The three remaining unknowns are found by distributing the residuals among all six equations by least squares. When this is done, we find for our original unknowns:

$$C_1 = 3.4727, \qquad C_2 = -13.4739, \qquad C_3 = 14.0244,$$
$$C_4 = -0.0003, \qquad C_5 = 0.0002, \qquad C_6 = 0.0001.$$

This method has been applied successfully to least-squares solutions in geometrical optics and to the colorimetric problem of finding polynomial reflection curves which will yield a prescribed set of tristimulus values under a given illuminant and fit other prescribed conditions.

In the latter problem, the small unknowns must be neglected prior to symmetrization and reduction to unit diagonal.

NEW FORMULATIONS OF THE EQUATIONS FOR COMPRESSIBLE FLOW*

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Introduction. A prominent aerodynamic effect of combustion in a moving gas stream is an alteration of the flow pattern owing to heat release within the fluid. This alteration occurs not only in the immediate neighborhood of heat sources but also downstream where the entropy and stagnation temperature vary from one streamline to another. As a background for combustion research, appropriate descriptions of these altered flow patterns have been investigated. This paper considers the downstream patterns, which are restricted to be the adiabatic and steady flows of an inviscid fluid. In a second paper,[1] diabatic (i.e., non-adiabatic) flows will be discussed.

Since one-dimensional flow theory[2,3] can be considerably condensed by use of the local Mach number M, it was natural to seek a corresponding condensation with the help of the Mach vector

$$\mathbf{M} = M\mathbf{V}/V$$

and the Crocco vector

$$\mathbf{W} = \mathbf{V}/V_t$$

in which V_t is the variable limiting velocity at each point of the fluid. The introduction of Mach and Crocco vectors into the compressible flow equations sufficiently simplified or altered their form that a number of further investigations were suggested including those of diabatic flow.

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[1] B. L. Hicks, *Diabatic flow of a compressible fluid*, submitted to Quarterly of Applied Mathematics.

[2] Neil P. Bailey, *The thermodynamics of air at high velocities*, Journ. Aero. Sci. **11**, 227–238 (1944).

[3] B. L. Hicks, D. J. Montgomery, and R. H. Wasserman, *The one dimensional theory of steady compressible fluid flow in ducts with friction and heat addition*, NACA TN, 1947.

1. Development of the new equations. The basis for our development is the following set of equations:

$$c_v\rho\mathbf{V}\cdot\nabla T + p\nabla\cdot\mathbf{V} = 0, \tag{1.1}$$

$$\rho\mathbf{V}\cdot\nabla\mathbf{V} + \nabla p = 0, \tag{1.2}$$

$$\nabla\cdot\rho\mathbf{V} = 0, \tag{1.3}$$

$$p = R\rho T. \tag{1.4}$$

It is appropriate to include (1.1), an expression of energy conservation, because the entropy S and stagnation temperature T_t are not constant throughout the field of flow. The Bernoulli relation,

$$c_pT_t = c_pT + \tfrac{1}{2}V^2, \tag{1.5}$$

then defines the stagnation temperature. In terms of M and W, equation (1.5) becomes

$$(T_t/T) = 1 + (\gamma - 1)M^2/2 = (1 - W^2)^{-1}. \tag{1.6}$$

The Mach vector equations are derived from (1.1), (1.2), (1.3), (1.4) by the substitution $\mathbf{V} = \sqrt{\gamma RT}\,\mathbf{M}$ and elimination of ρ and T. The new equation of motion resulting is

$$\mathbf{M}\cdot\nabla\mathbf{M} - \frac{\gamma - 1}{\gamma + 1}\mathbf{M}\nabla\cdot\mathbf{M} + \frac{1}{\gamma}\nabla\log p = 0, \tag{1.7}$$

and the new equation of continuity

$$\nabla\cdot\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-(\gamma+1)/2(\gamma-1)}\mathbf{M} = 0. \tag{1.8}$$

