# THE RESONATOR ACTION THEOREM*

BY

W. R. MACLEAN

*Polytechnic Institute of Brooklyn*

**1. The problem.** In the kinetic theory of gases there are a series of theorems relating to the "adiabatic invariants" of mechanical systems.[1] One such theorem, which is a special case[2] of a more general theorem of Boltzmann,[3] is contained in the statement:

*In a frictionless machine having a periodic motion, the kinetic action, i.e. the product of mean kinetic energy and period, is invariant against an adiabatic deformation.*

The word "adiabatic," relating to a machine, seems a strange one to use, but the meaning here bears an analogy to thermodynamics. If a machine moving in periodic motion has additional coordinates which are normally held fixed, but can, at will, be varied by outside influences, a variation of these will, in general, change the period. An example would be a violin string which is tightened while vibrating. *Adiabatic* means that the deformation is carried out with perfect uniformity and infinite slowness so that the energy so introduced has only d-c components and hence none near the frequency of oscillation.

If the machine is linear, that is, can be represented by linear differential equations (which will have constant coefficients when the machine is not being deformed), the theorem can be simplified to: *In a linear frictionless machine having a periodic motion, the action, i.e. the product of total energy and period, is invariant against an adiabatic deformation.* The simplification follows from the fact that in this type of system the mean potential and kinetic energies are equal.

The proof of this theorem, as carried out in Jeans and elsewhere, although quite general for a machine is still limited to a finite number of degrees of freedom. For the case of an electromagnetic resonator (where an infinity of degrees of freedom is involved) the proof for a cubical box has been indicated by Ehrenfest[4] in a paper on quantum problems. The theorem is, however, usually accepted as true in general,[5] but as far as the author is aware, a proof for a resonator of any shape and any distribution of dielectric constant as is needed for application to radio problems is still lacking.

It is the purpose of this paper to carry through such a proof for the specific case of a general lossless electro-magnetic resonator. Such a proof is considered worth while for engineers, since the theorem has certain uses in the design of equipment. For instance, if one is interested in the frequency change caused by a slight deformation

* Received June 12, 1944.

[1] For quick reference see for instance J. H. Jeans, *The dynamical theory of gases*, 4th ed. Cambridge University Press, 1925, p. 410 ff.

[2] P. Ehrenfest, Proceedings of the Amsterdam Academy, **16**, 591 (1914).

[3] L. Bolzmann, *Vorlesungen über Mechanik*, II. J. A. Barth, Leipzig, 1904, §48.

[4] P. Ehrenfest Annalen der Physik, **36**, 91 (1911).

[5] For instance, in L. Brillouin, *La Statistique Quantique*, Les Presses Universitaires de France, Paris, 1930, p. 39.

of an electro-magnetic structure and if this proves difficult to evaluate directly, one can instead try to determine the work done against the electro-magnetic forces, which by this theorem will give the desired result.

**2. Preliminary observations.** The basis of the proof is formed, of course, by Maxwell's equations. For an uncharged region without permanent magnetism, conductivity, or driving forces, they become[6,7]

Primary Equation: $\text{curl } \mathbf{H} = \mathbf{D}' + \mathbf{G}$. But here, $\mathbf{G} = 0$.
Secondary Equation: $\text{curl } \mathbf{E} = -\mathbf{B}'$.
Magnetic Equation: $\text{div } \mathbf{B} = 0$.
Electric Equation: $\text{div } \mathbf{D} = 0$.
Dielectric Equation: $\mathbf{D} = \epsilon \mathbf{E}$.
Core Equation: $\mathbf{B} = \mu \mathbf{H}$.

