PRESSURE FLOW OF A TURBULENT FLUID BETWEEN TWO INFINITE PARALLEL PLANES*

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1. Introduction. The solution of the Navier-Stokes differential equations for the steady laminar flow through a channel or a circular pipe is well known for its mathematical simplicity. The reason for this simplicity is that for such flows Prandtl's boundary layer equations hold rigorously for the entire region of the fluid. In other words the boundary layer extends up to the center of the channel, whereas in the case of the flow around a solid obstacle there is only a thin layer of viscous fluid attached to the surface of the obstacle.

The steady turbulent flow through a channel or a circular pipe is more complicated in the sense that all the equations of mean motion and the equations of double and triple correlation previously developed[1,2] have to be utilized to account for the mean velocity distribution in the entire region of the channel, and that they can not be further simplified by physical arguments as proposed, for example, by the boundary layer theory. However, if we examine the algebraic equation that represents the mean velocity distribution across the channel, we notice that it has functional behaviour similar to that of the formula for the mean velocity distribution within a turbulent boundary layer.[3] In other words, the turbulent flow in a channel bears some resemblance to the corresponding laminar flow on the whole, though its detailed structure is much more complicated as will be seen soon.

In what follows we shall first determine the mean velocity distribution based upon the equation of mean motion and the equations of double correlation, by giving the triple correlations their values in the middle of the channel. This procedure leads to good results in the theory of the spread of turbulent jets and wakes (references at the end of II), but in the present case it only agrees with the experiment in the central portion of the channel, while it fails when the side is approached. We shall also see that the mean squares of the three components of the velocity fluctuation agree qualitatively with observation in the corresponding region.

The second determination given below for the mean velocity distribution utilizes equations of mean motion and both the equations of double and triple correlation by neglecting terms involving quadruple correlations. It will be shown that the triple correlations which represent the transport of turbulent energy play a particularly important role in the vicinity of the wall of the channel, and therefore can not be dispensed with for a better representation of the mean velocity distribution.

From this second determination we shall find that neglect of terms involving quadruple correlations is justifiable as a first approximation. In other words the equations

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[1] P. Y. Chou, Chin. Journ. of Phys. **4**, 1–33 (1940). This paper will be referred to hereafter as I.

[2] P. Y. Chou, *On velocity correlations and the solutions of the equations of turbulent fluctuation*, Quart. of Appl. Math. **3**, 38–54 (1945). This paper will be referred to as II.

[3] N. Hu, *The turbulent flow along a semi-infinite plate* (unpublished).

of mean motion and of the double and triple correlations are sufficient in treating turbulent flow problems even though there is a wall present. Hence the mathematical procedure is comparatively simple in another sense that the building of equations satisfied by higher order correlations can be dropped up to the present degree of approximation.

The first determination reveals that the variation of the mean squares of the turbulent fluctuation is slower than the corresponding variation of the mean velocity distribution across the channel, which agrees qualitatively with experiment. In view of the fact that measurements of the mean squares of the components of turbulent fluctuation have not been reported systematically in the literature for the flow under consideration, we shall omit the quantitative comparison of the theory with the experimental data now available on these quantities.

In the second determination the mean velocity distribution will be calculated by assuming constant mean squares of turbulent velocity components across the channel. This is justifiable due to the slower variation of these functions across the channel, and furthermore the mean velocity distribution remains practically unchanged in the major portion of the channel—with the exception of the immediate neighborhood of the wall—when the constant values assumed for these functions are different from each other. This procedure of assigning constant values to the mean squares of the velocity fluctuations and then calculating the mean velocity distribution can be considered as the initial step in a method of iteration which will be explained in §3 below in greater detail.

In the final section we shall indicate the uncertainties connected with the correlation integrals pointed out before (II, §8). They are probably not important for the mean velocity distribution, because they involve possibly the mean squares of the turbulent fluctuation which are taken to be constant for the present calculation These uncertainties could be removed, if we had better experimental information on the variation of the turbulent level across the channel and on the velocity correlation between two distinct points in flows such as the one examined here. In other words the present theory is perhaps sufficient so far as the mean velocity distribution is concerned, and it points out the possibilities for future investigations in turbulence along both experimental and theoretical lines.

2. Mean velocity distribution based upon the solution of the equations of mean motion and of double correlation. As before (I, §4) we take the positive x-axis ($x=x^1$) as the direction of mean motion of the fluid, the y-axis ($y=x^2$) perpendicular to the two parallel planes forming the channel, and the z-axis ($z=x^3$) parallel to these planes. The plane in mid-channel is chosen as the xz-plane. From the equations of mean motion we have

$$\tau_{12}/\rho = -U_\tau^2\sigma - \nu dU/dy, \tag{2.1}$$

where

$$-\partial p/\rho\partial x = U_\tau^2/d, \qquad \sigma = y/d. \tag{2.2}$$

The quantity $2d$ represents the width of the channel and U_τ is the so-called friction velocity.

