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ON THE MOTION OF A SPINNING SHELL*

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1. **Introduction.** Next after the problem of the motion of a particle in a resisting medium, the problem of the motion of a spinning shell is the major problem of exterior ballistics. Many crude treatments have been given, but the problem was first discussed exhaustively by Fowler, Gallop, Lock and Richmond.[1,2] Reference may also be made to treatments by Cranz[3] and Moulton.[4]

An exact treatment of the motion of a spinning shell as a hydrodynamical problem is obviously out of the question. The problem must be treated aerodynamically. This means that the forces exerted on the shell by the air must be regarded as dependent only on the instantaneous motion of the shell. The connection between the aerodynamic force system and the motion cannot be deduced logically. It must appear in the mathematical theory as a hypothesis, preferably supported by experimental observations.

But although mathematical theory cannot supply the aerodynamic forces, it does give us some information about them. Two basic ideas are important here.

First, the shell has an axis of symmetry. This fact has been used in all existing theories.

The second idea is a little more subtle. It concerns the connection between the position of the mass center (or center of gravity) of the shell and the aerodynamic force system. In one manner of speaking, there is no such connection. For two shells, moving with identical motions but with different mass-distributions, the aerodynamic forces are the same. But we cannot introduce the aerodynamic force system into the mathematical argument without expressing that force system mathematically as a force and a couple (or something equivalent). To do this, we must use a base-point,

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[1] R. H. Fowler, E. G. Gallop, C. N. H. Lock, H. W. Richmond, *The aerodynamics of a spinning shell*, Phil. Trans. Roy. Soc. London (A) **221**, 295–387 (1920).

[2] R. H. Fowler, C. N. H. Lock, *The aerodynamics of a spinning shell*, Part II, Phil. Trans. Roy. Soc. London (A) **222**, 227–249 (1921).

[3] C. Cranz, *Lehrbuch der Ballistik*, J. Springer, Berlin, 1925, p. 358.

[4] F. R. Moulton, *New methods in exterior ballistics*, University of Chicago Press, Chicago, 1926, chap. 6.

and reduce the force system to a force at that base-point, together with a couple. It is well known that, for a given force system, the force is independent of the base-point, but the couple is not.

Also, to describe the motion of the shell mathematically, we must use a base-point. The motion is described by the velocity of that base-point and an angular velocity. The angular velocity is independent of the choice of base-point, but the velocity is not.

Now it is natural to use the mass center as base-point. If there are two shells, S_1 and S_2, with mass centers O_1 and O_2, we may use O_1 as base-point for S_1 and O_2 as base-point for S_2. Suppose that the two shells are of identical geometrical form (but O_1 and O_2 are not geometrically corresponding points) and that their motions at the instant are the same. (This means that geometrically corresponding points have equal velocities; the velocities of O_1 and O_2 are *not* the same.) Then the force systems on the two shells are the same. But the moments about O_1 and O_2 are *not* the same.

If we set out to formulate aerodynamic laws, using the mass center as base-point, we must exercise great care. We must ensure ***invariance with respect to shift of mass center***. We must make sure, in the case described above, that when we apply our law, first to S_1 and then to S_2, we get equivalent force systems.

Unfortunately, Fowler et al.[1] paid no attention to this fact in formulating their aerodynamic laws (pp. 302–305), although they draw attention to the necessity for invariance (p. 305), and in fact make use of it. By considering a special case, it is easy to see the fallacy in their basic laws.

Consider the two shells described above. Let the velocity of O_1 be directed along the axis of the shell, and let the shell have an angular velocity represented by a vector perpendicular to the axis (plane motion). The yaw is zero, and the effect of the air is a drag along the axis. But now consider S_2. On account of the angular velocity, the velocity of O_2 is not along the axis; there is a yaw, and hence a cross wind force in addition to a drag. It is easy to see that the force systems on the two shells are not equivalent, as they ought to be since the motions are the same.

Thus the theory of Fowler et al. contains a logical contradiction. It is very difficult to discuss critically a theory containing a logical contradiction, for from inconsistent hypotheses we may arrive almost anywhere (at $1=0$, for example.) It may well be, however, that the logical contradiction does not invalidate the physical conclusions of their paper. In the example given above, the yaw of S_2 may well be very small indeed in cases of practical interest, and the logical inconsistency may be no more serious than that involved in writing $\pi=3.14$. Used in one way, this statement leads to $1=0$; used in another way, it leads to important practical results.

Nevertheless it is sound policy, in building up a theory in applied mathematics to make it logically consistent as far as possible. In the present paper we shall take care to state the aerodynamic laws in such a way as to avoid logical inconsistency.

Apart from the thorough treatment of the theory of the aerodynamic force system in sections 3 and 4, the following features of the present paper may be summarized here.

The exact equations of motion of the shell (independent of any aerodynamic hypothesis) are given a very compact form in (2.6). In section 5 it is shown how the aerodynamic functions may be found from high frequency photographs of a shell. Such observations should provide the ultimate test of the validity of the aerodynamic method. In view of the success of the cruder jump card method of Fowler et al., it seems probable that the aerodynamic hypothesis is valid, and, if so, the proposed method of observation should give us all information required concerning the aerodynamic functions.

There are three conditions for the stability of a spinning shell (section 7), but they are too complicated to interpret in the general case. If Magnus effects are absent (section 8), they become much simpler, and in fact there is then just one stability condition (8.19). In this condition the effect of the position of the mass center is shown explicitly. The condition is stronger than the usual condition (8.13b) based on the stability factor; a shell which is considered stable on the basis of the usual condition may in fact be unstable. We are very much indebted to Professor E. J. McShane for his critical comments on this paper in its original form. He has informed us that the existence of second stability condition, stronger than the usual one, has already been pointed out by R. H. Kent (Report No. 85, Ballistic Research Laboratory). This condition is implicit in the paper by Fowler et al. (1.332, equation 3.6234, and 4.12); this is discussed in section 10, where their method is brought into line with the more general method of the present paper.

Some well known facts are confirmed by theory in section 9. For a stable shell, after the oscillations have been damped out, the axis of the shell always points above the trajectory and to the right if the spin is right-handed. The phenomenon of trailing is explained; the axis of the shell turns downward at a rate approximately equal to the rate of turning of the tangent to the trajectory.

Drift also is discussed in section 9. A general condition (9.17) is obtained for standard drift, i.e., drift to the right for right-handed spin. When we specialize to subsonic velocity and flat trajectory, this condition simplifies to (9.20). When the numerical values of Fowler et al. are inserted, this inequality is liberally satisfied, so that the present theory is in agreement with the observed facts.

2. Exact equations of motion. We shall now develop the equations of motion of a shell in convenient form. No assumption is made here regarding the aerodynamic forces, and the only assumption regarding the shell is that it has an axis of dynamic symmetry (i.e., the momental ellipsoid at the mass center is a spheroid). Thus our equations would apply, for example, to a homogeneous projectile of square section or to a bomb with three or more fins, placed symmetrically.

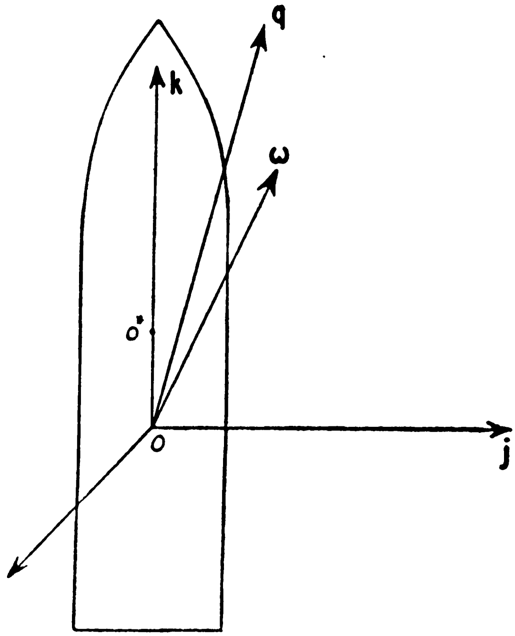

FIG. 1

We shall use the following notation, the motion being referred to a Newtonian reference system:

O = mass center of shell,
m = mass of shell,
A, C = transverse and axial moments of inertia at O,
$\mathbf{q}$ = velocity of O,
$\boldsymbol{\omega}$ = angular velocity of shell,
$\mathbf{h}$ = angular momentum of shell about O,
$\mathbf{F}$ = vector sum of aerodynamic forces acting on shell,
$\mathbf{G}$ = moment of aerodynamic forces about O,
$\mathbf{F}'$ = weight of shell.

Then the equations of motion are

$$m\dot{\mathbf{q}} = \mathbf{F} + \mathbf{F}', \qquad \dot{\mathbf{h}} = \mathbf{G}. \tag{2.1}$$

We introduce a right-handed unit orthogonal triad, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, fixed neither in space nor in the shell (Fig. 1). We take $\mathbf{k}$ along the axis of the shell, and $\mathbf{i}$, $\mathbf{j}$ perpendicular to $\mathbf{k}$, but the final choice of $\mathbf{i}$, $\mathbf{j}$ is deferred for the present. Let $\boldsymbol{\Omega}$ be the angular velocity of the triad.