For later purposes, it is convenient to introduce a vector potential in a manner somewhat different from the usual one. We suppose that there exists a field satisfying the above equations, and then by virtue of the electric equation, find a $\mathbf{U}$ such that:

$$\mathbf{D} = \text{curl } \mathbf{U}. \tag{1}$$

This designation still leaves $\mathbf{U}$ in large part arbitrary. From the Primary Equation, one can write:

$$\mathbf{H} - \mathbf{U}' = -\text{grad } \psi,$$

where $\psi$ is a suitable scalar. One now chooses

$$\text{div } \mathbf{U}' = \text{div } \mathbf{H}.$$

$\mathbf{U}$ is still arbitrary to the extent of the gradient of a harmonic (in the sense of potential theory) function. But from the last two:

$$\Delta\psi = 0;$$

hence $\psi$ is a harmonic function. So we can still adjust $\mathbf{U}$ so that:

$$\mathbf{H} = \mathbf{U}'. \tag{2}$$

The Secondary Equation finally subjects $\mathbf{U}$ to a differential equation:

$$\text{curl } [(1/\epsilon) \text{ curl } \mathbf{U}] = -\mu \mathbf{U}''. \tag{3}$$

Since we are not restricting ourselves to homogeneous media, this cannot be reduced to the usual Wave Equation. However, a step in the proof can be taken by the usual device of setting

$$\mathbf{U}(x, y, z, t) = \mathrm{S}(x, y, z) f(t)$$

which leads to

$$f(t) \text{ curl } [(1/\epsilon) \text{ curl } \mathrm{S}(x, y, z)] = -\mu \mathrm{S}(x, y, z) f''(t).$$

Since this is identically true in $x, y, z, t$, it can only hold if $f$ and $f''$ are proportional. In particular

[6] The author feels that Maxwell's equations should have names if for no other reason than to facilitate reference to them. "Primary" and "secondary" remind one of the phenomena in an unloaded transformer.

[7] Primes indicate derivatives with respect to time.

$$f'' = -\omega^2 f,$$

the solution of which is:

$$f = a \cos \omega t + b \sin \omega t.$$

Hence, individual solutions of (3) are sinusoidal in time. Since (3) is linear, solutions are made up by a superposition of such particular solutions. Substituting the last into (3) leads to:

$$\text{curl}\left[(1/\epsilon)\ \text{curl}\ \mathbf{S}\right] - \omega^2\mu\mathbf{S} = 0.$$

This will have solutions satisfying the boundary conditions (**E** normal and **B** tangent to the metal surfaces) only for certain values of $\omega$, the so-called *eigen*frequencies of the resonator. In general, these eigenfrequencies will not be harmonically related. Since the theorem requires that the resonator have a period both before and after the deformation it is necessary to restrict our considerations to the case where only one mode is excited at a time. We must have only one mode, i.e., one frequency also *after* the deformation. This precaution is made necessary by the existence of the special case of *degenerate modes.*

If a given resonator had a set of eigenfrequencies, one would expect after a very small deformation that it would then have a different set of eigenfrequencies which could be put in one to one correspondence with the old ones, and wherein corresponding ones would differ by only a very small amount. In particular, if the deformation were made small enough, these differences could be made small compared with the spacing of the eigenfrequencies of the original or final resonator.

This expectation is correct with one exception. Suppose that a particular mode $M_a$, having radian frequency $\omega_a$ were excited. Some field quantity, say the potential $\overline{\mathbf{U}}$ would be a particular function of position:[8]

$$\overline{\mathbf{U}} = \overline{\mathbf{U}}_a(x, y, z).$$

In general, this is the only function which can satisfy the differential equation and the boundary conditions. As such it is called an *eigen*function of the problem belonging to the eigenfrequency, $\omega_a$. As an exceptional case, however, it can occur that more than one eigenfunction, i.e., more than one *mode* exists for the same frequency. Then this particular frequency is called *degenerate.* The number of linearly independent modes corresponding to one frequency is called the degree of degeneracy. This phenomenon is mathematically similar to the Stark and Zeeman effects.

A degeneracy depends on the existence of some type of symmetry in the resonator. For instance, in a cylindrical resonator, the $TE_{111}$ mode[9] is degenerate.

A single $TE_{111}$ has a transverse axis of orientation, say the diameter along which **E** is radial. One can, however, superimpose thereon another $TE_{111}$ orientated at right angles to the first and having perhaps a time phase angle against the first. They are obviously linearly independent and have the same frequency. If now such a cylindrical resonator were deformed in such a manner as to destroy the symmetry, the degeneracy would in general be broken and the frequencies split. As a result, the condition that the resonator have *a* period after the deformation would not be met.

---

[8] The bar over a letter indicates complex vector phasor. This is the usual notation for simple harmonic fields.