Equation (2.1) defines the shearing stress τ_{12} in terms of y and dU/dy. Except in the immediate neighborhood of the wall the viscous stress is small, so τ_{12} is a linear function of y. On the wall $-\nu dU/dy$ should be equal to U_τ^2, and τ_{12} should tend toward zero as a limit.

The components τ_{23} and τ_{31} vanish due to symmetry, as pointed out before (I, §4).

From now on for simplicity we shall neglect the action of viscosity in the form of laminar stress in all the equations of motion. A physical condition mentioned previously is that all average values over time are functions of y only. Furthermore in the present special case in accordance with the definitions in II, Eqs. (5.3), those components of the slowly varying tensors a^n_{mik} and b_{ik} which have a single appearance of the index 3 either in i or k must be both identically zero. The vanishing of these functions is based upon the same argument as in the case of τ_{23}. The non-vanishing equations of the second order correlation (II, (8.2)) then become

$$-\frac{2}{\rho}\tau_{12}\frac{dU}{dy}+\frac{d}{dy}\overline{w_1^2w_2} = -a_{2111}\frac{dU}{dy}-b_{11}+\frac{2\nu}{3\lambda^2}(k-5)q^2-\frac{2\nu k}{\lambda^2}\overline{w_1^2}, \tag{2.3}$$

$$-\frac{1}{\rho}\tau_{22}\frac{dU}{dy}+\frac{d}{dy}\overline{w_1w_2^2} = -a_{2112}\frac{dU}{dy}-b_{12}-\frac{2\nu k}{\lambda^2}\overline{w_1w_2}, \tag{2.4}$$

$$\frac{d}{dy}\overline{w_2^3} = -a_{2122}\frac{dU}{dy}-b_{22}+\frac{2\nu}{3\lambda^2}(k-5)q^2-\frac{2\nu k}{\lambda^2}\overline{w_2^2}, \tag{2.5}$$

$$\frac{d}{dy}\overline{w_2w_3^2} = -a_{2133}\frac{dU}{dy}-b_{33}+\frac{2\nu}{3\lambda^2}(k-5)q^2-\frac{2\nu k}{\lambda^2}\overline{w_3^2}. \tag{2.6}$$

These are obtained by giving i and k the sets of values (1, 1), (1, 2), (2, 2), (3, 3), respectively.

In the above four equations q is the root-mean-square of velocity fluctuation defined by

$$q^2 = \overline{w_jw^j}, \tag{2.7}$$

and k is a constant. The slowly varying tensors a_{nmik} and b_{ik} should obey the divergence relations [II, (5.4)]

$$a_{2111}+a_{2122}+a_{2133} = 0, \qquad b_{11}+b_{22}+b_{33} = 0. \tag{2.8}$$

The equation of vorticity decay [II, (7.11)] satisfied by Taylor's scale of microturbulence λ becomes in the present case

$$-14G\overline{w_1w_2}dU/dy - 70Fq^3/3\sqrt{3} = -2\nu Eq^2/3\lambda^2, \tag{2.9}$$

where E, F and G are regarded as constants.

The constant G in (2.9) is probably not important, for in the center of the channel the term involving G in (2.9) is zero due to the vanishing of dU/dy there, and in the immediate neighborhood of the wall $\overline{w_1w_2}$ vanishes although $-dU/dy$ is large. Hence for simplicity we choose G to be zero. In fact the presence of G would only change our results slightly, as will be seen. The physical meaning of neglecting G in (2.9) is that the term that represents the creation of vorticity by deformation of the mean motion is negligible when compared with those due to transport and decay.

If G were set equal to zero, Eq. (2.9) yields

$$\lambda q/\nu = \sqrt{3}E/35F = R_0, \tag{2.10}$$

where R_0 is a constant number.

Now we shall substitute the values of the triple correlations at the center of the channel into Eqs. (2.3)–(2.6) according to their odd and even properties as functions of y. We can set

$$\overline{w_1^2 w_2} = U_\tau^3 \alpha_1 \sigma, \quad \overline{w_2^3} = U_\tau^3 \alpha_2 \sigma, \quad \overline{w_2 w_3^2} = U_\tau^3 \alpha_3 \sigma, \quad \overline{w_1 w_2^2} = U_\tau^3 \alpha_4, \tag{2.11}$$

where α_1, α_2, α_3 and α_4 are four dimensionless constants. Although the factor U_τ^2 in (2.11) is introduced for dimensional reasons, it is possible that these four constants are all independent of the Reynolds number of the mean flow.

If we substitute from (2.10) and (2.11) into (2.3), (2.5) and (2.6), add the three together and take into account the conservation relations (2.8), we find that the mean square of the velocity fluctuation q^2 satisfies the relation

$$\frac{q^4}{U_\tau^4} = \frac{R_0^2}{10R_\tau}\left[\alpha - \frac{2\sigma}{U_\tau}\frac{dU}{d\sigma}\right], \tag{2.12}$$

where

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3, \qquad R_\tau = U_\tau d/\nu; \tag{2.13}$$

R_τ is called the friction Reynolds number.