We may now resolve the vectors as follows:

$$\left.\begin{aligned}
\mathbf{q} &= u\mathbf{i} + v\mathbf{j} + w\mathbf{k},\\
\boldsymbol{\omega} &= \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k},\\
\boldsymbol{\Omega} &= \Omega_1\mathbf{i} + \Omega_2\mathbf{j} + \Omega_3\mathbf{k},\\
\mathbf{h} &= A\omega_1\mathbf{i} + A\omega_2\mathbf{j} + C\omega_3\mathbf{k},\\
\mathbf{F} &= F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k},\\
\mathbf{G} &= G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k},\\
\mathbf{F}' &= F_1'\mathbf{i} + F_2'\mathbf{j} + F_3'\mathbf{k}.
\end{aligned}\right\} \tag{2.2}$$

Clearly $\Omega_1 = \omega_1$, $\Omega_2 = \omega_2$.

In scalar form the equations of motion (2.1) then read

$$\left.\begin{aligned}
m(\dot{u} - v\Omega_3 + w\omega_2) &= F_1 + F_1',\\
m(\dot{v} - w\omega_1 + u\Omega_3) &= F_2 + F_2',\\
m(\dot{w} - u\omega_2 + v\omega_1) &= F_3 + F_3',
\end{aligned}\right\} \tag{2.3}$$

$$\left.\begin{aligned}
A(\dot{\omega}_1 - \omega_2\Omega_3) + C\omega_3\omega_2 &= G_1,\\
A(\dot{\omega}_2 + \omega_1\Omega_3) - C\omega_3\omega_1 &= G_2,\\
C\dot{\omega}_3 &= G_3.
\end{aligned}\right\} \tag{2.4}$$

It is now convenient to introduce complex variables. We write

$$\left.\begin{aligned}
u + iv &= \xi,\\
\omega_1 + i\omega_2 &= \eta,\\
F_1 + iF_2 &= F,\\
G_1 + iG_2 &= G,\\
F_1' + iF_2' &= F'.
\end{aligned}\right\} \tag{2.5}$$

We multiply the second equation of (2.3) by i and add it to the first, and deal similarly with the equations (2.4). Thus we reduce the equations of motion to the form:

$$\left.\begin{aligned} \dot{\xi} + i\xi\Omega_3 - iw\eta &= (F + F')/m, \\ \dot{\eta} + i\eta\Omega_3 - iC'\omega_3\eta &= G/A, \qquad (C' = C/A), \\ \dot{w} - u\omega_2 + v\omega_1 &= (F_3 + F_3')/m, \\ \dot{\omega}_3 &= G_3/C. \end{aligned}\right\} \tag{2.6}$$

These equations are exact; no approximations have been made.

3. **The general aerodynamic hypothesis.** What is here set down is probably a little more general and explicit than previous statements about aerodynamic force systems. There is no implication that the hypothesis is physically accurate in all cases. All we can hope is that deductions from these assumptions lead in suitable cases to results in fair agreement with observation. But it seems best to make the hypothesis mathematically clear.

First we consider a fluid, at rest or in motion. We are not particularly conerned with the properties of the fluid. The important thing is that it defines

(i) a scalar field of density ρ;

(ii) a scalar field of local sound velocity c;

(iii) a vector field of velocity $\mathbf{W}$.

This last field defines two other vector fields, vorticity ($\mathbf{V}=1/2$ rot $\mathbf{W}$) and acceleration ($\mathbf{a}=d\mathbf{W}/dt$).

Usually in ballistics we deal with the static case in which $\mathbf{W}=0$ and ρ, c are functions of height only. A more accurate model is that in which $\mathbf{W}$ is horizontal, but in different directions at different heights to allow for changes in the direction of the wind with variation of height.

Now suppose we wish to investigate the motion of a solid through this fluid. To treat the problem adequately we should of course consider the disturbance produced in the fluid by the solid. But we do not do this. We use the fluid merely to compute from its undisturbed motion the aerodynamic forces acting on the solid.

Let O^* be the centroid of the solid, i.e., the position its mass center would occupy were the solid of uniform density. Let the motion of the solid be described by the velocity $\mathbf{q}^*$ of O^* and the angular velocity $\boldsymbol{\omega}^*$.

The basic hypothesis is then as follows:

Aerodynamic hypothesis: The aerodynamic force system exerted on the solid by the fluid consists of

(i) an aerostatic force;

(ii) an aerokinetic force system.

The aerostatic force acts at O^* and equals

$$\rho V_0(\mathbf{a} - \mathbf{P}) \tag{3.1}$$

where ρ is the density of the fluid at O^*, V_0 is the volume of the solid, and $\mathbf{P}$ is the body force per unit mass acting on the fluid at O^*. (Note that if $\mathbf{a}=0$ and $\mathbf{P}$ is gravity, this is simply the Archimedean buoyancy.) *The aerokinetic force sys-*

tem is represented by a force $\mathbf{F}^*$ at O^* and a couple $\mathbf{G}^*$; these are functions of ρ and c at O^* and of the vectors

$$\mathbf{q}^* - \mathbf{W}, \qquad \boldsymbol{\omega}^* - \mathbf{V}. \tag{3.2}$$

If $\mathbf{q}^* = \mathbf{W}$ and $\boldsymbol{\omega}^* = \mathbf{V}$, then $\mathbf{F}^* = 0$ and $\mathbf{G}^* = 0$.

Henceforth we shall assume $\mathbf{W} = 0$, and so $\mathbf{F}^*$, $\mathbf{G}^*$ depend only on ρ, c, $\mathbf{q}^*$, $\boldsymbol{\omega}^*$, while the aerostatic force is $-\rho V_0 \mathbf{P}$. If we were discussing the aerodynamics of a dirigible, the aerostatic force would be very important. For a shell it is quite trivial and we shall omit it altogether.

Thus for our purposes the aerodynamic force system consists of the force $\mathbf{F}^*$ at O^* and the couple $\mathbf{G}^*$; they are functions of ρ, c, $\mathbf{q}^*$, and $\boldsymbol{\omega}^*$.

It will be observed that our base-point O^* has been chosen in a definite way with respect to the *geometry* of the solid, and not with respect to its *mass-distribution*. This frees our laws from the objection raised in the Introduction to the laws of Fowler et al.

It is to be noted that it is by no means essential to select the centroid as base point. But it is least confusing to choose, once and for all, a point simply related to the geometry of the solid, and the centroid seems the most natural point to take.

4. **The aerodynamic force system for a shell with an axis of symmetry.** We now consider a shell with an axis of aerodynamic symmetry. By this we mean that its exterior is a surface of revolution. We might proceed for the present without introducing the mass-distribution of the shell, but it seems simpler to proceed at once to the case of complete symmetry. We shall therefore suppose that the shell has a common axis of aerodynamic and dynamic symmetry. All that is stated in section 2 is then valid and we shall use the same notation.

The mass center of the shell is at O and its centroid at O^*. Let us write

$$\overrightarrow{OO^*} = r\mathbf{k}, \tag{4.1}$$

and

$$\left.\begin{aligned} \mathbf{q}^* &= \text{velocity of } O^*,\\ \boldsymbol{\omega}^* &= \text{angular velocity of shell},\\ \mathbf{F}^* &= \text{vector sum of aerodynamic forces},\\ \mathbf{G}^* &= \text{moment of aerodynamic forces about } O^*. \end{aligned}\right\} \tag{4.2}$$

Then

$$\left.\begin{aligned} \mathbf{q}^* &= \mathbf{q} + \boldsymbol{\omega} \times r\mathbf{k}, \qquad \boldsymbol{\omega}^* = \boldsymbol{\omega},\\ \mathbf{F}^* &= \mathbf{F}, \qquad \mathbf{G}^* = \mathbf{G} + \mathbf{F} \times r\mathbf{k}. \end{aligned}\right\} \tag{4.3}$$

In the notation of (2.5) with asterisks attached to the symbols referring to O^*, we have in consequence

$$\left.\begin{aligned} \xi^* &= \xi - ir\eta, & w^* &= w,\\ \eta^* &= \eta, & \omega_3^* &= \omega_3,\\ F^* &= F, & F_3^* &= F_3,\\ G^* &= G - irF, & G_3^* &= G_3. \end{aligned}\right\} \tag{4.4}$$

Now $\mathbf{F}^*$ and $\mathbf{G}^*$ depend on $\mathbf{q}^*$ and $\boldsymbol{\omega}^*$. It follows from the aerodynamic symmetry that if the pair of vectors $\mathbf{q}^*$, $\boldsymbol{\omega}^*$ is given a rigid body rotation about the axis of symmetry, then the pair of vectors $\mathbf{F}^*$, $\mathbf{G}^*$ is also rotated rigidly about the axis through the same angle. Hence the following ten scalar quantities are unaltered by such a rotation:

$$F_3, \qquad G_3,$$

$$\left.\begin{array}{llll} u^*F_1^* + v^*F_2^*, & v^*F_1^* - u^*F_2^*, & u^*G_1^* + v^*G_2^*, & v^*G_1^* - u^*G_2^*, \\ \omega_1^*F_1^* + \omega_2^*F_2^*, & \omega_2^*F_1^* - \omega_1^*F_2^*, & \omega_1^*G_1^* + \omega_2^*G_2^*, & \omega_2^*G_1^* - \omega_1^*G_2^*. \end{array}\right\} \tag{4.5}$$

But, to within such a rotation, the vectors $\mathbf{q}^*$, $\boldsymbol{\omega}^*$ are determined by the quantities

$$w, \omega_3, \quad u^{*2} + v^{*2}, \quad \omega_1^{*2} + \omega_2^{*2}, \quad u^*\omega_1^* + v^*\omega_2^*, \quad u^*\omega_2^* - v^*\omega_1^*, \tag{4.6}$$

between which there exists the identity

$$(u^{*2} + v^{*2})(\omega_1^{*2} + \omega_2^{*2}) - (u^*\omega_1^* + v^*\omega_2^{*2})^2 = (u^*\omega_2^* - v^*\omega_1^*)^2. \tag{4.7}$$

Therefore the quantities (4.5) are functions of the quantities (4.6); in fact, for a shell of given size and shape, (4.5) are functions only of (4.6) and the air scalars ρ, c at O^*.