The equations of motion and continuity in terms of the Crocco vector are derived similarly, the initial transformations being $\mathbf{V} = V_t\mathbf{W}$ and $T = V_t^2(1 - W^2)/2c_p$. It is in addition necessary to use the fact that T_t and therefore also V_t are constant on streamlines for adiabatic, inviscid flow (cf., for example, Vaszonyi's proof[4]). The Crocco vector equations are then

$$\mathbf{W}\cdot\nabla\mathbf{W} + \frac{\gamma - 1}{2\gamma}(1 - W^2)\nabla\log p = 0, \tag{1.9}$$

$$\nabla\cdot(1 - W^2)^{1/\gamma-1}\mathbf{W} = 0. \tag{1.10}$$

2. General comments. The local behavior of a compressible fluid is characterized by its Mach number. Therefore the vectors **M** and **W**, from whose magnitudes M can be determined, supply more useful descriptions of a flow than the vector **V** from whose magnitude M cannot be computed without knowledge of V_t. The equations possess equal generality in all three languages but are simpler in **M** and **W** language for these representations have permitted elimination of all but one vector variable from the equation of continuity and elimination of all but this vector and p from the equation of motion. Such reduction in the number of independent variables has heretofore been accomplished only for iso-energetic flow.

[4] A. Vazsonyi, *On rotational gas flows*, Quarterly Appl. Math. **3**, 29–37 (1945).

Rotational flows are most aptly described by the **W** language; the added term $\mathbf{M}\nabla\cdot\mathbf{M}$ in the **M** equation of motion brings with it no advantage. (The **W** language is still the best when *diabatic* rotational flows are considered.) However when either **M** or **W** is irrotational, the physical characteristics of the resulting flows make it appropriate to retain each language.

3. Geometry of stream tubes and transition through the speed of sound. In the one-dimensional approximation, isentropic compressible flows have minimum flow area at sonic velocity. The same property has been suggested by Prandtl[5] for irrotational three-dimensional flows. We shall prove that this property is valid for all continuous, steady adiabatic flows when an appropriate description of stream-tube area is introduced.

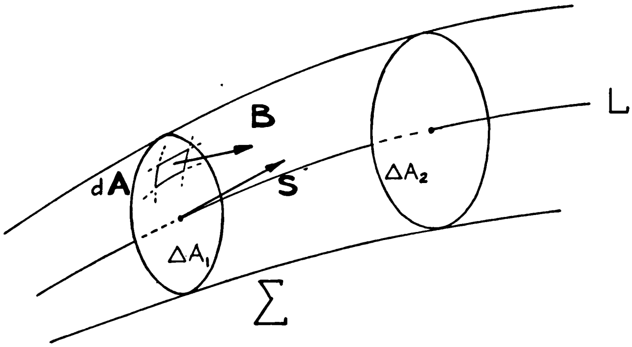

FIG. 1.

Let **B** be any continuous and solenoidal vector parallel to **M**. Consider any two plane sections ΔA_1, ΔA_2 of the stream-tube Σ which are normal to the streamline L contained in Σ. Since $\int_{\Delta A_1}\mathbf{B}\cdot d\mathbf{A}_1=\int_{\Delta A_2}\mathbf{B}\cdot d\mathbf{A}_2=\int_{\Delta A}B_n dA$ a position (m) can be found for each section such that $[B_n(m)\Delta A]_1=[B_n(m)\Delta A]_2$ or

$$\frac{\partial}{\partial s}\log B_n(m)+\frac{1}{\Delta A}\frac{\partial\Delta A}{\partial s}=0, \tag{3.1}$$

in which $\partial/\partial s$ denotes differentiation along L. As $\Delta A\rightarrow 0$, $B_n(m)\rightarrow B$, the magnitude of B on L. Also if $\mathbf{s}=\mathbf{B}/B$ is the unit vector in the direction of flow

$$\nabla\cdot\mathbf{s}=\lim_{\Delta A\rightarrow 0}\frac{1}{\Delta A}\frac{\partial\Delta A}{\partial s}\cdot \tag{3.2}$$

The fractional rate of change of stream-tube area with respect to streamline arc is therefore measured by $\nabla\cdot\mathbf{s}$ for flow in both two and three dimensions. Similar relations derived geometrically have been used by von Kármán[6] for two-dimensional flows.