Relation (2.12) is very significant, for it tells us that for large values of $-dU/d\sigma$, q^2 varies as the square root of $-\sigma dU/d\sigma$. Within a large portion of the channel, $dU/d\sigma$ is proportional to σ, so the dependence of q^2 upon σ is fairly linear. This linear dependence has been observed to some extent for $\overline{w_1^2}$ by Wattendorf and Kuethe[4] and by Wattendorf and Baker,[5] and has been anticipated in the light of von Kármán's law of similarity.

If G were different from zero, the above procedure would lead to

$$\frac{q^2}{U_\tau^2} = R_0\left[c - 2(1+35G)\frac{\sigma}{U_\tau}\frac{dU}{d\sigma}\right]\Big/\sqrt{10R_\tau}\left(c - \frac{2\sigma}{U_\tau}\frac{dU}{d\sigma}\right)^{1/2} \tag{2.14}$$

which has a functional behaviour similar to that of (2.12) for large values of $-dU/d\sigma$.

It is apparent that Eqs. (2.3), (2.5) and (2.6) will determine w_1, w_2 and w_3 separately. Here we encounter the uncertainty pointed out in II, §8 that the slowly varying functions a_{nmik} and b_{ik} may contain powers or even more complicated functions of q as factors, and the existing experimental data do not provide enough evidence for a quantitative comparison with these theoretical formulae. If, for the sake of mathematical convenience, we assume b_{11}, b_{22} and b_{33} to be constant, and a_{2111}, a_{2122} and a_{2133}, which are odd functions of σ, to be proportional to σ, then $\overline{w_1^2}$, $\overline{w_2^2}$ and $\overline{w_3^2}$ will behave very much like q^2, that is, when σ is near zero, $\overline{w_1^2}$, $\overline{w_2^2}$ and $\overline{w_3^2}$ are constants, and when σ is large and near unity, they are all proportional to the square root of $-\sigma dU/d\sigma$.

The equation for determining the mean velocity distribution is given by (2.4) which can now be written, on account of (2.10) and the condition that $\overline{w_1 w_2}$ is constant, in the form

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[4] F. L. Wattendorf and A. M. Kuethe, Physics 5, 153–164 (1934).

[5] Th. von Kármán, Proceedings of the Fifth International Congress for Applied Mechanics (Cambridge, Mass. 1938), p. 349.

$$\frac{1}{U_\tau^2}(\overline{w_2^2}+a_{2112})\frac{1}{U_\tau}\frac{dU}{d\sigma}=-\frac{2kR_\tau a^2}{R_0^2U_\tau^2}\sigma; \tag{2.15}$$

here we have set the odd function b_{12} equal to zero for simplicity.

As pointed out before the dependence of a_{2112} upon q in the above equation is also not known. If $\overline{w_2^2}$ and q^2 are both regarded as constants, the mean velocity distribution according to (2.15) is parabolic, which agrees with experimental data fairly well for the range of σ from 0 to 0.8, and fails near the walls of the channel. This parabolic law of velocity distribution has been suggested by Stanton[6] in his measurements of flows through a circular pipe of which the channel is a limiting case.

It has been calculated, though details will not be shown here, that this parabolic distribution of the mean velocity for constant q^2 and $\overline{w_2}$ is not essentially changed if we solve for $\overline{w_1^2}$, $\overline{w_2^2}$, $\overline{w_3^2}$ and $dU/d\sigma$ simultaneously under the further assumption that both $a_{m\alpha}^n$ and b_{ik} are equal to zero. This condition is equivalent to the vanishing of $(\overline{\tilde{\omega}_{,i}w_k+\tilde{\omega}_{,k}w_i})/\rho$, which means that the shearing interaction between the pressure gradient and velocity fluctuations is zero; it has been used in jets and wakes, as mentioned before. The reason why the velocity distribution is parabolic even for this more rigorous treatment is not difficult to see without going into detailed calculations. For in the neighborhood of $\sigma=0$, both $\overline{w_2^2}$ and q^2 are constants, so $dU/d\sigma$ is proportional to σ. When the values of σ are near unity, both q^2 and $\overline{w_2^2}$ are proportional to the square root of $-\sigma dU/d\sigma$, and hence mutually proportional; consequently Eq. (2.15) again shows that $dU/d\sigma$ is proportional to σ even in the vicinity of the channel wall. We should anticipate, by the same argument, that similar simultaneous solutions for $\overline{w_1^2}$, $\overline{w_2^2}$, $\overline{w_3^2}$ and $dU/d\sigma$ would hold true even under the more general condition that b_{11}, b_{22} and b_{33} be constants and a_{2111}, a_{2122} and a_{2133} be proportional to σ as mentioned previously.

In §4 below we shall compare the numerical values of R_0, R_τ and α of (2.12) with available measurements.

