$$\phi(z) = z + f(z)\frac{f(z)h(z) - 1}{f'(z)}\cdot$$

Thus we may impose more rigid conditions further strengthening the convergence, viz.,

$$\phi'(\alpha) = 0,\ \phi''(\alpha) = 0, \cdots, \phi^{(n)}(\alpha) = 0, \tag{6}$$

where n is any positive integer.[5]

A function $\phi(z)$ satisfying the conditions (5) and (6) can be obtained in the following way. The conditions (6) will be satisfied if the derivative of $\phi(z)$ appears in the form

$$\phi'(z) = (f(z))^n g(z) f'(z),$$

the undetermined function $g(z)$ being regular at $z=\alpha$. It remains to adapt $g(z)$ to the condition $\phi(\alpha)=\alpha$. One has

$$\phi(z) = \int (f(z))^n g(z) f'(z) dz = \int w^n g(f^{-1}(w)) dw$$

whence, by repeated integration by parts, it follows that

$$\phi(z) = n!\sum_{\nu=0}^{n} \frac{(-1)^\nu}{(n-\nu)!}(f(z))^{n-\nu} g_{\nu+1}(z)$$

where $g_\mu(z)$ is the μ-fold iterated indefinite integral of $g(f^{-1}(w))$ for $w=f(z)$. Thus, for $z=\alpha$ one has

$$\phi(\alpha) = (-1)^n n!\, g_{n+1}(\alpha).$$

Therefore

$$g_{n+1}(z) = \frac{(-1)^n}{n!} z$$

will give a function $\phi(z)$ which has all the desired properties. In this way one obtains the function $\Phi_n(z)$ of (3), and it is evident that this function has the properties stated in Theorem II.

[5] From a letter of Professor V. A. Bailey we have learnt (in May 1941) that this problem has been dealt with in some special cases by E. Netto in his *Vorlesungen über Algebra* vol. I, Teubner, Leipzig, 1896, p. 300. In the same letter Bailey has given an elegant solution of the problem which, however, does not suit our present purpose. Further he has drawn our attention to the paper by L. Sancery, *De la méthode des substitutions successives pour le calcul des racines des équations*, Nouvelles Annales d. Math. (2) **1**, 305–315 (1862), which, however, was not accessible to us.

THE CAPACITY OF TWIN CABLE*

By J. W. CRAGGS and C. J. TRANTER (*Military College of Science, Stoke-on-Trent, England*)

1. Introduction. The problem of determining the capacity of two long parallel cylindrical conductors can be easily solved by the use of a conformal transformation.[1] A simple extension of the method gives the result for the case in which each conductor

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[1] F. B. Pidduck, *A treatise on electricity*, Cambridge University Press, 1916, p. 77.

is surrounded by a dielectric sheath whose boundary is a member of the coaxial system of circles defined by the boundaries of the conductors. The case in which the sheaths are concentric with the conductors is of much greater practical importance and in many types of cable the sheaths are actually touching. In this paper we give the derivation of the potential distribution for this latter case together with a practical method for the calculation of the capacity.

2. Statement of the problem. We consider the symmetrical problem of two circular wires each of radius R_1 surrounded by concentric touching sheaths of radius R_2 and dielectric constant K_1, the whole being immersed in an infinite medium of dielectric constant K_2.

For infinitely long straight wires, the problem reduces to the determination of potentials V_1, V_2 satisfying: (*i*) the equations

$$\nabla^2 V_1 = 0, \tag{1}$$

for $R_1 \leq r \leq R_2$, and

$$\nabla^2 V_2 = 0, \tag{2}$$

in the region between the circle $r = R_2$ and the line $x = 0$, where ∇^2 is the two dimensional form of Laplace's operator, the polar coordinates r, θ are based on the centre of one of the conductors and the cartesian coordinates x, y have origin at the point

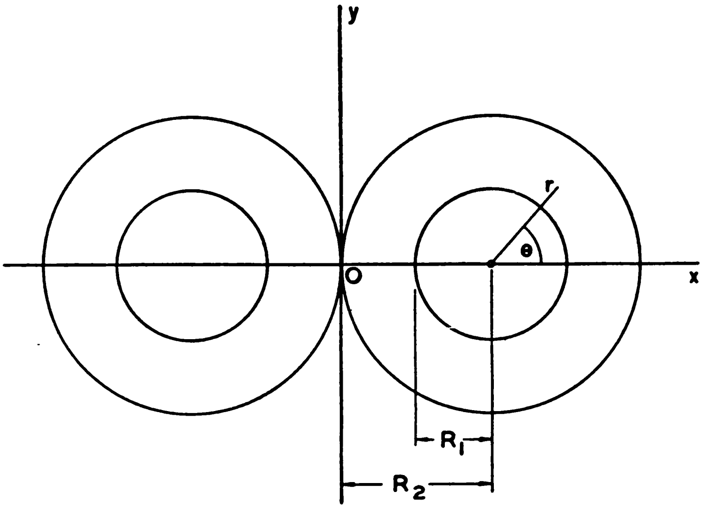

FIG. 1.