[9] This designation of modes is now common. See, for example, Sarbacher and Edson, *Hyper and ultra-high frequency engineering,* J. Wiley and Sons, 1943, p. 387.

In this special case, the theorem applies to each component separately. The proper component in the degenerate state can always be chosen as follows: one deforms the resonator splitting the frequencies, then quenches all but one frequency, then relaxes the deformation. Repetition of this deformation will then yield only one frequency.

For purposes of the theorem, it is necessary to demonstrate that the mean electric and magnetic energies of a resonator oscillating in one mode are equal. In complex amplitude notation one has for the magnetic energy:[10]

$$2\langle W_M \rangle = \int_V dv\, \bar{\mathbf{H}}^* \cdot \bar{\mathbf{B}}.$$

Applying the Secondary Equation and the vector identity for the divergence of a cross product:

$$\text{div}\,(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl}\,\mathbf{A} - \mathbf{A} \cdot \text{curl}\,\mathbf{B} \tag{4}$$

one has

$$2\langle W_M \rangle = \int_V dv\, \mathbf{H}^* \cdot j(\text{curl}\,\mathbf{E})/\omega = \int_V (j dv/\omega)\, \text{div}\,(\bar{\mathbf{E}} \times \bar{\mathbf{H}}^*) + \int_V (j dv/\omega) \bar{\mathbf{E}} \cdot \text{curl}\,\bar{\mathbf{H}}^*.$$

The first integral on the right transforms to the surface by Gauss' theorem and vanishes by the rule of exchanging dot and cross in the triple product since $\bar{\mathbf{E}}$ is normal to the surface. In the second integrand we use the primary equation and have

$$2\langle W_M \rangle = \int_V dv\, \bar{\mathbf{E}} \cdot \bar{\mathbf{D}}^* = 2\langle W_E \rangle.$$

If one solves for the field, say for $\bar{\mathbf{U}}$, in a resonator oscillating at one frequency, it is conceivably possible for $\bar{\mathbf{U}}$ to be a *complete* complex vector, i.e., have non-colinear real and imaginary parts. If this is so, however, then the mode is certainly degenerate, since the real and imaginary parts of $\bar{\mathbf{U}}$ each satisfy the differential equations. The fields derived from each part of $\bar{\mathbf{U}}$ individually satisfy Maxwell's equations and the boundary equations. So as long as we restrict ourselves to non-degenerate modes it suffices to consider only field vectors which swing on a line. Moreover, since only pure real or imaginary $\bar{\mathbf{U}}$ is needed, $\mathbf{U}$ is everywhere in phase, as are $\mathbf{E}$ and $\mathbf{H}$. The latter two are, however, in exact time quadrature with each other as is seen from Maxwell's Equations.

To compute the work done in deforming the walls of the resonator or moving the pieces of material within it, one needs the expression for the density of force exerted. For the material, this is found by taking the divergence of the Maxwell tension tensor,[11] $T$:

$$T_{ij} = E_i D_j - \tfrac{1}{2}\delta_{ij}\mathbf{E} \cdot \mathbf{D} + H_i B_j - \tfrac{1}{2}\delta_{ij}\mathbf{H} \cdot \mathbf{B}. \tag{5}$$

The density of force, $\mathbf{k}$, is then found from

$$k_i = \sum_{j=1}^{3} \partial T_{ij}/\partial x_j$$

[10] The marks < > designate time average over a cycle. Asterisk means complex conjugate. The complex amplitudes are *rms*.

[11] $\delta_{ij}$ is the Kronecker delta: equals unity if the indices are alike, otherwise zero. Note we are now working with instantaneous values.

which can be evaluated to give

$$\mathbf{k} = -\tfrac{1}{2}\mathbf{E}^2 \operatorname{grad} \epsilon - \tfrac{1}{2}\mathbf{H}^2 \operatorname{grad} \mu + \epsilon\mu \frac{d}{dt}(\mathbf{E} \times \mathbf{H})$$

for an uncharged, passive, and lossless region.

Since the deformations to be made are to be *adiabatic*, i.e., carried out uniformly with extreme slowness, the work done can be obtained from the mean value of the force. Averaging the last over a cycle, one has:

$$\langle \mathbf{k} \rangle = -\tfrac{1}{2}\langle \mathbf{E}^2 \rangle \operatorname{grad} \epsilon - \tfrac{1}{2}\langle \mathbf{H}^2 \rangle \operatorname{grad} \mu. \tag{6}$$

This is the density of force exerted *by* the field *on* the material.