3. Equations of triple correlation and the mean velocity distribution. The non-vanishing equations of the triple correlation [II, (8.3)] for the present problem can be written in the form

$$3\overline{w_1^2w_2}\frac{dU}{dy}+\frac{d}{dy}\overline{w_1^3w_2}=-b_{21111}\frac{dU}{dy}-c_{111}+\frac{3}{\rho^2}\tau_{11}\frac{d\tau_{12}}{dy}\cdot \tag{3.1}$$

$$2\overline{w_1w_2^2}\frac{dU}{dy}+\frac{d}{dy}\overline{w_1^2w_2^2}=-b_{21112}\frac{dU}{dy}-c_{112}+\frac{1}{\rho^2}\left(2\tau_{12}\frac{d\tau_{12}}{dy}+\tau_{11}\frac{d\tau_{22}}{dy}\right), \tag{3.2}$$

$$\overline{w_2^3}\frac{dU}{dy}+\frac{d}{dy}\overline{w_1w_2^3}=-b_{21122}\frac{dU}{dy}-c_{122}+\frac{1}{\rho^2}\left(\tau_{22}\frac{d\tau_{12}}{dy}+2\tau_{12}\frac{d\tau_{22}}{dy}\right), \tag{3.3}$$

$$\overline{w_2w_3^2}\frac{dU}{dy}+\frac{d}{dy}\overline{w_1w_2w_3^2}=-b_{21133}\frac{dU}{dy}-c_{133}+\frac{1}{\rho^2}\tau_{33}\frac{d\tau_{12}}{dy}, \tag{3.4}$$

$$\frac{d}{dy}\overline{w_2^4}=-b_{21222}\frac{dU}{dy}-c_{222}+\frac{3}{\rho^2}\tau_{22}\frac{d\tau_{22}}{dy}, \tag{3.5}$$

[6] T. E. Stanton, Proc. Roy. Soc. London. (A) **85**, 366–376 (1911).

$$\frac{d}{dy}\overline{w_2^2 w_3^2} = -b_{21233}\frac{dU}{dy} - c_{233} + \frac{1}{\rho^2}\tau_{33}\frac{d\tau_{22}}{dy}\cdot \tag{3.6}$$

These are obtained by giving i, k and l the sets of values (1, 1, 1), (1, 1, 2), (1, 2, 2), (1, 3, 3), (2, 2, 2), (2, 3, 3), respectively. The other component tensor equations in which the index 3 appears an odd number of times, namely, (1, 1, 3), (1, 2, 3), (2, 2, 3) and (3, 3, 3), are all identically zero, as are the corresponding equations of the second order correlation.

From the discussions in the previous section it is apparent that Eqs. (2.3), (2.5) and (2.6) are used to determine the mean squares of the fluctuation components, and elimination of the triple correlations between these three equations and (3.1), (3.3) and (3.4) respectively will give a more accurate determination of them. As pointed out before, existing experimental data are not accurate enough to give a quantitative comparison with the theory, and we shall not go into these detailed calculations here. Furthermore Eqs. (3.5) and (3.6) lead to quantities which are still beyond experimental proof; discussions of them will also be omitted for the present.

The elimination of the triple correlation $\overline{w_1 w_2^2}$ between (2.4) and (3.2) leads to the equation for the mean velocity distribution. Before writing down this equation we shall introduce a few more simplifications. In the first place the even function b_{21112}, which may depend upon q as mentioned previously (II, §8), is assumed to be a constant; likewise the odd function c_{112} is taken to be proportional to y and is put in the form,

$$c_{112} = 2cU_\tau^4\sigma/d. \tag{3.7}$$

It is also possible that the dimensionless number c may be a function of q and therefore an implicit function of the coordinate y.

The quadruple correlation $\overline{w_1^2 w_2^2}$ in (3.2) is of the same order of magnitude as $(\overline{w_1 w_2})^2$ and $\overline{w_1^2}\,\overline{w_2^2}$. As a first approximation we shall neglect all of these terms and it will be shown afterwards in §4 that this approximation is justifiable. In short, (3.2) defines the triple correlation $w_1 w_2$ approximately by

$$\overline{w_1 w_2^2} = -\tfrac{1}{2}b_{21112} - cU_\tau^4\sigma \Big/ \frac{dU}{d\sigma}\cdot \tag{3.8}$$

Utilizing the above relation and (2.10) which is derived from the equation of vorticity decay, we find, after setting b_{12} in (2.4) equal to zero for mathematical convenience, that

$$\frac{a}{U_\tau}\frac{dU}{d\sigma} + c\frac{d}{d\sigma}\frac{U_\tau\sigma}{\dfrac{dU}{d\sigma}} = b\sigma, \tag{3.9}$$

where

$$a = -(\overline{w_2^2} + a_{2112})/U_\tau^2, \qquad b = 2kR_\tau q^2/R_0^2 U_\tau^2. \tag{3.10}$$

The physical meaning of the three terms in the above equation is as follows: the term in a represents the creation of turbulent energy partly due to deformation of the mean flow (I, §3(a)) and on account of a_{2112} partly contributed by the shear due to the pressure fluctuation $(\overline{\tilde{\omega}_{,i}w_k + \tilde{\omega}_{,k}w_i})/\rho$; the term in b denotes the decay of turbulence; the term in c denotes the transport of turbulent energy.