of contact of the sheaths and axes as shown in Fig 1; (*ii*) the boundary conditions

$$V_1 = 1 \tag{3}$$

when $r = R_1$, and

$$V_1 = V_2, \quad (4) \qquad K_1 \partial V_1/\partial r = K_2 \partial V_2/\partial r, \tag{5}$$

when $r = R_2$, and

$$V_2 = 0 \tag{6}$$

when $x = 0$. Condition (6) is a result of the symmetry of the problem provided that the potential on the left hand conductor is taken as -1.

The capacity C per unit length of wire is then given by $C=\frac{1}{2}Q$, where Q is the charge per unit length on either conductor.

The capacity per unit length is unaltered if we replace R_2 by unity and R_1 by $R_1/R_2(=a)$ and we shall do this in the subsequent work.

3. The analytical solution. It is natural to express the potential V_1 in the polar coordinates defined above. We therefore write

$$V_1 = 1 + B \log \frac{r}{a} + \sum_{n=1}^{\infty} \left\{ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n \right\} b_n \cos n\theta, \tag{7}$$

this being the most general solution of (1), symmetrical in θ and satisfying (3).

Conformal representation by the use of

$$\xi - i\eta = \frac{1}{x + iy} = \frac{1}{1 + re^{i\theta}} \tag{8}$$

transforms the region $r>1$, $x>0$ into $0<\xi<\frac{1}{2}$, the boundaries $x=0$, $r=1$ becoming $\xi=0$, $\xi=\frac{1}{2}$ respectively. The general solution of (2) satisfying (6) and the conditions of symmetry is

$$V_2 = \int_0^\infty f(t) \sinh 2\xi t \cos 2\eta t dt. \tag{9}$$

The constants B, b_n of (7) and the function $f(t)$ of (9) are now to be determined from the boundary conditions (4) and (5).

Now on $r=1$ $(\xi=\frac{1}{2})$ the relation (8) gives

$$\eta = \tfrac{1}{2} \tan \tfrac{1}{2}\theta = \tfrac{1}{2}\beta \tag{10}$$

say, and

$$\frac{\partial V}{\partial r} = - \tfrac{1}{4} \sec^2 \tfrac{1}{2}\theta \frac{\partial V}{\partial \xi} = - \tfrac{1}{4}(1 + \beta^2) \frac{\partial V}{\partial \xi}\cdot \tag{11}$$

Thus (4) and (5) become

$$1 - B \log a + \sum_{n=1}^{\infty} \frac{1 - a^{2n}}{a^n} b_n \cos n\theta = \int_0^\infty f(t) \sinh t \cos \beta t dt, \tag{12}$$

$$KB + K\sum_{n=1}^{\infty} \frac{1 + a^{2n}}{a^n} nb_n \cos n\theta = - \tfrac{1}{2}(1 + \beta^2) \int_0^\infty tf(t) \cosh t \cos \beta t dt, \tag{13}$$

where $K=K_1/K_2$.

Multiplying (12) by $\cos n\theta$ $(n=0, 1, 2, \cdots)$ and integrating with respect to θ from 0 to π, we have

$$1 - B \log a = \frac{1}{\pi} \int_0^\pi \int_0^\infty f(t) \sinh t \cos \beta t dt d\theta = \int_0^\infty e^{-t} f(t) \sinh t dt \tag{14}$$

since

$$\int_0^\pi \cos \beta t d\theta = 2 \int_0^\infty \frac{\cos \beta t}{1 + \beta^2} d\beta = \pi e^{-t}$$

and

$$\frac{1-a^{2n}}{a^n}b_n = \frac{2}{\pi}\int_0^\pi\int_0^\infty f(t)\sinh t\cos\beta t\cos n\theta dt d\theta = \int_0^\infty e^{-t}f(t)\sinh t\, I_n(t)dt, \tag{15}$$

where

$$I_n(t) = \frac{2e^t}{\pi}\int_0^\pi \cos n\theta\cos\beta t d\theta = \frac{4e^t}{\pi}\int_0^\infty \frac{\cos\beta t}{1+\beta^2}\cos(2n\tan^{-1}\beta)d\beta$$
$$= \sum_{p=0}^{n-1}(-1)^p\frac{{}^{n-1}C_p}{(n-p)!}(2t)^{n-p}. \tag{16}$$

