are the Fourier transforms (in three dimensions) of the \(\phi_j\), we may write
\[\phi_j(x, y, z) = \frac{1}{8\pi^3} \int \int \int_{-\infty}^{\infty} F_j(\xi, \eta, \zeta) e^{i(\xi x + \eta y + \zeta z)} d\xi d\eta d\zeta\]
\[= \frac{1}{8\pi^3} \int \int \int_{-\infty}^{\infty} F_j(\xi, \eta, \zeta) e^{i(\xi x + \eta y + \zeta z)} d\xi d\eta d\zeta \int \int \int_{-\infty}^{\infty} T(r, s, t) e^{-i(\xi x + \eta y + \zeta z)} dr ds dt,\]

and the order of the indicated integrations may be changed to give,
\[\phi_i = \frac{1}{8\pi^3} \int \int \int_{-\infty}^{\infty} T(r, s, t) dr ds dt \int \int \int_{-\infty}^{\infty} F_j(\xi, \eta, \zeta) e^{i((x-r)xi + (y-s)\eta \eta + (z-t)\zeta \zeta)} d\xi d\eta d\zeta.\]

Since each \(F_j\) (as defined by equation 7) is a ratio of second order polynomial in \(\xi^2, \eta^2,\) and \(\zeta^2,\) to one of third order, we may write,
\[F_j = B \frac{R_1^2 R_2^2}{R_1^2 R_2^2},\]
where \(R_2^2 = \lambda_2^2 \xi^2 + \mu_2^2 \eta^2 + \zeta^2,\) and where the \(\lambda_k, \mu_k,\) and \(B,\) are constants depending on the values of the constants appearing in the determinants defining \(F_j,\) and hence, may be considered as known. Note that the \(\lambda_k^2, \mu_k^2,\) for \(k = 1, 2, 3,\) must be non-negative, since no singularity may exist except at the origin.

In many cases, the expressions for the \(F_j\) may be reduced to the form,
\[F_j = \sum_{k=1, 2, 3} \frac{A_{jk}}{R_k^2}.\]

This will always be true when the problem involves a material which is isotropic in a certain plane (for example, a laminated plastic) unless identical values of \(R^2\) recur in the denominator. This may be seen by noting that since the denominator of \(F_j\) must be invariant under a rotation about the \(z\) axis due to this isotropy (the plane of isotropy is here taken as the \(x, y\) plane), \(\xi^2\) and \(\eta^2\) must occur in the combination \(\xi^2 + \eta^2,\) and hence, \(\lambda_k = \mu_k,\) and the \(R_k^2\) become essentially binomials. The reduction of \(F_j\) to the form of equation (10) is, in this case, merely a matter of evaluating \(A_{jk}.\)
When equation (10) does hold, the integration proceeds as follows: Using the conventional vector notation and the new coordinates with the subscript \( k \) (where \( \xi_k = \lambda_k \xi, \ x_k = x \xi_k^{-1}, \ r_k = r \xi_k^{-1}, \) etc. and where \( \vec{m}_k = i(x_k - r_k) + j(y_k - s_k) + k(z_k - t_k) \)), the integral over \( \xi, \eta, \) and \( \zeta, \) of equation (9) defining Green's function \( G \), may be written,

\[
G_j(x, y, z, r, s, t) = \int \int \int_{-\infty}^{\infty} \sum_{k=1,2,3} A_{jk} \frac{e^{i\vec{m}_k \cdot \vec{R}_k}}{R_k^2} \frac{d\xi_k}{\lambda_k} \frac{d\eta_k}{\mu_k} d\zeta_k. \tag{11}
\]