The definitions of a and b in (3.10) show that they depend upon q^2 and $\overline{w_2^2}$, and are therefore functions of σ. Since q^2 as well as $\overline{w_2^2}$ as shown in the previous section varies much more slowly than $dU/d\sigma$ itself across the channel, we shall regard them as constants as the initial step to solve for $dU/d\sigma$. This initial process can also be regarded as the first step in the method of iteration in solving the present problem of turbulent flow. The second step will be to substitute this expression obtained for the mean velocity into (2.3), (2.5) and (2.6) after eliminating the triple correlations by means of (3.1), (3.3) and (3.4), and to solve for $\overline{w_1^2}$, $\overline{w_2^2}$ and $\overline{w_3^2}$. As the third step in this procedure, we utilize these values of the mean squares of the fluctuation components and solve (3.9) again for $dU/d\sigma$, and see whether the new result agrees with the solution obtained in the first step. Obviously this procedure of obtaining alternately the mean velocity and mean squares of the turbulent fluctuation can be extended indefinitely.

In the present paper we shall not follow this refined method of approach; instead we shall solve (3.9) by assigning constant values to a, b and c, or to q^2 and $\overline{w_2^2}$, and compare the different solutions by varying these constants. The result will be that except in the immediate neighborhood of the wall of the channel, the different mean velocity distributions according to (3.9) for the different sets of a, b and c respectively agree well with each other and with experiment, showing that the variation of the mean squares of the turbulent fluctuation across the channel does not influence the mean velocity distribution very much.

The solution of (3.9) with constant a, b and c is

$$\tfrac{1}{2}\sigma^2 = aU/bU_\tau + cA_1 e^{bU/cU_\tau} + A_2, \tag{3.11}$$

where A_1 and A_2 are two constants of integration.

If A_1 in (3.11) is zero, then (3.10) gives a parabolic law of velocity distribution and a must be negative, since b according to its definition in (3.10) is positive. The presence of the term in A_1 gives the so-called "logarithmic law" of velocity distribution which holds true especially in the neighborhood of the wall of the channel. Hence the product cA_1 can not be zero. It is apparent that this exponential term in U/U_τ is due to the presence of the triple correlation in Eq. (3.9).

The boundary conditions used to determine the constants cA_1, A_2 and the ratio a/b are:

$$\text{when } \sigma = 0, \quad U = U_c; \qquad \text{when } \sigma = 1, \quad U = 0, \quad -dU/U_\tau d\sigma = \infty. \tag{3.12}$$

The value U_c denotes the maximum velocity of the flow in mid-channel. We have chosen the derivative $-dU/U_\tau d\sigma$ on the wall of the channel to be infinite. In fact, it should be R_τ which is a fairly large number. Since we are interested in the mean velocity distribution within the channel proper, substituting infinity for the friction Reynolds number R_τ gives a good approximation.

The boundary conditions (3.12) render (3.11) into the following final form,

$$(e^{\kappa U_c/U_\tau} - \kappa U_c/U_\tau - 1)\sigma^2 = -\kappa(U_c - U)/U_\tau + e^{\kappa U_c/U_\tau}[1 - e^{-\kappa(U_c-U)/U_\tau}], \tag{3.13}$$

where $\kappa = b/c$ and $b/a = 2[\exp(\kappa U_c/U_\tau) - \kappa U_c/U_\tau - 1]/\kappa$.

Equation (3.13) expresses the mean velocity defect $(U_c - U)/U_\tau$ as a function of σ with two parameters κ and U_c/U_τ. The presence of these two constants may appear at the first sight to contradict the experimental velocity defect law formulated by von Kármán,[7] according to which $(U_c - U)/U_\tau$ should be independent of the Reynolds number of the mean flow which, in turn, is a function of the ratio U_c/U_τ. A close examination of the experimental data shows, however, that this discrepancy is not serious. In the first place von Kármán's velocity defect law can only hold true in the central portion of the channel and there is a dependence of the velocity defect upon the Reynolds number in the vicinity of the channel wall. It has been shown that for flows in circular pipes U_c/U_τ increases from about 19 to 33 when the friction Reynolds number $2aU_\tau/\nu$ changes from $\sqrt{10^5}$ to 10^5, $2a$ being the diameter of the pipe.[8] In the second place even formula (3.13), which does indicate the dependence of $(U_c - U)/U_\tau$ upon U_c/U_τ, can only account for the mean velocity distribution in the interior of the channel for a given set of constants a, b and c in (3.11), and these constants have to take another set of values in the turbulent boundary layer on the wall, although the same functional behaviour of (3.11) still prevails within the layer.[3] This point will be discussed in greater detail in the following section.