We now write

$$u^*F_1^* + v^*F_2^* = s_1, \qquad v^*F_1^* - u^*F_2^* = s_2. \tag{4.8}$$

Multiplying the second equation by i and subtracting it from the first, we get

$$\bar{\xi}^*F^* = s_1 - is_2, \tag{4.9}$$

the bar denoting the complex conjugate. Dealing similarly with the other quantities in (4.5), we see that

$$\left.\begin{array}{ll} \bar{\xi}^*F^*, & \bar{\xi}^*G^*, \\ \bar{\eta}^*F^*, & \bar{\eta}^*G^*, \end{array}\right\} \tag{4.10}$$

are complex functions of the real quantities in (4.6).

We cannot proceed further without an additional hypothesis. *We shall assume that*

$$F_1^*, \qquad F_2^*, \qquad G_1^*, \qquad G_2^*$$

are linear functions of

$$u^*, \qquad v^*, \qquad \omega_1^*, \qquad \omega_2^*.$$

This is certainly a reasonable assumption when the latter quantities are small.

We can then write

$$\left.\begin{array}{l} F^* = \alpha_1 u^* + \alpha_2 v^* + \beta_1\omega_1^* + \beta_2\omega_2^*, \\ G^* = \gamma_1 u^* + \gamma_2 v^* + \delta_1\omega_1^* + \delta_2\omega_2^*, \end{array}\right\} \tag{4.11}$$

where the eight complex coefficients are functions of w, ω_3, ρ and c. When we form the quantities (4.10) and use the fact that these must be functions of the quantities (4.6), we find $\alpha_2 = i\alpha_1$, $\beta_2 = i\beta_1$, etc., and so

$$\left.\begin{array}{l} F^* = \xi^*P^* + \eta^*Q^*, \\ G^* = \xi^*P'^* + \eta^*Q'^*, \end{array}\right\} \tag{4.12}$$

where P^*, Q^*, P'^*, Q'^* are complex functions of w, ω_3, ρ, c.

The components F_3, G_3 are functions of the quantities (4.6). *We shall assume that they are functions only of* w, ω_3, ρ, c. This also is a plausible assumption when u^*, v^*, ω_1^*, ω_2^* are small.

To sum up: *There are ten real aerodynamic functions of* w, ω_3, ρ, c, *contained in the set*

$$P^*, \quad Q^*, \quad P'^*, \quad Q'^*, \quad F_3^*, \quad G_3^*. \tag{4.13}$$

Let us see what these assumptions amount to in the case of a shell in a wind-tunnel. We think of the shell as moving and the air at rest. We put

$$v^* = 0, \qquad \omega_1^* = \omega_2^* = \omega_3^* = 0,$$

and (4.12) gives

$$F_1^* + iF_2^* = u^*P^*, \qquad G_1^* + iG_2^* = u^*P'^*.$$

In this simple case we must have, by symmetry since $\omega_3^* = 0$,

$$F_2^* = G_1^* = G_3^* = 0,$$

and so we have

$$F_1^* = u^*P^*, \qquad iG_2^* = u^*P'^*. \tag{4.14}$$

It is easy to see that these equations imply that (for small yaw), the cross wind force and the moment are proportional to the yaw. This is the usual assumption.

We now pass from the centroid O^* to the mass center O by the transformation (4.4). We get for the force system $\mathbf{F}$, $\mathbf{G}$ on the shell

$$\left.\begin{aligned} F &= F_1 + iF_2 = \xi P + \eta Q, \qquad & F_3 &= F_3^*, \\ G &= G_1 + iG_2 = \xi P' + \eta Q', \qquad & G_3 &= G_3^*, \end{aligned}\right\} \tag{4.15}$$

where P, Q, P', Q' are complex functions of w, ω_3, ρ, c, given by

$$\left.\begin{aligned} P &= P^*, \qquad & Q &= Q^* - irP^*, \\ P' &= P'^* + irP^*, \qquad & Q' &= Q'^* - irP'^* + ir(Q^* - irP^*). \end{aligned}\right\} \tag{4.16}$$

This gives the transformation of the aerodynamic functions when we pass from the centroid O^* to the mass center O. Actually this is the transformation for passage from any base-point to any other, provided of course that both lie on the axis.

To show the real and imaginary parts of the aerodynamic functions, we shall write (with similar equations in asterisked form)

$$\left.\begin{aligned} P &= P_1 + iP_2, \qquad & Q &= Q_1 + iQ_2, \\ P' &= P_1' + iP_2', \qquad & Q' &= Q_1' + iQ_2'. \end{aligned}\right\} \tag{4.17}$$

The transformation (4.16) then gives

$$\left.\begin{aligned} P_1 &= P_1^*, \\ P_2 &= P_2^*, \\ Q_1 &= Q_1^* + rP_2^*, \\ Q_2 &= Q_2^* - rP_1^*, \\ P_1' &= P_1'^* - rP_2^*, \\ P_2' &= P_2'^* + rP_1^*, \\ Q_1' &= Q_1'^* + rP_2'^* + r(-Q_2^* + rP_1^*), \\ Q_2' &= Q_2'^* - rP_1'^* + r(Q_1^* + rP_2^*). \end{aligned}\right\} \tag{4.18}$$

The method used above for the resolution of the aerodynamic force system is not the usual one. Three important vectors are involved: **k** the axis of the shell, **q** the velocity of the mass center, **ω** the angular velocity. In resolving vectors, it is necessary to pick out one of these three as a fundamental vector and build a basic triad on it. The traditional plan is to pick out **q** as fundamental and take **k** as a secondary vector, so that **q** and **k** together give one of the planes of the basic triad. Resolution of **F** along **q** and perpendicular to **q** in this plane gives the usual drag and lift. However convenient this may be for wind-tunnel work in which **q** is fixed while **k** is altered, it certainly appears less convenient than the method of the present paper for a simple mathematical formulation of the problem of the spinning shell. There is a further objection to the usual plan; the direction of **q** depends on the mass center.

The conventional terminology does not suit the present resolution. The following is suggested. The asterisk indicates that the centroid is used as base-point. The same notation without asterisks refers to the mass center.

$$\left.\begin{aligned} u^*\mathbf{i} + v^*\mathbf{j} &= \text{cross velocity,} \\ w\mathbf{k} &= \text{axial velocity,} \\ \omega_1\mathbf{i} + \omega_2\mathbf{j} &= \text{cross spin,} \\ \omega_3\mathbf{k} &= \text{axial spin.} \end{aligned}\right\} \tag{4.19}$$

$$\left.\begin{aligned} P_1^* \mid \xi^* \mid &= \text{cross force due to cross velocity } (-), \\ P_2^* \mid \xi^* \mid &= \text{Magnus force due to cross velocity } (+), \\ Q_1^* \mid \eta^* \mid &= \text{Magnus force due to cross spin } (+), \\ Q_2^* \mid \eta^* \mid &= \text{cross force due to cross spin } (+), \\ F_3 &= \text{axial force } (-). \end{aligned}\right\} \tag{4.20}$$

$$\left.\begin{aligned} P_1'^* \mid \xi^* \mid &= \text{Magnus torque due to cross velocity } (-), \\ P_2'^* \mid \xi^* \mid &= \text{cross torque due to cross velocity } (-), \\ Q_1'^* \mid \eta^* \mid &= \text{cross torque due to cross spin } (-), \\ Q_2'^* \mid \eta^* \mid &= \text{Magnus torque due to cross spin } (+), \\ G_3 &= \text{Magnus axial torque } (-). \end{aligned}\right\} \tag{4.21}$$

It is a consequence of symmetry that where the word "Magnus" is included above, the quantity in question changes sign with ω_3; where the word "Magnus" does not occur, the quantity in question does not change sign with ω_3. For uniformity, we have called the axial (viscous) torque "Magnus"; there is justification for this in the fact that it is the viscous torque that sets up the circulation which is responsible for the other Magnus effects. The signs in parentheses indicate probable signs of the various quantities when ω_3 is positive, assuming a center of pressure in front of the centroid.

Since

$$\mid \xi^* \mid = q^* \sin (\mathbf{q}^*, \mathbf{k}), \qquad \mid \eta^* \mid = \omega \sin (\boldsymbol{\omega}, \mathbf{k}), \tag{4.22}$$

it is clear that the usual sine law of variation is implicit in (4.20), (4.21). But since we suppose the angles in question to be small, the sine, tangent and circular measure are not distinguishable.