To find the force exerted by the field on the walls of the resonator, pick a certain point on the walls and there make a coordinate transformation bringing the $x$ axis coincident with the *inward* normal. **E** has then only an $x$ component. At the same time make the $y$ axis coincide with the direction of the magnetic field which lies in the boundary. The tension then takes on the form

$$T_{ij} = \begin{pmatrix} w_E & 0 & 0 \\ 0 & -w_E & 0 \\ 0 & 0 & -w_E \end{pmatrix} + \begin{pmatrix} -w_M & 0 & 0 \\ 0 & w_M & 0 \\ 0 & 0 & -w_M \end{pmatrix},$$

where the $w$'s are the electric and magnetic energy densities. Since the vector element of surface has only an (positive) $x$ component (for inward normal) the surface density of force, **f**, becomes

$$\mathbf{f} = (w_E - w_M),\ 0,\ 0.$$

Since the $x$ component is directed inwardly, we can say that the field exerts either pure pressure or pure suction on the walls. In particular, if we designate as the density of the Lagrangian function:

$$l = w_M - w_E, \tag{7}$$

the field exerts an (algebraic) pressure, $p$, on the walls, given by:

$$p = l. \tag{8}$$

For slow uniform (adiabatic) movements of the walls, the work done can be found from the mean pressure

$$\langle p \rangle = \frac{1}{\tau}\int_0^\tau l\,dt = \langle l \rangle, \tag{9}$$

where $\tau$ is the period of oscillation.

**3. The resonator theorem.** *In a lossless electromagnetic resonator, the action of each mode, i.e., the product of total energy and period, is invariant against an adiabatic deformation.* In case a symmetry creating a degenerate frequency is involved, the separation of modes is to be carried out as described above.

Since the mean electric and magnetic total energies are equal, the integral of $l$ over the volume $V$ and the period $\tau$ vanishes:

$$h = \int_0^\tau dt \int_V l\,dv = 0. \tag{10}$$

Since this is true for any resonator, it is also true *after* we have deformed our resonator slightly. Hence, the variation of $h$, i.e., the difference between a new and an old $h$, is zero.

To see most clearly what is meant, it is best to imagine two editions of the original resonator at hand, one of which is varied. After a slight variation, they beat ever so slowly with each other. Since throughout each, **U** is in phase, one waits for a cycle when the **U** in both resonators pass through zero together. This instant we take as $t=0$. Since they are almost alike, the two **U**'s will differ by only a small amount over the following cycle.

The variation in $h$, $\delta h$, which is zero, is brought about by four effects: (a) the change in period, (b) the change in volume, (c), the change in the field, (d) the change in dielectric constant and permeability due to movement of the materials.[12] These four increments together lead to

$$\delta h = \delta\tau[L]_\tau + \tau\int_{\delta v}\langle l\rangle dv + \int_0^\tau dt\int_V \delta l\, dv = 0,$$

where $L$ is the volume integral of $l$, i.e. $L$ is the Lagrangian function. Since $L$ is periodic, the initial rather than final value may be used in the first term. The second term is integrated over $\delta V$ which consists of a thin shell adjoining the envelope, hence the value of $<l>$ to be used is that obtaining on the closest surface point. One sees, therefore, that the second term is proportional to the work done by the field on the walls. Hence, in terms of the field energy, $W$, it is $\tau$ times the negative increment of $W$ due to the movement of conductors. So the last becomes:

$$0 = L_0\delta\tau - \tau(\delta W)_c + \int_0^\tau dt\int_V dv\,\delta\left(\frac{1}{2}\mu\mathbf{H}^2 - \frac{1}{\epsilon}\mathbf{D}^2\right).$$

In the expression for $l$, the magnetic and electric energy densities are written in inverse form, so to speak, for reasons of future convenience. Carrying out the indicated variation in the last term above one gets

$$0 = L_0\delta\tau - \tau(\delta W)_c + \int_0^\tau dt\int_V dv(\mathbf{B}\cdot\delta\mathbf{H} - \mathbf{E}\cdot\delta\mathbf{D}) + \frac{1}{2}\int_0^\tau dt\int_V dv(\mathbf{H}^2\delta\mu + \mathbf{E}^2\delta\epsilon).$$