The quantity U_c/U_τ in (3.13) is given by experiment; then the constant κ is fixed, for instance, by passing the theoretical curve through the experimental point at $\sigma = 0.7$. In view of the variation of the ratio U_c/U_τ with the Reynolds number of the mean flow, we shall choose a few different values of κ and calculate the mean velocity

TABLE 1. $(U_c - U)/U_\tau$

	(1)	(2)	(3)	(4)	(5)
σ \ κ	Obs.	−0.1	0.0	+0.1	0.2151
0.00	0.00	0.00	0.00	0.00	0.00
0.10	0.16	0.06	0.05	0.05	0.05
0.20	0.38	0.23	0.22	0.21	0.19
0.30	0.66	0.52	0.50	0.48	0.44
0.40	1.10	0.95	0.91	0.88	0.81
0.50	1.64	1.50	1.47	1.42	1.34
0.60	2.33	2.22	2.19	2.15	2.08
0.70	3.13	3.13	3.13	3.13	3.13
0.80	4.28	4.31	4.38	4.50	4.75
0.90	6.30	5.92	6.17	6.60	7.72
0.93	—	6.58	6.91	7.51	9.30
0.96	8.81	7.39	7.88	8.74	11.84
0.98	—	8.12	8.76	9.93	15.01
0.99	—	8.64	9.40	10.82	18.21
1.00	—	9.86	10.94	13.07	∞

distribution. This will lead to different values of U_c/U_τ. But we shall see that for all these cases the mean velocity distributions agree with each other and with experiment within the channel proper.

Let us calculate the mean velocity distribution for the values of κ equal to -0.1, 0, 0.1 and 0.2151, and determine the corresponding values of U_c/U_τ by passing the theoretical curves through the experimental point at $\sigma = 0.7$. The equations that determine $(U_c - U)/U_\tau$ for $\kappa = -0.1$ and 0.1 are given respectively by

[7] Th. von Kármán, Proceedings of the Fourth International Congress for Applied Mechanics, Cambridge 1934, p. 70.

[8] S. Goldstein, *Modern developments in fluid dynamics*, vol. 2, The Clarendon Press, Oxford, 1938, p. 338.

$$\sigma^2 = 0.2785(U_c - U)/U_\tau - 1.039[e^{0.1(U_c-U)/U_\tau} - 1], \tag{3.14}$$

$$\sigma^2 = -1.5870(U_c - U)/U_\tau + 2.662[1 - e^{-0.1(U_c-U)/U_\tau}]. \tag{3.15}$$

For the case $\kappa=0$, we can get a limiting equation by letting κ approach zero in (3.13), or we can solve (3.9) directly by setting b equal to zero. The latter procedure leads to

$$(U_c - U)/U_\tau = 10.94[1 - (1 - \sigma^2)^{1/2}], \tag{3.16}$$

where 10.94 is the value of $(c/a)^{1/2}$.

The case when $\kappa=0.2151$ is represented by

$$\sigma^2 = 1 - e^{-0.2151(U_c-U)/U_\tau}. \tag{3.17}$$

This is the solution of (3.11) with a set equal to zero; the numerical value 0.2151 stands for the ratio b/c.

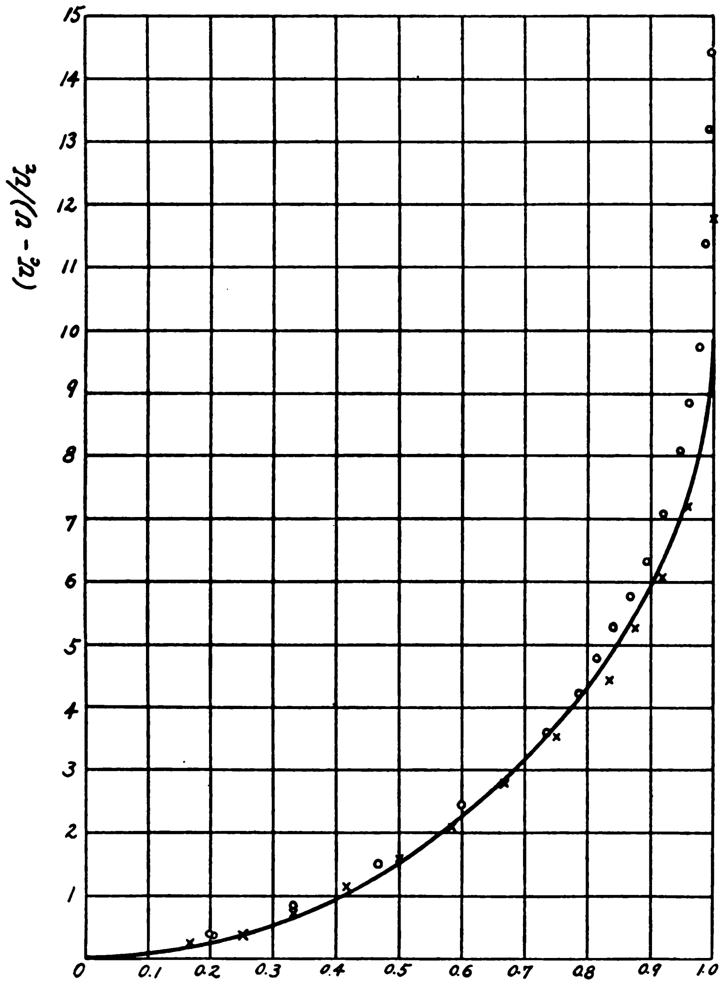

FIG. 1. Velocity distributions in a channel.