It is convenient to introduce positive dimensionless aerodynamic functions, as is done by Fowler et al. So we write, paying attention to dimensions and signs,

$$\left.\begin{aligned} P_1^* &= -\rho a^2 w f_1^*, & P_2^* &= \rho a^3 \omega_3 f_2^*, \\ Q_1^* &= \rho a^4 \omega_3 g_1^*, & Q_2^* &= \rho a^3 w g_2^*, \\ P_1'^* &= -\rho a^4 \omega_3 f_1'^*, & P_2'^* &= -\rho a^3 w f_2'^*, \\ Q_1'^* &= -\rho a^4 w g_1'^*, & Q_2'^* &= \rho a^5 \omega_3 g_2'^*. \end{aligned}\right\} \quad (4.23)$$

Here ρ is the air-density and a the radius of the cross section of the shell. The functions (f^*, g^*) depend certainly on w/c, and possibly also on $a\omega_3/c$ and the Reynolds number. The above equations may be regarded as definitions of the eight aerodynamic functions (f^*, g^*), which are analogous to the f_L, f_M, etc. of Fowler et al. To the above equations we may add

$$F_3 = -\rho a^2 w^2 f_3, \qquad G_3 = -\rho a^4 w \omega_3 g_3, \quad (4.24)$$

where f_3 and g_3 are dimensionless; f_3 is the usual drag except for the slight difference that we resolve along the axis of the shell and treat w as basic instead of q^*.

As the notation is necessarily somewhat complicated, let us summarize as follows: Askerisked quantities refer to the centroid, unasterisked to the mass center.

The aerodynamic force system is denoted by

$$F^* = F_1^* + iF_2^*, \qquad G^* = G_1^* + iG_2^*, \qquad F_3, \qquad G_3.$$

There are ten real aerodynamic functions contained in the set

$$P^*, \qquad Q^*, \qquad P'^*, \qquad Q'^*, \qquad F_3, \qquad G_3,$$

and these may be expressed in terms of the ten positive dimensionless aerodynamic functions

$$f_1^*, \quad f_2^*, \quad g_1^*, \quad g_2^*, \quad f_1'^*, \quad f_2'^*, \quad g_1'^*, \quad g_2'^*, \quad f_3, \quad g_3.$$

The same notation may be used with reference to the mass center, but since the aerodynamic force system has nothing to do with the mass center as such, the asterisked quantities are the more fundamental. If we wish to pass from O^* to O, we must transform by (4.18) and (4.23). Thus $f_1^* = f_1$, $f_2^* = f_2$, $f_3^* = f_3$, $g_3^* = g_3$, but the other functions change.

One more notation will be introduced for convenience in (6.4).

It is clear from (4.20), (4.21), (4.23) that if the dimensionless aerodynamic functions (f^*, g^*) are constants, we have the following proportionalities, δ denoting the small yaw:

$$\left.\begin{aligned} \text{cross force due to cross velocity} &\propto w^2\delta, \\ \text{cross torque due to cross velocity} &\propto w^2\delta, \\ \text{axial force} &\propto w^2, \\ \text{axial torque} &\propto w\omega_3. \end{aligned}\right\} \quad (4.25)$$

The first three of these are in agreement with experiment for subsonic velocities—the effects vary as the square of the velocity. The last (axial torque) requires comment.

The form of G_3 in (4.24) agrees with Fowler et al., but one may ask why (apart from the theory of dimensions) the factor w should be present. The following is a possible explanation. The rotation of the shell generates a rotating wake. If this wake has, throughout, the same spin as the shell, it has angular momentum $\frac{1}{2}\pi\rho a^4\omega_3$ per unit length. In unit time a length w of wake is generated, and so, by the conservation of angular momentum, the rate of loss of angular momentum of the shell is

$$-G_3 = \tfrac{1}{2}\pi\rho a^4 w\omega_3.$$

This argument not only confirms the form G_3 of (4.24); it gives

$$g_3 = \tfrac{1}{2}\pi. \tag{4.26}$$

A crude argument of this sort must be accepted only provisionally in the absence of experimental check.

5. **Determination of the aerodynamic functions by observation.** Fowler et al. stressed the importance of avoiding the simple empirical assumptions previously employed. As in the case of the drag function, it is necessary to determine the aerodynamic functions experimentally. What follows is a refinement and generalization of the jump card method of Fowler et al. Unless there are technical difficulties, or unless the basic aerodynamic hypothesis is wrong, the following method should yield *all* the aerodynamic functions quite simply, except perhaps g_3, and no doubt a method could be devised for it also.

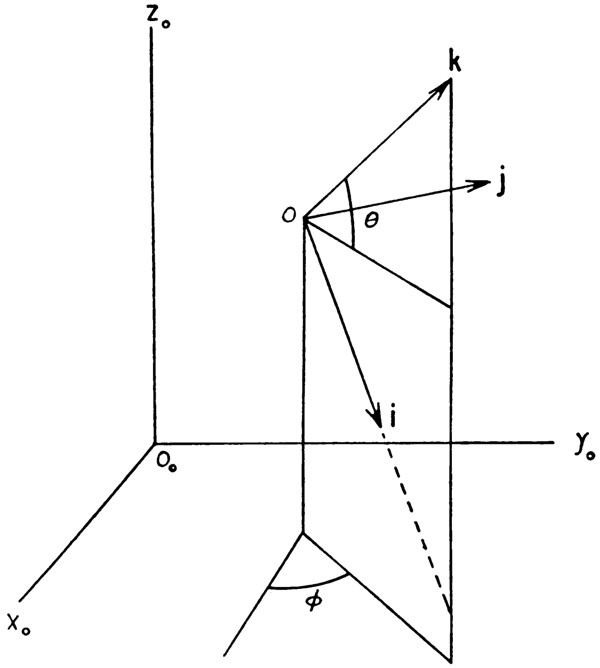

FIG. 2

Let a shell be fired horizontally and observations made of it not long after it leaves the muzzle. These observations consist of high-frequency photographs, one set of photographs being taken vertically and the other set horizontally from the side. These photographs show successive positions of the shell at short intervals of time.

We now turn to the exact equations of motion (2.6). There is some indeterminacy in these because we have not yet chosen the vector **i** definitely. Let us choose it in the vertical plane through the axis of the shell (**k**), pointing downward (Figure 2).

Then

$$F' = mg \cos \theta, \tag{5.1}$$

and the first two equations of (2.6) may be written

$$\left.\begin{aligned} F &= m(\dot{\xi} + i\xi\Omega_3 - iw\eta) - mg \cos \theta, \\ G &= A(\dot{\eta} + i\eta\Omega_3 - iC'\omega_3). \end{aligned}\right\} \tag{5.2}$$

These equations are exact. We may put $\cos \theta = 1$, since the axis of the shell is approximately horizontal. Then $\Omega_3 = 0$ by (6.2).

Now m, A, C' are known for the shell; w may be found from the observations or otherwise (muzzle velocity), and ω_3 deduced from the rifling. To find ξ, η as functions of t, it is merely necessary to measure on the photographic plates the linear displacements of the mass center and the angular displacements of the axis of the shell, corresponding to the short intervals between successive photographs. Smooth graphs might be made showing u, v, ω_1, ω_2 as functions of t or the complex quantities ξ, η might be plotted on an Argand diagram with the values of t marked in. In any case it should not be difficult to obtain $\dot{\xi}$ and $\dot{\eta}$ also as functions of t from these graphs.

When these functions are inserted in the right-hand sides of (5.2), we have F and G as functions of t. By (4.15) we have

$$\xi P + \eta Q = F, \qquad \xi P' + \eta Q' = G. \tag{5.3}$$

If we use two values of t, each of these equations yields two complex equations, and from them P, Q, P', Q' can be found. Here we have a good test of the aerodynamic hypothesis, for the values of P, Q, P', Q' should be independent of the particular instants chosen.

It may be advisable, as a refinement, to allow for the decrease in w between the two instants in question. This can easily be done from our knowledge of the drag function.

By repeating the experiment on the same shell, but using different muzzle velocities and riflings, we obtain P, Q, P', Q' as functions of w and ω_3.

The next step is to transform from the mass center to the centroid. This is done by (4.16), and we obtain P^*, Q^*, P'^*, Q'^* as functions of w and ω_3. Finally, the dimensionless aerodynamic functions (f^*, g^*) are found from (4.23).

It should be stressed that these last functions are characteristic of the *form* of the shell and completely independent of the mass distribution. Indeed, to a certain extent they will be independent of the size of the shell, but this must be accepted with caution.