Equation (6) gives an expression for the mean value of the force density on the material. If one multiplies (6) by $\delta\mathbf{x}$, i.e. by the displacement of the material, and integrates over the volume, one has the work done by the field on the material. This is proportional to the last term above, the latter being just $\tau$ times the negative increment of field energy due to movement of the material. Also by (2) the quantity **H** in the first integral can be expressed in terms of the potential, **U**, leading to:

$$0 = L_0\delta\tau - \tau(\delta W)_c + \int_0^\tau dt\int_V dv\mathbf{B}\cdot\delta\mathbf{U}' - \int_0^\tau dt\int_V dv\mathbf{E}\cdot\delta\mathbf{D} - \tau(\delta W)_{\text{mat}}.$$

The variations of $W$ due to conductor and material movement combine to form

[12] The question of surfaces of discontinuity on material boundaries is ignored. This is permissible, since they may be replaced by large though finite values of the gradient of the material constants. It is merely a matter of mathematical convenience which method is adopted.

the total variation of field energy. In the first integral we replace the integrand by the time derivative of both factors less the correction term and integrate the former with respect to time. We also express **D** in terms of **U** by (1), giving

$$0 = L_0\delta\tau - \tau\delta W + \int_V dv[\mathbf{B}\cdot\delta\mathbf{U}]_0^\tau - \int_0^\tau dt \int_V dv(\mathbf{B}'\cdot\delta\mathbf{U} + \mathbf{E}\cdot\text{curl }\delta\mathbf{U}). \tag{11}$$

Since **B** has period $\tau$, it can be taken outside the time limits in the first integrand, which leaves $\delta\mathbf{U}$ between those limits. However, $\delta\mathbf{U}=0$ at $t=0$ by our choice of time origin. At $t=\tau$, the old **U** is also zero, hence $\delta\mathbf{U}$ at that time is simply the new **U**, say $\tilde{\mathbf{U}}$, at $t=\tau$. The $x$ component of the latter would be given, for example, by

$$\tilde{U}_x(t) = A \sin(\omega + \delta\omega)t.$$

The amplitude can be expressed in terms of the initial derivative, and if we also substitute $t=\tau$, we have:

$$[\delta U_x]_\tau = \tilde{U}_x(\tau) = \frac{1}{\omega + \delta\omega} U_x'(0) \sin \delta\omega\,\tau.$$

For small variations, the *sin* may be dropped, and if $\delta\omega$ is expressed in terms of $\delta\tau$, one has to first order:

$$[\delta U_x]_\tau = -\tilde{U}_x'(0)\delta\tau = -[U_x']_0\delta\tau.$$

In the last step it is permissible to equate the old to the new since we have already obtained the variation.

Substituting this result into (11) one has:

$$0 = L_0\delta\tau - \tau\delta W - \delta\tau\int_V dv[\mathbf{B}\cdot\mathbf{U}']_0 - \int_0^\tau dt \int dv(\mathbf{B}'\cdot\delta\mathbf{U} + \mathbf{E}\cdot\text{curl }\delta\mathbf{U}).$$

Since $\mathbf{U}'=\mathbf{H}$, the first integral is the negative of twice the magnetic energy at $t=0$ times the variation in period. This combines with the first term to give $-W_0\delta\tau$. But since the *total* energy does not vary in time, one has by changing sign in the last:

$$0 = W\delta\tau + \tau\delta W + I, \tag{12}$$

where $I$ is the last integral above which we will now show to be zero. By the use of the identity (4) the last term can be integrated by parts which gives

$$I = \int_0^\tau dt \int_S (\delta\mathbf{U}\times\mathbf{E})\cdot d\mathbf{s} + \int_0^\tau dt \int_V dv(\mathbf{B}' + \text{curl }\mathbf{E})\cdot\delta\mathbf{U}.$$

The second integrand is zero by the Secondary Equation, and the first is seen to be zero by interchanging dot and cross since **E** is normal to the surface.

Using this in (12) we have simply:

$$\delta(\tau W) = 0$$

which is the theorem.