If we now change to a spherical coordinate system in which \( \gamma \) is the angle between \( \vec{m}_k \) and \( \vec{R}_k \) and \( \delta \) is the polar angle about \( \vec{m}_k \), this integral becomes,

\[
G_j = \int \int \int \sum_{k=1,2,3} \frac{A_{jk}}{\lambda_k \mu_k} e^{im_k \cdot R_k} \cos \gamma \sin \gamma \ d\gamma \ d\delta \ dR_k,
\]

where the integration now takes place over, \( 0 \leq \gamma \leq \pi, \ 0 \leq \delta \leq 2\pi, \ 0 \leq R_k < \infty \). The elementary integrations over \( \gamma \) and \( \delta \) produce

\[
G_j = \sum_{k=1,2,3} \frac{4\pi A_{jk} m_k R_k}{\lambda_k \mu_k} \int_{0}^{\infty} \sin m_k R_k dR_k,
\]

which is known to have the value,

\[
G_j = 2\pi^2 \sum_{k=1,2,3} \frac{A_{jk}}{\lambda_k \mu_k} \frac{1}{m_k}.
\]

Now transforming the remaining terms of equation (9) to the coordinates with the subscript \( k \), and substituting the above value for Green's function, we obtain,

\[
\phi_j = \frac{1}{4\pi} \int \int \int \sum_{k=1,2,3} \frac{A_{jk} T(\lambda_k r_k, \mu_k s_k, t_k)}{\sqrt{(x_k - r_k)^2 + (y_k - s_k)^2 + (z_k - t_k)^2}} \ d\tau_k \ d\delta_k \ dt_k. \tag{12}
\]

Hence, the problem, wherein \( T(x, y, z) \) represents the temperature distribution, becomes the problem of evaluating the Newtonian potential function corresponding to a mass distribution of,

\[
\rho = \frac{A_{jk}}{4\pi} T(\lambda_k x_k, \mu_k y_k, z_k).
\]

For an isotropic material, the \( \phi_k \) become alike, and are given by,

\[
\phi_i = \frac{\alpha}{4\pi} \frac{1 + v}{1 - v} \int \int \int \frac{T(r, s, t)}{\sqrt{(x - r)^2 + (y - s)^2 + (z - t)^2}} \ dr \ ds \ dt.
\]

In the evaluation of Green's function for those cases where the denominator of \( F_j \) has a multiple root, it is convenient to introduce the notation

\[
G_i = \sum_n G_{jn}, \quad \Delta_k = \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} + \frac{\partial^2}{\partial z_k^2}.
\]

In this case, integrals of the form,

\[ G_{jn} = \int \int \int_{-\infty}^{\infty} \frac{A_{jn} R_1^2}{R_2^4} e^{i\left[(x-r)\xi + (y-s)\eta + (z-t)\zeta\right]} d\xi \, d\eta \, d\zeta \] (13)

must be evaluated, provided \( R_1^2 \neq R_2^2 \). Using the above notation, the equivalence of the following to equation (13) may be verified by substitution:

\[ \Delta_x G_{jn} = \Delta_1 \int \int \int \frac{A_{jn}}{R_2^4} e^{i\left[(x-r)\xi + (y-s)\eta + (z-t)\zeta\right]} d\xi \, d\eta \, d\zeta. \] (13a)

The integral involved in this equation is, however, the same as that appearing in equation (11), so (13a) becomes,

\[ \Delta_2 G_{jn} = \Delta_1 \frac{2C_{jn}}{\sqrt{(x_2 - r_2)^2 + (y_2 - s_2)^2 + (z_2 - t_2)^2}}, \]

where \( C_{jn} \) is an easily evaluated constant. Substitution will again show that,

\[ G_{jn} = C_{jn} \Delta_1 \sqrt{(x_2 - r_2)^2 + (y_2 - s_2)^2 + (z_2 - t_2)^2}, \]

is equivalent to the above equation, and hence the \( \phi_j \) are given by,

\[ \phi_j = \frac{1}{8\pi^3} \sum_n \int \int \int T(r, s, t) G_{jn}(r, s, t, x, y, z) dr \, ds \, dt. \] (14)

In those cases where \( F_j \) cannot be reduced to one of the foregoing convenient forms, \( G_j \) is more difficult to evaluate. Since no explicit form has been found for this function, other than complicated definite integrals, it is believed best to leave it in the form defined by equation (9).

3. The body-force problem. As in dealing with isotropic materials, the solution of the body force problem may be shown to reduce to a form analogous to that of the thermo-elastic problem. To show this, we shall consider only the problem where the body force is directed parallel to the \( x \) axis, noting that the general solution is obtained by the superposition of three such problems.