6. **Plan of solution and partial linearization of the equations.** We now introduce fixed axes $O_0x_0y_0z_0$, O_0z_0 being directed vertically upward. Let θ be the inclination of $\mathbf{k}$ to the horizontal (Figure 2), and ϕ the inclination of the horizontal projection of $\mathbf{k}$ to O_0x_0. We have already made the vector $\mathbf{i}$ definite in section 5. We have then

$$\left.\begin{aligned} \mathbf{F}' &= mg \cos \theta\mathbf{i} - mg \sin \theta\mathbf{k}, \\ \mathbf{\Omega} &= -\dot{\phi} \cos \theta\mathbf{i} - \dot{\theta}\mathbf{j} + \dot{\phi} \sin \theta\mathbf{k}. \end{aligned}\right\} \tag{6.1}$$

Hence

$$\eta = -(\dot{\phi} \cos \theta + i\dot{\theta}), \qquad \Omega_3 = \dot{\phi} \sin \theta. \tag{6.2}$$

We substitute from (4.15) in (2.6), and the equations of motion become

$$\left.\begin{aligned} \dot{\xi} + i\xi\Omega_3 - iw\eta &= \xi X + \eta Y + g \cos\theta, \\ \dot{\eta} + i\eta\Omega_3 - iC'\omega_3\eta &= \xi X' + \eta Y', \\ \dot{w} - u\omega_2 + v\omega_1 &= F_3/m - g \sin\theta, \\ \dot{\omega}_3 &= G_3/C, \end{aligned}\right\} \tag{6.3}$$

where

$$X = P/m, \qquad Y = Q/m, \qquad X' = P'/A, \qquad Y' = Q'/A. \tag{6.4}$$

If we substitute from (6.2) for η, Ω_3 and regard X, Y, X', Y', F_3, G_3 as known functions of w, ω_3, ρ, c, we have six real equations for the dependent variables u, v, w, θ, ϕ, ω_3. But unless we assume ρ, c to be constants, we must bring in further equations. Let us assume them to be functions of height (z_0) only. By resolution of velocity we have

$$\left.\begin{aligned} \dot{x}_0 + i\dot{y}_0 &= (u \sin\theta + iv + w \cos\theta)e^{i\phi}, \\ \dot{z}_0 &= -u \cos\theta + w \sin\theta. \end{aligned}\right\} \tag{6.5}$$

When the last of these equations is associated with (6.3), we have seven real equations for seven unknowns, namely, those stated above and z_0. When they have been solved, the trajectory of the mass center is given by (6.5).

We now make the following two assumptions: (i) the vertical plane through the axis of the shell turns slowly; (ii) the angle of yaw is small. The first assumption implies that $\dot{\phi}$ and hence Ω_3 is small; the second implies that ξ/w is small. On account of the smallness of Ω_3 we reject the second terms in the first two equations of (6.3), and on account of the smallness of ξ/w we reject the second and third terms in the third equation.

Our partially linearized equations now read

$$\left.\begin{aligned} \dot{\xi} - iw\eta &= \xi X + \eta Y + g \cos\theta, \\ \dot{\eta} - iC'\omega_3\eta &= \xi X' + \eta Y', \\ \dot{w} &= F_3/m - g \sin\theta, \\ \dot{\omega}_3 &= G_3/C, \end{aligned}\right\} \tag{6.6}$$

where

$$\eta = -(\dot{\phi} \cos\theta + i\dot{\theta}). \tag{6.7}$$

7. **The stability of a spinning shell.** In discussing rapid oscillations of the shell, we treat w and ω_3 as constants in the first two equations of (6.6). Consequently X, Y, X', Y' are constants. In rapid oscillations differentiation with respect to t greatly increases the importance of a term. Hence we shall treat $\cos\theta$ as a constant in the first equation of (6.6); the term corresponding to a small change in θ will be negligible in comparison with the terms in η.

We have then linear equations with constant coefficients, which have solutions of the form

$$\left.\begin{aligned} \xi &= A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + A_3, \\ \eta &= B_1 e^{\alpha_1 t} + B_2 e^{\alpha_2 t} + B_3, \end{aligned}\right\} \tag{7.1}$$

where α_1, α_2 are the roots of the equation

$$\alpha^2 - (iC'\omega_3 + X + Y')\alpha + i(C'\omega_3 X - wX') + XY' - X'Y = 0, \tag{7.2}$$

and

$$\left.\begin{aligned} A_3 &= -\frac{g}{E}\cos\theta(iC'\omega_3 + Y'), \\ B_3 &= \frac{g}{E}\cos\theta\cdot X', \\ E &= i(C'\omega_3 X - wX') + XY' - X'Y. \end{aligned}\right\} \tag{7.3}$$

The condition for stability is that both roots of (7.2) should have non-positive real parts.

If we write

$$\left.\begin{aligned} K_1 &= X_1 + Y_1', \\ K_2 &= C'\omega_3 + X_2 + Y_2', \\ K_3 &= -C'\omega_3 X_2 + wX_2' + X_1Y_1' - X_2Y_2' - X_1'Y_1 + X_2'Y_2, \\ K_4 &= C'\omega_3 X_1 - wX_1' + X_1Y_2' + X_2Y_1' - X_1'Y_2 - X_2'Y_1, \end{aligned}\right\} \tag{7.4}$$

then (7.2) becomes

$$\alpha^2 - (K_1 + iK_2)\alpha + (K_3 + iK_4) = 0. \tag{7.5}$$

The condition for stability may be written

$$K_1 + \zeta\cos\chi \leqq 0, \tag{7.6}$$

where ζ, χ are defined by

$$\left.\begin{aligned} \zeta^4 &= (K_1^2 - K_2^2 - 4K_3)^2 + 4(K_1K_2 - 2K_4)^2, \qquad \zeta \geqq 0, \\ \zeta^2 \sin 2\chi &= 2(K_1K_2 - 2K_4), \\ \zeta^2 \cos 2\chi &= K_1^2 - K_2^2 - 4K_3, \qquad -\tfrac{1}{2}\pi \leqq \chi \leqq \tfrac{1}{2}\pi. \end{aligned}\right\} \tag{7.7}$$

It is immediately evident that there is instability if $K_1 > 0$. If $K_1 \leqq 0$, then the condition (7.6) is equivalent to

$$K_1^2 \geqq \zeta^2 \cos^2\chi, \tag{7.8}$$

or

$$2K_1^2 \geqq \zeta^2(1 + \cos 2\chi). \tag{7.9}$$

On substituting for $\zeta^2\cos 2\chi$ from (7.7), this becomes

$$K_1^2 + K_2^2 + 4K_3 \geqq \zeta^2. \tag{7.10}$$

Thus there is instability if $K_1 \leqq 0$, $K_1^2 + K_2^2 + 4K_3 < 0$. If $K_1 \leqq 0$, $K_1^2 + K_2^2 + 4K_3 \geqq 0$, the condition (7.10) is equivalent to

$$(K_1^2 + K_2^2 + 4K_3)^2 \geqq \zeta^4, \tag{7.11}$$

and, on substitution from (7.7), this becomes

$$K_1^2 K_3 + K_1 K_2 K_4 - K_4^2 \geqq 0. \tag{7.12}$$

To sum up, *the motion of the shell is stable if, and only if, the following three conditions are all satisfied*:

$$K_1 \leqq 0, \tag{7.13a}$$

$$K_1^2 + K_2^2 + 4K_3 \geqq 0, \tag{7.13b}$$

$$K_1^2 K_3 + K_1 K_2 K_4 - K_4^2 \geqq 0. \tag{7.13c}$$

The K's are given by (7.4).

These conditions are more general than any given previously.

If there is strong stability (i.e., if the real parts of α_1, α_2 are negative and large), then the first terms in (7.1) die away quickly. In fact, the rapid oscillations are damped out, and we are left with

$$\left.\begin{aligned} \xi &= -\frac{g}{E}\cos\theta\cdot(iC'\omega_3 + Y'), \\ \eta &= \frac{g}{E}\cos\theta\cdot X'. \end{aligned}\right\} \tag{7.14}$$

With these we associate the last two equations of (6.6), viz.

$$\left.\begin{aligned} \dot{w} &= F_3/m - g\sin\theta, \\ \dot{\omega}_3 &= G_3/C, \end{aligned}\right\} \tag{7.15}$$

and also $\eta = -(\dot{\phi}\cos\theta + i\dot{\theta})$.

In (7.14), (7.15) and the last of (6.5) we have seven real equations for the seven quantities u, v, w, θ, ϕ, ω_3, z_0. E is a function of w and ω_3 as in (7.3); it also involves z_0, since the properties of the air depend on z_0 and aerodynamic functions X, Y, X', Y' depend on the properties of the air. The above equations determine the motion of the stable shell.

We note that the equations (7.14), (7.15) are simply (6.6) with the terms $\dot{\xi}$, $\dot{\eta}$ deleted. To test whether this treatment is valid, we should solve (7.14), (7.15) for ξ, η, calculate $\dot{\xi}$, $\dot{\eta}$ by differentiating these solutions, and compare these calculated values with the other terms in (6.6). They should, of course, turn out to be small.

8. **Stability in the absence of Magnus effects.** If we accept the linear law (4.11), the aerodynamic force system (4.13) is the most general possible. As we shall see in section 10, the force system of Fowler et al. is a special case. The system (4.13) contains ten real functions, and it appears impossible to make any deductions of physical interest without introducing some simplifications. We shall retain a force system a little more general than that of Fowler et al.; our system satisfies the fundamental condition of invariance with respect to shift of mass center, whereas theirs does not.

Let us refer to (4.20), (4.21), and assume that *all Magnus effects vanish, except G_3*; this means that

$$P_2^* = Q_1^* = P_1'^* = Q_2'^* = 0. \tag{8.1}$$

This leaves us with four real aerodynamic functions, in addition to F_3 and G_3

$$P_1^* < 0, \qquad Q_2^* > 0, \qquad P_2'^* < 0, \qquad Q_1'^* < 0. \tag{8.2}$$

There can be no doubt that these inequalities are physically valid.