The experimental values of $(U_c - U)/U_\tau$ which are taken from a paper by Goldstein[9] are given in column (1) of Table 1; the corresponding theoretical values according to (3.14), (3.16), (3.15) and (3.17) are tabulated in columns (2), (3), (4) and (5), respectively.

From this table we see that as the value of κ increases from -0.1 to $+0.2151$, U_c/U_τ changes from 9.86 to ∞. Hence 0.2151 is the maximum limiting value of κ for constant a, b, c in (3.9). Equation (3.17) shows in this limiting case that the mean velocity distribution in the whole channel is "logarithmic."

In order to avoid confusion, only the solution (3.14) for $\kappa = -0.1$ is plotted in Fig. 1. The circles represent Dönch's measurements[10] found for $U_m d/\nu$ equal to 8.7×10^4, U_m being the average value of U over a cross section of the channel. The crosses reproduce Nikuradse's results[11] for $U_m d/\nu$ equal to 3.3×10^4. It is seen that apart from the immediate neighborhood of the channel wall, agreement between theory and experiment is satisfactory.

4. Relation between the present theory and some known experimental data. From the foregoing calculations we see that we can subject to experimental test not only the mean velocity defect distribution $(U_c - U)/U_\tau$ and the mean squares of the fluctuation components, but also the relation (2.10) between λ and q and the relation (3.8) which approximates the triple correlation $\overline{w_1 w_2^2}$. Let us examine relations (2.10) and (2.12) first.

The experimental data used by Taylor in his statistical theory[12] are, in c.g.s. units; $U_c = 114$ cm/sec, $U_\tau = 5.39$ cm/sec, $\rho = 0.00123$, $\nu = 0.14$, $d = 12.3$ cm. Since according to (2.10) $R_0 = \lambda q/\nu$ is a constant, we can compute R_0 from the values of λ and q in the center of the channel. In Taylor's table λ^2 in mid-channel is equal to 2.9 cm^2, so λ is 1.7 cm. The mean magnitude of the velocity fluctuation q at this point is roughly $1.2U_\tau$ [cf. (4.2) below]. These values then give

$$R_0 = 78.5, \qquad R_\tau = 474. \tag{4.1}$$

We have shown above in §2 that $\overline{w_1^2}$ behaves very much like q^2. The experimental values of $\overline{w_1^2}/U_\tau^2$ across the channel, determined by Wattendorf and Baker[5] for the flow with Reynolds number 109,000, can be represented by

$$\overline{w_1^2}/U_\tau^2 = 0.412(1 + 27.2\sigma^2)^{1/2}. \tag{4.2}$$

As far as the order of magnitude is concerned, $\overline{w_1^2}$ can be put equal to $q^2/3$. By comparing (4.2) with (2.12), we find that

$$R_0\sqrt{\alpha}/\sqrt{10R_\tau} \sim 3 \times 0.4 = 1.2. \tag{4.3}$$

If we use a parabolic representation of the mean velocity distribution that goes through the experimental point at $\sigma = 0.7$, we have

$$-2\sigma dU/U_\tau d\sigma = 25.6\sigma^2. \tag{4.4}$$

Then α in (2.10) becomes $25.6/27.2 = 0.94$. Putting this value of α in (4.3), we obtain $R_0^2/R_\tau \sim 10 \times 1.4/0.94 = 15$, while from (4.1) we find that $R_0^2/R_\tau \sim 13$.

[9] S. Goldstein, Proc. Roy. Soc. London (A) **159**, 473–496 (1937).

[10] F. Dönch, Forschungsarbeiten des Ver. Deutsch. Ing. no. **282** (1926).

[11] J. Nikuradse, Forschungsarbeiten des Ver. Deutsch. Ing. no. **289** (1929).

[12] G. I. Taylor, Proc. Roy. Soc. London (A) **151**, 456 (1935).

This shows the order of agreement between the two sets of values obtained from two entirely different experimental sources. It must be pointed out, however, that the number $R_\tau=474$ for the experimental value of λ used in the above calculation may be too low for the flow to be in a fully developed turbulent state.

Equation (2.10) also shows the dependence of R_0 upon the quantities E and F which occur in the definitions of the double and triple correlation functions between two distinct points [II, (6.8), (6.11)]. The measurement of these functions separately will give another check on the value of the number R_0 discussed above.

We next study the values of the three constants a, b and c in Eqs. (3.9) and their physical significance.

(1) $\kappa=-0.1$. According to (3.10) b must be positive, so c in this case must be negative. The definition of c/a from (3.13) and (3.14) gives

$$c/a = 20/0.2785 = 72. \tag{4.5}$$

Hence a must be also negative. If a_{2112} in (3.10) were zero, a becomes of the order of 0.4 and c is equal to 29. From the definition of c in (3.7), we find that

$$c_{112} = 58U_\tau^4/d, \tag{4.6}$$

which is 29 times larger than $2\tau_{12}d\tau_{12}/\rho^2dy$, a term of the same order of magnitude as the one in the quadruple correlation $\overline{dw_1^2w_2^2}/dy$. Hence all these terms are negligible as a first approximation.