Equations (1) and (2) are modified to contain the body-force function, \( X \), and to eliminate the temperature terms. Equations (4) are then obtained again, where now the right hand sides are replaced respectively by, \( X \), 0, and 0.

The \( \phi_j \) will not, in general, vanish at infinity in this problem, hence the procedure needs a slight modification. The second and third of these equations are integrated with respect to \( y \) and \( z \) respectively and then differentiated with respect to \( x \). This yields equations (4a), where again, \( X \), 0, 0, appear on the right and where the \( \phi_j \) are replaced by \( \partial \phi_j / \partial X \). The procedure is now identical with that of the thermal problem, and the \( \phi_j \) are found by the expressions analogous to equation (14).

4. The two-dimensional problem. If we carry through in two dimensions the procedure used in the previous sections of this paper, we arrive at an equation which is identical to equation (8) except that \( z \), \( t \), and \( \xi \), no longer appear. The expressions for \( F_j \) are now simpler in form, being given by,

\[ F_j = \frac{\lambda \xi^2 + \mu \eta^2}{(\lambda \xi^2 + \mu \eta^2)(\lambda \xi^2 + \mu \eta^2)}. \]
which may always be reduced to the form

\[ F_i = \sum_{k=1,2} \frac{A_{jk}}{R^2} \]

unless \( R_2 = R_3 \). \((R_2^2 = \lambda_2^2 \xi_2^2 + \mu_2^2 \eta_2^2)\).

Before changing the order of integration, we differentiate equation (8) with respect to \( y \). The integral form of Green’s function becomes then

\[ \frac{\partial G_j}{\partial y} = \int \int \frac{A_{jk}}{\lambda_k \xi_k^2 + \mu_k \eta_k^2} \, d\xi \, d\eta, \quad (15a) \]

unless \( R_2 = R_3 \), in which case,

\[ \frac{\Delta_2}{\partial y} = \Delta_1 \int \int A_i \frac{i\eta e^{i(x \xi + y \eta)}}{\lambda \xi^2 + \mu \eta^2} \, d\xi \, d\eta. \quad (15b) \]

This latter expression is, of course, derived by the same reasoning used in the three dimensional problem.

Equation (15a), after the introduction of the coordinates with the subscript \( k \), can be written in the iterated integral form,

\[ \frac{\partial G_{jk}}{\partial y} = \int_0^\infty 4A_{jk} \frac{\xi_k}{\lambda_k} \sin \eta_k \, d\eta_k \int_0^\infty \frac{\cos \eta_k x_k \eta_k}{1 + \left( \frac{\xi_k}{\eta_k} \right)^2} \, d\left( \frac{\xi_k}{\eta_k} \right) \]

which is known to be equivalent to,

\[ \frac{\partial G_{jk}}{\partial y} = \int_0^\infty 4A_{jk} \frac{\sin \eta_k \eta_k \pi}{2} e^{-r x \eta k} \, d\eta_k \]

and this integral yields,

\[ \frac{\partial G_{jk}}{\partial y} = \frac{2\pi A_{jk}}{\lambda_k} \frac{y_k}{x_k^2 + y_k^2} \]

or

\[ G_{jk} = \frac{\pi A_{jk}}{\lambda_k \mu_k} \ln (x_k^2 + y_k^2), \quad (16) \]

and we obtain the familiar two-dimensional logarithmic potential.

Equation (15b), then becomes, in an analogous manner,

\[ \Delta_2 G_i = \frac{\pi A_i}{\lambda_k \mu_k} \Delta_1 \ln (x_i^2 + y_i^2) \]

or,

\[ G_i = \frac{\pi A_i}{2\lambda_k \mu_k} [x_i^2 + y_i^2 \ln (x_i^2 + y_i^2)]. \quad (17) \]

Hence, Green’s functions are determined for each two-dimensional problem involving thermal stress or body forces in the infinite plate. The usual methods of superimposing plane stress (or strain) solutions may be utilized, of course, to solve the corresponding problems for the finite body.