Applying Fourier's integral theorem to (13), we obtain

$$-\frac{t}{K}f(t)\cosh t = \frac{4B}{\pi}\int_0^\infty\frac{\cos\beta t}{1+\beta^2}d\beta + \frac{4}{\pi}\sum_{n=1}^\infty\frac{1+a^{2n}}{a^n}nb_n\int_0^\infty\cos n\theta\frac{\cos\beta t}{1+\beta^2}d\beta$$
$$= 2Be^{-t} + e^{-t}\sum_{n=1}^\infty\frac{1+a^{2n}}{a^n}nb_nI_n(t). \tag{17}$$

Equations (15) and (17) lead to

$$-\frac{1-a^{2p}}{Ka^p}b_p = B\alpha_p + \sum_{n=1}^\infty\frac{1+a^{2n}}{a^n}nb_nA_{np}, \tag{18}$$

where

$$\alpha_p = 2\int_0^\infty e^{-2t}\frac{\tanh t}{t}I_p(t)dt, \tag{19}$$

and

$$A_{np} = \int_0^\infty e^{-2t}\frac{\tanh t}{t}I_p(t)I_n(t)dt. \tag{20}$$

Finally (14) and (17) give

$$\frac{1}{K}(B\log a - 1) = 2B\int_0^\infty e^{-2t}\frac{\tanh t}{t}dt + \frac{1}{2}\sum_{n=1}^\infty\frac{1+a^{2n}}{a^n}nb_n\alpha_n. \tag{21}$$

The infinite set of equations (18) gives the values of the coefficients b in terms of B. Substitution in (21) yields an equation for B and the capacity per unit length C can then be determined from $C = -\frac{1}{4}K_1B$, since

$$2C = Q = -\frac{K_1}{4\pi}\int_0^{2\pi}\left(\frac{\partial V_1}{\partial r}\right)_{r=a}a d\theta.$$

This completes the analytical solution.

4. Method of computation. In practice, a good approximation to the capacity may be obtained by retaining only a finite number m of the coefficients b_n. Eliminating the m quantities $[(1+a^{2n})/a^n]nb_n$ $(n=1, 2, \cdots, m)$, from the $(m+1)$ equations (18) and (21), and writing

$$\gamma_n = \frac{1-a^{2n}}{nK(1+a^{2n})}, \tag{22}$$

we obtain

$$\begin{vmatrix} A_{11}+\gamma_1 & A_{12}\cdots & A_{1m} & \alpha_1 \\ A_{21} & A_{22}+\gamma_2\cdots & A_{2m} & \alpha_2 \\ \cdot & \cdot & \cdot & \cdot \\ A_{m1} & A_{m2}\cdots & A_{mm}+\gamma_m & \alpha_m \\ \alpha_1 & \alpha_2\cdots & \alpha_m & \frac{2}{K}\left(\frac{1}{B}-\log a\right)+4\log\frac{\pi}{2} \end{vmatrix} = 0, \qquad (23)$$

since

$$\int_0^\infty e^{-2t}\frac{\tanh t}{t}dt = \log\frac{\pi}{2}\cdot$$

The quantities α_n, A_{np} can be easily computed from Eqs. (19), (20) and (16) with the help of the result[2]

$$\int_0^\infty (2t)^n e^{-2t}\tanh t\,dt = n!\left\{\left(1-\frac{1}{2^n}\right)\zeta(n+1)-\tfrac{1}{2}\right\},$$

where $\zeta(n)$ is the Riemann Zeta-function, tabulated for integral n in J. Edwards, "The integral calculus," vol. 2, Macmillan, London, 1922, p. 144.[3]

The solution for $K=1$ differs from the well-known exact solution for this case by less than 0.2 per cent, when only the first three of b_n are retained, provided that $a\leq\frac{1}{2}$. For larger values of K and a it may be necessary to retain more terms to achieve the desired accuracy but for practical values the amount of computation required is not excessive.

[2] When $n=0$, the result reduces to $\int_0^\infty e^{-2t}\tanh t\,dt=\log 2-\frac{1}{2}$.

[3] A four-figure table is given in E. Jahnke and F. Emde, *Tables of functions*, Dover Publications, New York, 1943, p. 273.

LARGE DEFLECTION OF CANTILEVER BEAMS*

By K. E. BISSHOPP and D. C. DRUCKER (*Armour Research Foundation*)

The solution for large deflection of a cantilever beam[1] cannot be obtained from elementary beam theory since the basic assumptions are no longer valid. Specifically, the elementary theory neglects the square of the first derivative in the curvature formula and provides no correction for the shortening of the moment arm as the loaded end of the beam deflects. For large finite loads, it gives deflections greater than the length of the beam! The square of the first derivative and correction factors for the shortening of the moment arm become the major contribution to the solution of

* Received April 6, 1945.

[1] This problem was considered by H. J. Barten, "On the Deflection of a Cantilever Beam," Quarterly of Applied Math., **2**, 168–171 (1944). Previously an approximate solution had been obtained by Gross und Lehr in *Die Federn*, Berlin VDI Verlag, 1938.