The continuity equation in **M** language can be rewritten in terms of $\nabla\cdot\mathbf{s}$ as

$$M^{-1}\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}(1-M^2)\frac{\partial M}{\partial s}+\nabla\cdot\mathbf{s}=0. \tag{3.3}$$

For $\nabla\cdot\mathbf{s}>0$, $|1-M|$ cannot decrease but for $\nabla\cdot\mathbf{s}<0$, $|1-M|$ must decrease in the di-

[5] L. Prandtl, *General considerations on the flow of compressible fluids*, NACA TM No. 805, 1936.

[6] Th. von Kármán, *Compressibility effects in aerodynamics*, Jour. Aero. Sci. **8**, 337–356 (1941).

rection of flow i.e. sonic velocity is approached. As $M \rightarrow 1$ however, $\partial M/\partial s$ diverges like $-(\gamma+1)(\nabla \cdot \mathbf{s})/4(1-M)$ and $\partial p/\partial s$ diverges similarly. Consequently if M is to pass through the value unity continuously, $\nabla \cdot \mathbf{s}$ must be zero there. Consideration of the signs shows that the indicated extremum corresponds to a minimum of stream-tube area at $M=1$.

4. Thermodynamic considerations. The equation

$$\mathbf{V} \times (\nabla \times \mathbf{V}) = c_p \nabla T_t - T \nabla S \tag{4.1}$$

which is similar to Vazsonyi's,[4] contains the thermodynamic implications of the $\mathbf{V}$ equations. (Less general equations than (4.1) have also been derived by Crocco,[7] Tollmien[8] and Emmons.[9]) The corresponding equation in terms of $\mathbf{W}$ and the stagnation pressure p_t is easily obtained from (1.9)

$$\mathbf{W} \times (\nabla \times \mathbf{W}) = \frac{\gamma - 1}{2\gamma} (1 - W^2) \nabla \log p_t \tag{4.2}$$

showing that there is one fundamental thermodynamic quantity in adiabatic flow, namely p_t, and not two separate ones S and T_t. Thus for a given $p_t = p_t'$ distribution, the same $\mathbf{W}$ (or $\mathbf{M}$ but not $\mathbf{V}$) field obtains no matter what the variation of S and T_t so long as they combine in the form $[T_t \exp(-S/c_p)]^{\gamma/\gamma-1}$ to give p_t'. There seems to be no corresponding theorem for equation (4.1).

Differentiation of (4.1) yields a differential form of Bjerknes' theorem[8]

$$\nabla \times [\mathbf{V} \times (\nabla \times \mathbf{V})] = - \nabla T \times \nabla S. \tag{4.3}$$

Differentiation of (4.2) results in Crocco's equation[7] containing $\mathbf{W}$ only:

$$\nabla \times \left[\frac{\mathbf{W} \times (\nabla \times \mathbf{W})}{1 - W^2} \right] = 0. \tag{4.4}$$

Equations (1.10), (4.3) and (4.4) were originally derived for isoenergetic flow, a restriction now seen to be unnecessary.