We now transform to the mass center O by (4.18). We find

$$P_2 = Q_1 = P_1' = Q_2' = 0. \tag{8.3}$$

Thus the Magnus effects do not reappear with change of base-point; in fact, *the vanishing of Magnus effects is an invariant condition.* For base-point O there are again just four real aerodynamic functions in addition to F_3 and G_3:

$$\left.\begin{aligned} P_1 &= P_1^*, \\ Q_2 &= Q_2^* - rP_1^*, \\ P_2' &= P_2'^* + rP_1^*, \\ Q_1' &= Q_1'^* + rP_2'^* + r(-Q_2^* + rP_1^*). \end{aligned}\right\} \tag{8.4}$$

Then by (6.4), (7.4) and (8.3),

$$\left.\begin{aligned} K_1 &= X_1 + Y_1' = P_1/m + Q_1'/A, \\ K_2 &= C'\omega_3, \\ K_3 &= wX_2' + X_1Y_1' + X_2'Y_2 = \frac{wP_2'}{A} + \frac{1}{mA}(P_1Q_1' + P_2'Q_2), \\ K_4 &= C'\omega_3X_1 = \frac{C'\omega_3P_1}{m}. \end{aligned}\right\} \tag{8.5}$$

The stability conditions (7.13) read

$$X_1 + Y_1' \leqq 0, \tag{8.6a}$$

$$(C'\omega_3)^2 + 4wX_2' + (X_1 + Y_1')^2 + 4(X_1Y_1' + X_2'Y_2) \geqq 0, \tag{8.6b}$$

$$X_1Y_1'(C'\omega_3)^2 + (X_1 + Y_1')^2(wX_2' + X_1Y_1' + X_2'Y_2) \geqq 0. \tag{8.6c}$$

These are the stability conditions in the absence of Magnus effects. Now by (4.23), (6.4), (8.4), we have (since $A^* = mr^2 + A$)

$$\left.\begin{aligned} X_1 &= -\frac{\rho a^2 w}{m} f_1^*, \\ Y_2 &= \frac{\rho a^3 w}{m}\left(g_2^* + \frac{r}{a} f_1^*\right), \\ X_2' &= -\frac{\rho a^3 w}{A}\left(f_2'^* + \frac{r}{a} f_1^*\right) \\ Y_1' &= -\frac{\rho a^4 w}{A}\left[g_1'^* + \frac{r}{a}(g_2^* + f_2'^*) + \frac{r^2}{a^2} f_1^*\right], \\ X_1 + Y_1' &= -\frac{\rho a^4 w}{A}\left[g_1'^* + \frac{r}{a}(g_2^* + f_2'^*) + \frac{A^*}{ma^2} f_1^*\right], \\ X_1Y_1' + X_2'Y_2 &= \frac{\rho^2 a^6 w^2}{mA}(f_1^* g_1'^* - g_2^* f_2'^*). \end{aligned}\right\} \tag{8.7}$$

If we substitute these expressions in (8.6) we get stability conditions in terms of the functions (f^*, g^*). However, these conditions are somewhat complicated, and we shall make approximations.

The f's of Fowler et al. hardly exceed 10 in value. Our (f^*, g^*) functions are defined in a slightly different way, but it certainly seems legitimate to assert that the dimensionless quantities

$$\epsilon = \frac{\rho a^3}{m} f \tag{8.8}$$

are much less than unity, f standing for any one of the (f^*, g^*) functions. Then it is clear that

$$(X_1 + Y_1')^2, \qquad X_1 Y_1' + X_2' Y_2$$

are both small relative to wX_2'. Consequently our stability conditions (8.6) may be simplified to

$$X_1 + Y_1' \leqq 0, \tag{8.9a}$$

$$(C'\omega_3)^2 + 4wX_2' \geqq 0, \tag{8.9b}$$

$$X_1 Y_1' (C'\omega_3)^2 + (X_1 + Y_1')^2 w X_2' \geqq 0. \tag{8.9c}$$

It will be noticed that Y_2 has disappeared from the stability conditions in the last approximation. This aerodynamic function corresponds to cross force due to cross spin relative to the mass center [cf. (6.4) and (4.20)]. Thus it might be asserted that, for the discussion of stability in the absence of Magnus effects, cross force due to cross spin may be neglected. But this statement is not entirely correct, because this cross force contributes to the moment Y_1', and Y_1' remains in the stability conditions.

Let us examine the first stability condition (8.9a). On substitution from (8.7) it reads

$$\frac{r}{a}(g_2^* + f_2'^*) + g_1'^* + \frac{A^*}{ma^2} f_1^* \geqq 0. \tag{8.10a}$$

If r is positive (so that the mass center lies behind the centroid), this inequality is certainly satisfied; it is also satisfied for some negative range of r. But an interesting question arises: Can we make the shell unstable by pushing its mass center forward towards the nose? This is hardly to be expected on physical grounds, and it may well be that (8.10a) is satisfied for all permissible values of r, i.e., all values which place the mass center inside the shell.

It is tedious (and perhaps of little physical interest) to discuss the other stability conditions for sufficiently large negative values of r. We shall therefore assume either that r is positive, or, if it is negative, it is such that (8.10a) is satisfied and also

$$X_2' < 0, \qquad Y_1' < 0. \tag{8.11}$$

Let us write

$$s = \frac{(C'\omega_3)^2}{-4wX_2'}. \tag{8.12}$$

This is essentially the same as the usual stability factor.[5] Then the second stability condition (8.9b) takes the familiar form

$$s \geqq 1, \tag{8.13b}$$

while the third condition (8.9c) may be written

$$s \geqq \frac{(X_1 + Y_1')^2}{4X_1Y_1'}. \tag{8.13c}$$

Since the fraction on the right is never less than unity, this condition replaces (8.13b).

Let us substitute in (8.13c) from (8.7) and sum up as follows:

STABILITY CONDITION. The following assumptions are made:

(i) Magnus effects are negligible (except that G_3 may exist).

(ii) The quantities ϵ of (8.8) are very small.

(iii) The mass center is behind the centroid, or, if in front, its negative coordinate r is such that (8.10a) is satisfied and also

$$\left.\begin{aligned} &f_2'^* + \frac{r}{a} f_1^* > 0, \\ &g_1'^* + \frac{r}{a}(g_2^* + f_2'^*) + \frac{r^2}{a^2} f_1^* > 0. \end{aligned}\right\} \tag{8.14}$$

Then the motion of the shell is stable if, and only if,

$$s \geqq \frac{ma^2}{4A} \frac{[g_1'^* + (r/a)(g_2^* + f_2'^*) + (A^*/ma^2)f_1^*]^2}{f_1^*[g_1'^* + (r/a)(g_2^* + f_2'^*) + (r^2/a^2)f_1^*]}, \tag{8.15}$$

where s is as in (8.12), or equivalently

$$s = \frac{C^2\omega_3^2}{4\rho a^3 A w^2[f_2'^* + (r/a)f_1^*]}, \qquad A = A^* - mr^2. \tag{8.16}$$

To show the dependence on r more explicitly, we introduce the dimensionless quantity

$$p = \frac{C^2\omega_3^2}{4\rho a^5 m w^2}, \tag{8.17}$$

so that

$$p = s\frac{A}{ma^2}\left(f_2'^* + \frac{r}{a} f_1^*\right). \tag{8.18}$$

Then *the sole condition for stability reads*

$$p \geqq \frac{(f_2'^* + (r/a)f_1^*)[g_1'^* + (r/a)(g_2^* + f_2'^*) + (A^*/ma^2)f_1^*]^2}{4f_1^*[g_1'^* + (r/a)(g_2^* + f_1'^*) + (r^2/a^2)f_1^*]}. \tag{8.19}$$

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[5] T. J. Hayes, *Elements of ordnance*, J. Wiley and Sons, New York, 1938, p. 418.

Since A^* is the transverse moment of inertia at the centroid, the position of the mass center is involved in this formula only in the symbol r.

We see therefore that the usually accepted criterion for stability (8.13b) is not the true one; it must be replaced by one of the inequalities (8.13c), (8.15) or(8. 19), which are of course equivalent to one another. As we remarked in the Introduction, the existence of a second condition for stability has been noticed by R. H. Kent. We shall refer to stability again in section 10.

9. **The trajectory of a stable shell in the absence of Magnus effects.** Let us assume, as in the preceding section, that Magnus effects are absent, except that G_3 may exist. Then, using (8.3) and (6.4) with (7.14), we get for the trajectory of a stable shell, after the disturbance has been damped out,

$$\left.\begin{aligned} \xi &= -\frac{g}{E}\cos\theta(iC'\omega_3 + Y_1'),\\ \eta &= i\frac{g}{E}\cos\theta\cdot X_2', \qquad \eta = -\dot{\phi}\cos\theta - i\dot{\theta}. \end{aligned}\right\} \tag{9.1}$$

Here E is as in (7.3); let us make the approximation indicated above (8.9), so that

$$E = wX_2' + iC'\omega_3X_1. \tag{9.2}$$

Splitting (9.1) into real and imaginary parts we get

$$\left.\begin{aligned} u &= -\frac{g}{|E|^2}\cos\theta[X_1(C'\omega_3)^2 + wX_2'Y_1'] \\ v &= -\frac{g}{|E|^2}\cos\theta\cdot C'\omega_3wX_2', \end{aligned}\right\} \tag{9.3}$$

(where we have dropped a term X_1Y_1' in comparison with wX_2') and

$$\left.\begin{aligned} \dot{\phi} &= -\frac{g}{|E|^2}C'\omega_3X_2'X_1, \\ \dot{\theta} &= -\frac{g}{|E|^2}\cos\theta\cdot w(X_2')^2. \end{aligned}\right\} \tag{9.4}$$

We shall assume, as in section 8, that X_1, X_2', Y_1' are all negative. Further, since the shell is stable, we have as in (8.9c)

$$X_1Y_1'(C'\omega_3)^2 + (X_1 + Y_1')^2wX_2' \geqq 0. \tag{9.5}$$

But

$$(X_1 + Y_1')^2 > Y_1'^2, \qquad X_2' < 0,$$

and therefore

$$X_1Y_1'(C'\omega_3)^2 + Y_1'^2wX_2' \geqq 0. \tag{9.6}$$

It follows at once from (9.3) that u is positive. This means that *the nose of the shell points above the trajectory.*

From (9.4) we see that $\dot{\phi}<0$ if $\omega_3>0$. Thus *for positive (right-handed) spin the vertical plane through, the axis of the shell turns to the right.*[6] For negative spin it turns to the left.