After the value of c is known, the triple correlation function $\overline{w_1w_2^2}$ is determined uniquely according to (3.8); the constant term $\frac{1}{2}b_{21112}$ is fixed by the value of $\overline{w_1w_2^2}$ at $\sigma=0$.

By means of $b=-0.1c$, the definition of b in (3.10) and the values of R_τ/R_0^2 and q^2/U_τ^2 given before, we find the numerical value of k to be of the order of 16.

(2) $\kappa=0$, $(c/a)^{1/2}=10.94$, $c/a=120$. This gives results similar to those in case (1) and consequently the terms in the quadruple correlations in (3.2) are still negligible. In this case c and a must be negative as in the foregoing example. The meaning of $b=0$ is that the term due to the decay of turbulence in (3.9) is negligible when compared with the other two.

(3) $\kappa=0.1$. In this case c should be positive. Then the definition of c/a from (3.13) and (3.15) gives $c/a=20/1.5870\sim 13$, and a must be also positive. The condition that a be greater than zero changes the picture a great deal, for then a_{2112} in (3.10) is negative and its magnitude is greater than $\overline{w_2^2}$. Nevertheless the terms in the quadruple correlations are still negligible, if the absolute value of a_{2112} is, say, a few multiples of $\overline{w_2^2}$.

(4) $\kappa=0.2151$, $a=0$. Here we have $b/c=0.2151$. Putting this value into the definition of b from (3.10), we have $c\sim 10\times k\times 1.2/15$, which is about 12, if k is of the order of 15. This gives c_{112} of (3.7) equal to $24U_\tau^4\sigma/d$, a quantity still about 10 times greater than the terms involving the quadruple correlations in (3.2). The physical significance of $a=0$ means that in (3.9) the term due to deformation is small when compared with the terms due to transport of and the decay of the turbulent energies. Obviously from the equations of double correlation (2.3)–(2.6) the magnitude of k

can be determined by measurement of the mean squares of the velocity fluctuation components.

From the four alternative cases discussed above we see that although the measurement of the mean velocity distribution alone will not single out which one is the correct theoretical mean velocity distribution, measurements of the variations of the higher order correlation functions across the channel will decide this question. For example, experiment on the triple correlation $\overline{w_1 w_2^2}$ in (3.8) will decide whether c is negative or positive, and the theoretical pattern for the mean velocity distribution can thus be determined.

From another angle the above four special cases can also be considered to represent the mean velocity distribution in four parts of the channel. In the central portion, we have negative a and negative c [cf. (3.10), (3.8)]. Since both c and a_{2112} can be functions of the coordinate σ, their values may change at the various points of the channel. It is possible that a may eventually become positive as σ increases near the wall, while c which was negative in mid-channel, increases to zero and finally becomes positive on the wall of the channel.

According to its definition in (3.10), b is positive and is a monotonically increasing function of the distance from the center; likewise κ increases monotonically with σ, if c has already become positive. This increasing property of κ as the wall is approached is substantiated experimentally. In Hu's theory of the turbulent flow along a semi-infinite plate,[3] the mean velocity distribution in the turbulent boundary layer can be represented by an equation analogous to (3.11), and the value of b/c is equal to 0.4 instead of 0.1 as in case (3). Hence our present solution for mean motion only covers the channel proper; if the boundary layer on the channel wall is approached, the solution should be replaced by Hu's result. In fact it is well-known experimentally that the turbulent boundary layer on the wall covers the region $30 < R_\tau(1-\sigma) < 250$. In Dönch's measurement[10] cited above, R_τ is equal to 3630, so the range $0.931 < \sigma < 0.992$ represents approximately the turbulent boundary layer on the wall and we should expect formula (3.13) to fail in this region. A rigorous theory to explain the mean velocity distribution for the entire channel including the boundary layer might not be impossible according to present indications, but the actual mathematical manipulation involved would be much more complicated than that in the present treatment.

5. Conclusion. Based upon the foregoing analysis in the cases of the four values of κ for the motion of a turbulent fluid through a channel, we may conclude that the velocity defect distribution $(U_c - U)/U_\tau$, which is practically independent of the Reynolds number of the mean flow within the channel proper according to von Kármán, is also independent of the magnitudes of the turbulent fluctuation when the flow has reached the steady turbulent state. The question as to whether the above conclusion can be generalized to state that the double and triple correlation distributions across the channel when expressed in terms of the frictional velocity U_τ, namely, the ratios $\overline{w_i w_j}/U_\tau^2$ and $\overline{w_i w_j w_k}/U_\tau^3$, are also independent of the Reynolds number of the mean flow and of the correlations of still higher orders remains to be seen theoretically as well as experimentally. In any event, the friction velocity U_τ probably plays an important role for turbulent flow problems involving the presence of a wall, as in the present problem.