5. Irrotational flows. We now consider in turn the consequences of irrotationality of $\mathbf{M}$, $\mathbf{W}$ and $\mathbf{V}$ fields. If $\mathbf{M} = \nabla \varphi_M$, an integrability condition

$$\nabla^2 \varphi_M = f(\varphi_M) \tag{5.1}$$

must be satisfied if the $\mathbf{M}$ equation of motion (1.7) is to be integrable. (A similar equation for the stream function occurs in theory of rotational incompressible flow.)[10]. Since the potential function must also satisfy the continuity equation (1.8) it is to be expected that the manifold of permissible functions φ_M will be somewhat restricted. This is illustrated by the fact that $p \exp(\gamma M^2/2)$ is a function of φ_M alone, or that M, φ_M and p_t are related by the expression

[7] L. Crocco, *Eine neue Stromfunktion für die Erforschung der Bewegung der Gase mit Rotation*, ZaMM **17**, 1–7 (1937).

[8] W. Tollmien, *Ein Wirbelsatz für stationäre isoenergetische Gasströmungen*, Luftfahrtforschung **19**, 145–147 (1942), British R.T.P. Trans. No. 1744, Ministry Aircraft Prod.

[9] H. W. Emmons, *The numerical solution of compressible fluid flow problems*, NACA TN No. 932, 1944.

[10] M. Lagally, *Ideale Flüssigkeiten*, Handbuch der Physik, Julius Springer (Berlin), vol. 7, ch. I, Art. 19, p. 29, and Art. 32, p. 49.

$$\Xi(M) = \Phi(\varphi_M) - \frac{1}{\gamma} \log p_t, \tag{5.2}$$

in which

$$\Xi(M) = -\frac{\gamma-1}{2} \int M^3 \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1} dM,$$

$$\Phi(\varphi_M) = \frac{\gamma-1}{\gamma+1} \int f(\varphi_M) d\varphi_M. \tag{5.3}$$

(In uniplanar flow, explicit formulae for potential and stream functions can be derived for both irrotational **M** and some irrotational **W** flows. We have not ascertained whether these formulae represent more than the well-known elementary radial and vortex flows.)

If $\mathbf{W} = \nabla\varphi_W$, then $\nabla p_t = 0$, a case discussed by Vazsonyi[4] in the less appropriate **V** language. Flows with constant stagnation pressure but variable entropy and stagnation temperature from one streamline to another may be of immediate practical interest. For example the effect of variable chamber gas temperature upon thrust coefficient of a rocket with "perfect" nozzle can be computed. This calculation is possible because irrotational **V** flow is the special case of irrotational **W** flow which occurs when not only p_t but also S and therefore T_t are constant throughout the flow ($\nabla S = \nabla T_t = 0$ is implied by $\mathbf{V} = \nabla\varphi_V$). Accordingly the same partial differential equation (derived from the continuity equation (1.10)) is satisfied by both φ_W and its isentropic form φ_V,

$$\sum_{i=1}^{3} \left[1 - W^2 - \frac{2}{\gamma-1}\left(\frac{\partial\varphi_w}{\partial x_i}\right)^2\right] \frac{\partial^2\varphi_w}{\partial x_i^2} - \frac{4}{\gamma-1} \sum_{i>j=1}^{3} \frac{\partial\varphi_w}{\partial x_i} \frac{\partial\varphi_w}{\partial x_j} \frac{\partial^2\varphi_w}{\partial x_i \partial x_j} = 0. \tag{5.4}$$

LINEARIZATION OF SOLUTIONS IN SUPERSONIC FLOW*

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1. Introduction. The equations governing flow at supersonic speeds are believed to be well known, but the difficulties of calculating exact solutions are so great that approximate solutions are the aim of the present and foreseeable future. Two main approaches to such approximate solutions are commonly considered:

(a) the calculation of numerical approximations by high-speed calculators,

(b) simplification of the equations and explicit solution of the simplified equations.

Of course the first route requires the high-speed calculating machines which are now in sight, but not yet available.

It is the purpose of this note to propose and exemplify a third approach which may partly replace (b) and frequently supplement (a), namely

(c) simplification of the dependence of the answers on one or more parameters.

2. Is linearization of solutions mathematically justified? The first objection which many would raise against "linearizing the solutions" as a substitute for "linearizing

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