These two facts are well known to be true in practice.

There remain two outstanding physical facts to explain. These are (i) the trailing of the shell along the trajectory, (ii) the drift.

We see from (9.4) that $\dot{\theta}$ is negative, i.e., the inclination of the axis of the shell to the horizontal decreases steadily. But does it decrease at that rate required for trailing? We must be careful to avoid a circular argument. We have *assumed* that trailing takes place—otherwise the yaw is not small, and all our arguments are based on the smallness of the yaw. We must now *verify* that $\dot{\theta}$, as given by (9.4), is approximately equal to the rate of turning of the tangent to the trajectory of the mass center. The theory of the plane particle-trajectory gives, on resolution along the normal,

$$\dot{\theta}_0 = -\frac{g\cos\theta_0}{w}, \tag{9.7}$$

where θ_0 is the inclination of the tangent to the horizontal. To establish the required result, we must compare this with (9.4), and show that

$$\frac{|E|^2}{(wX_2')^2} = 1, \tag{9.8}$$

approximately. Now by (9.2), (8.12), (8.7), this fraction is

$$1 + X_1^2\left(\frac{C'\omega_3}{wX_2'}\right)^2 = 1 - \frac{4sX_1^{\ 2}}{wX_2'}$$

$$= 1 + 4s\,\frac{\rho a^3}{m}\,\frac{A}{ma^2}\,\frac{f_1^{*2}}{f_2^{*\prime} + (r/a)f_1^*}\,. \tag{9.9}$$

The last expression here is of the order of $s\epsilon$, where ϵ is as in (8.8). Hence, unless the stability factor s is very great, this expression is very small, and the condition of trailing is approximately fulfilled.

It is interesting that if s is very great the verification breaks down, for this is just what we would expect. If, by some mechanism, an enormous spin were imparted to a shell, the gyroscopic stability would be so great that the direction of the axis would remain fixed and the shell would not trail.

To discuss the drift, we write down (6.5) again:

$$\dot{x}_0 + i\dot{y}_0 = (u\sin\theta + iv + w\cos\theta)e^{i\phi}. \tag{9.10}$$

This is the horizontal velocity of the mass center in complex form. Consider the complex quantity

$$\alpha + i\beta = \frac{\ddot{x}_0 + i\ddot{y}_0}{\dot{x}_0 + i\dot{y}_0}\,. \tag{9.11}$$

It is obvious that the vector $\dot{x}_0+i\dot{y}_0$ turns to the left if β is positive, and to the right if β is negative. It is our business to investigate the sign of β.

[6] Hayes, op. cit., 420.

We differentiate (9.10) logarithmically and simplify the result by the fact that u/w and v/w are small. This gives

$$\beta = \frac{d}{dt}\left(\frac{v}{w\cos\theta}\right) + \dot{\phi}. \tag{9.12}$$

With the approximation (9.8), we have from (9.3), (9.4)

$$\frac{v}{w\cos\theta} = -\frac{gC'\omega_3}{w^2X_2'}, \qquad \dot{\phi} = -\frac{gC'\omega_3}{w^2X_2'}X_1, \tag{9.13}$$

and so

$$\frac{\beta}{Z} = \frac{d}{dt}\log|Z| + X_1, \tag{9.14}$$

where

$$Z = -\frac{gC'\omega_3}{w^2X_2'}. \tag{9.15}$$

As a terminology, let us say that a shell has standard *drift* when it goes to the right ($\beta<0$) for right-handed spin ($\omega_3>0$), and vice versa. Now Z has the same sign as ω_3. Hence we get a standard drift if

$$\frac{\beta}{Z} \equiv \frac{d}{dt}\log|Z| + X_1 < 0. \tag{9.16}$$

Substituting from (8.7), we see that this *condition for standard drift* reads

$$f_1^* > \frac{m}{\rho a^2 w}\frac{d}{dt}\log\frac{\omega_3}{\rho w^3(f_2'^* + (r/a)f_1^*)}. \tag{9.17}$$

Let us look into the meaning of this inequality, assuming that the dimensionless aerodynamic functions are constants. This corresponds to a subsonic velocity [cf. (4.25)]. Further, let the trajectory be flat, so that ρ is constant and θ so small that it may be neglected.

Then by (6.6) and (4.24)

$$\dot{w} = -\frac{\rho a^2}{m}w^2 f_3, \qquad \dot{\omega}_3 = -\frac{\rho a^4}{C}w\omega_3 g_3. \tag{9.18}$$

Let s be the arc length of the trajectory (do not confuse with the stability factor). Then $w=ds/dt$, $\dot{w}=w\,dw/ds$, and so we have

$$\frac{1}{w}\frac{dw}{ds} = -\frac{\rho a^2}{m}f_3, \qquad \frac{1}{\omega_3}\frac{d\omega_3}{ds} = -\frac{\rho a^4}{C}g_3. \tag{9.19}$$

The right-hand side of (9.17) becomes

$$\frac{m}{\rho a^2}\frac{d}{ds}\log\frac{\omega_3}{w^3} = 3f_3 - \frac{ma^2}{C}g_3,$$

and so the condition for standard drift reads

$$f_1^* + \frac{ma^2}{C} g_3 > 3f_3. \tag{9.20}$$

We may write f_1 in place of f_1^*—they are equal. We note that ma^2/C will lie between 1 and 2.

We observe from (9.20) that cross wind force and axial torque tend to give standard drift, but axial force acts the other way. Let us use the numerical values of Fowler et al. in (9.20). We have ([1], pp. 306, 309)

$$\left.\begin{aligned} f_1 &= f_L + f_R = f_N = 3.34, \\ f_3 &= f_R = 0.34. \end{aligned}\right\} \tag{9.21}$$

We see that (9.20) is liberally satisfied, even if $g_3 = 0$. Thus the present theory appears adequate to explain drift without bringing in Magnus effects.

10. **The aerodynamic force system of Fowler, Gallop, Lock, and Richmond.**[1] In the preparation of this section we are very much indebted to Professor E. J. McShane, who read our paper in its original form and pointed out in detail the connections between our work and that of Fowler et al.

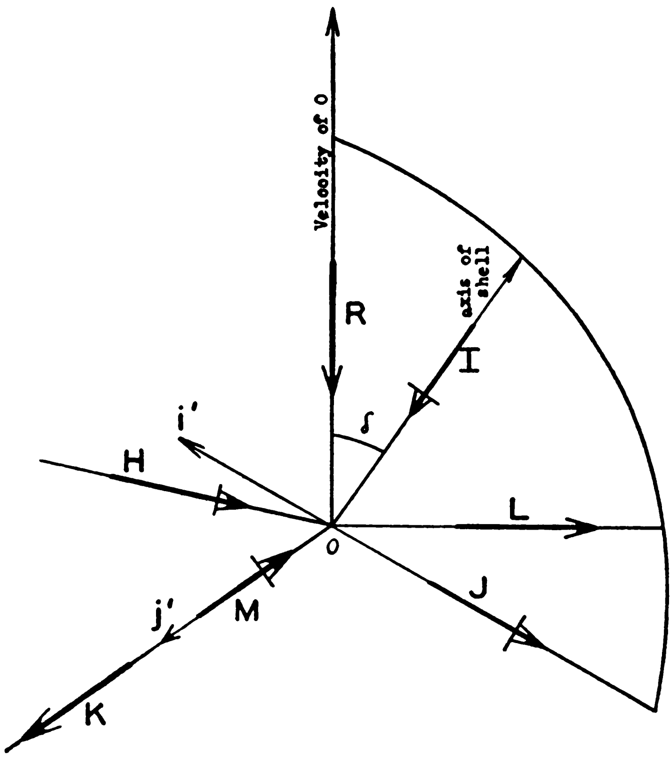

FIG. 3

The axis of the shell is indicated in Figure 3; O is the mass center and δ the yaw. The aerodynamic force system of Fowler et al. is represented by seven vectors—three forces (plain arrows) and four couples (arrows with crossbars). Their terminology is as follows:

$$\left.\begin{aligned} \mathbf{R} &= \text{drag},\\ \mathbf{L} &= \text{cross wind force},\\ \mathbf{K} &= \text{swerving force},\\ \mathbf{M} &= \text{moment tending to increase yaw},\\ \mathbf{H} &= \text{yawing moment due to yawing},\\ \mathbf{I} &= \text{axial moment},\\ \mathbf{J} &= \text{swerving moment}.\end{aligned}\right\} \qquad (10.1)$$

We shall use the notation of the present paper for velocity, angular velocity and the radius of cross section of the shell (a), and consider only the case of small yaw ($\delta = |\xi|/w$). Then the dimensionless aerodynamic functions of Fowler et al. are defined by

$$\left.\begin{aligned} R &= \rho a^2 w^2 f_R,\\ L &= \rho a^2 w^2 \delta f_L = \rho a^2 w|\xi| f_L,\\ K &= \rho a^3 w\omega_3 \delta f_K = \rho a^3 \omega_3 |\xi| f_K,\\ M &= \rho a^3 w^2 \delta f_M = \rho a^3 w |\xi| f_M,\\ H &= \rho a^4 w |\eta| f_H,\\ I &= \rho a^4 w\omega_3 f_I,\\ J &= \rho a^4 w\omega_3 \delta f_J = \rho a^4 \omega_3 |\xi| f_J.\end{aligned}\right\} \qquad (10.2)$$

Let $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}$ be an orthogonal triad of unit vectors, with $\mathbf{k}$ along the axis of the shell. The vector $\mathbf{i}'$ lies as shown in the plane containing $\mathbf{k}$ and the velocity of O. Then, to the first order in δ,

$$\left.\begin{aligned} \mathbf{R} &= -R\delta\mathbf{i}' - R\mathbf{k},\\ \mathbf{L} &= -L\mathbf{i}',\\ \mathbf{K} &= K\mathbf{j}',\\ \mathbf{M} &= -M\mathbf{j}',\\ \mathbf{H} &= -H\frac{\omega_1'}{|\eta|}\mathbf{i}' - H\frac{\omega_2'}{|\eta|}\mathbf{j}',\\ \mathbf{I} &= -I\mathbf{k},\\ \mathbf{J} &= -J\mathbf{i}',\end{aligned}\right\} \qquad (10.3)$$

where ω_1', ω_2' are the components of $\boldsymbol{\omega}$ along $\mathbf{i}'$, $\mathbf{j}'$.

Let $\mathbf{i}$, $\mathbf{j}$ be any orthogonal unit vectors, perpendicular to $\mathbf{k}$, so that the triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is that considered in the present paper. It does not matter at present whether $\mathbf{i}$ lies in the vertical plane through $\mathbf{k}$. We have

$$\mathbf{i}' = \frac{u\mathbf{i} + v\mathbf{j}}{|\xi|}, \qquad \mathbf{j}' = \frac{-v\mathbf{i} + u\mathbf{j}}{|\xi|}, \qquad \omega_1'\mathbf{i}' + \omega_2'\mathbf{j}' = \omega_1\mathbf{i} + \omega_2\mathbf{j}. \qquad (10.4)$$

The total aerodynamic force is

$$\begin{aligned}\mathbf{F} = \mathbf{R} + \mathbf{L} + \mathbf{K} = \mathbf{i}\left(-R\delta\frac{u}{|\xi|} - L\frac{u}{|\xi|} - K\frac{v}{|\xi|}\right) \\ + \mathbf{j}\left(-R\delta\frac{v}{|\xi|} - L\frac{v}{|\xi|} + K\frac{u}{|\xi|}\right) \\ - \mathbf{k}R,\end{aligned} \tag{10.5}$$

and so, since $w\delta = |\xi|$,

$$\left.\begin{aligned}F = F_1 + iF_2 = \xi\left(-\frac{R}{w} - \frac{L}{|\xi|} + \frac{iK}{|\xi|}\right), \\ F_3 = -R.\end{aligned}\right\} \tag{10.6}$$

The total aerodynamic couple is

$$\begin{aligned}\mathbf{G} = \mathbf{M} + \mathbf{H} + \mathbf{I} + \mathbf{J} = \mathbf{i}\left(M\frac{v}{|\xi|} - H\frac{\omega_1}{|\eta|} - J\frac{u}{|\xi|}\right) \\ + \mathbf{j}\left(-M\frac{u}{|\xi|} - H\frac{\omega_2}{|\eta|} - J\frac{v}{|\xi|}\right) \\ - \mathbf{k}I,\end{aligned} \tag{10.7}$$

and so

$$\left.\begin{aligned}G = G_1 + iG_2 = -iM\frac{\xi}{|\xi|} - H\frac{\eta}{|\eta|} - J\frac{\xi}{|\xi|}, \\ G_3 = -I.\end{aligned}\right\} \tag{10.8}$$

Certain quantities are defined as follows:

$$\left.\begin{aligned}\kappa = \frac{L}{m|\xi|}, \quad \lambda = \frac{K}{m\omega_3|\xi|}, \quad \mu = \frac{Mw}{|\xi|}, \\ \nu = \kappa + \frac{R}{mw}, \quad h = \frac{H}{A|\eta|}, \quad \Gamma = \frac{I}{C\omega_3}, \quad \gamma = \frac{Jw}{C\omega_3|\xi|}.\end{aligned}\right\} \tag{10.9}$$

Then (10.6), (10.8) give

$$\left.\begin{aligned}F = \xi(i\lambda m\omega_3 - m\nu), \qquad F_3 = -mw(\nu - \kappa), \\ G = \xi\left(-i\frac{\mu}{w} - \frac{\gamma C\omega_3}{w}\right) - \eta A h, \qquad G_3 = -C\omega_3\Gamma.\end{aligned}\right\} \tag{10.10}$$

Comparing these with (4.15), we see that the force system of Fowler et al. is a particular case of our general system, with

$$\left.\begin{aligned}P = -m\nu + im\lambda\omega_3, \\ Q = 0, \\ P' = -\frac{\gamma C\omega_3}{w} - i\frac{\mu}{w}, \\ Q' = -Ah.\end{aligned}\right\} \tag{10.11}$$

The general system has eight real parts in these terms; the system of Fowler et al. has only five:

$$\left.\begin{aligned} P_1 &= -m\nu, & P_2 &= m\lambda\omega_3, \\ P_1' &= -\frac{\gamma C\omega_3}{w}, & P_2' &= -\frac{\mu}{w}, \\ Q_1' &= -Ah. \end{aligned}\right\} \tag{10.12}$$

It is clear from the transformation (4.16) that $Q=0$ is not invariant with respect to shift of mass center. Thus Eqs. (10.11) describing the aerodynamic force system cannot be valid in general. It may happen of course that they are true for one particular mass center, but they cannot remain true when we shift the mass center.

Fowler et al. find little evidence for the existence of the Magnus effects J, K, or equivalently γ, λ. If we put them equal to zero, the survivors in (10.12) are

$$P_1 = -m\nu, \qquad P_2' = -\frac{\mu}{w}, \qquad Q_1' = -Ah. \tag{10.13}$$

These should be compared with (8.4), which are the general survivors in the absence of Magnus effects. We note that Q_2 is absent from (10.13), which means that the mass center is chosen so that $Q_2^* - rP_1^*$ is zero, or at least negligible.

By (6.4) we obtain from (10.13)

$$X_1 = -\nu, \qquad X_2' = -\frac{\mu}{Aw}, \qquad Y_1' = -h, \tag{10.14}$$

and so the stability condition (8.13c) reads

$$s \geqq \frac{(\nu + h)^2}{4\nu h}. \tag{10.15}$$

We have referred in the Introduction to a second stability condition implicit in the work of Fowler et al.; it is

$$s \geqq \frac{(\kappa + h)^2}{4\kappa h}. \tag{10.16}$$

The difference between (10.15) and (10.16) does not appear to be very great in practice. It is a question of replacing ν by κ, and by (10.9), (10.2)

$$\frac{\nu - \kappa}{\kappa} = \frac{R}{L}\frac{|\xi|}{w} = \frac{f_R}{f_L} = \frac{1}{10}, \tag{10.17}$$

roughly.

There are very simple relationships between the dimensionless aerodynamic functions in the two theories. We take the mass center O as base-point, and use (4.23) without asterisks, together with (10.12), (10.9), (10.2); we find

$$\left.\begin{aligned} f_1 &= f_R + f_L = f_N, & f_2 &= f_K, \\ f_1' &= f_J, \quad f_2' = f_M, & g_1' &= f_H. \end{aligned}\right\} \tag{10.18}$$

The functions g_1, g_2, g_2' are zero in the theory of Fowler et al.

As the paper of Fowler et al. is one of the basic papers of modern ballistics, it will be useful to summarize our criticisms as follows:

(i) Their aerodynamic force system is not the most general system consistent with

(a) the aerodynamic hypothesis,

(b) linear dependence on the cross components in the case of small yaw,

(c) the symmetry of the shell.

(ii) Their system does not satisfy the fundamental requirement of invariance with respect to shift of mass center.

(iii) If only shells with mass centers near their centroids are considered, it may be that the above theoretical objections are of small practical importance.

We believe that our exact dynamical equations (6.3) provide a clearer approach to the problem of the spinning shell than do the dynamical equations of Fowler et al. But it is frankly admitted that our simple treatment of the equations of motion in section 7 does not appear to be as satisfactory mathematically as their method. We have made the plausible but rather crude assumption that it is permissible to regard $\cos\theta$, w, ω_3 as constant during the oscillation. It would be interesting to apply their more refined methods to our differential equations, but this we must defer for the present.