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APPROXIMATIONS IN ELASTICITY BASED ON THE CONCEPT OF FUNCTION SPACE*

BY

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1. Introduction. The theory developed in the present paper applies to any elastic body (in general anisotropic) possessing a positive-definite strain-energy function, quadratic in the components of stress. This function provides us with a metric in a function space in which the point or vector represents a state of stress. The geometry of the function space follows Euclidean analogies closely, and is powerful in suggesting methods of approximation. The aim is to obtain approximate solutions of elastic boundary value problems with errors which are calculable, the error being measured in terms of distance in function space, or, equivalently, in terms of strain-energy.

After dealing with notation and basic concepts in Sec. 2, we discuss vectors in function space in Sec. 3. Sections 4 and 5 are intended to introduce the reader to the ideas which lie behind the general method; only a first approximation is discussed, and only the simplest types of boundary conditions.

In Sec. 6 more general types of boundary conditions are introduced, and higher approximations under these boundary conditions are treated in Secs. 7–11. Throughout, the basic plan is to locate the solution of the elastic boundary-value problem (considered as a point in function space) on a hypercircle of determinable center and radius. As a practical test of the method, it is used in Sec. 12 to obtain approximate solutions for the torsion of a prism of square cross section. In Sec. 13 reference is made to other work, and some known results are strengthened by use of the present method.

2. Notation and basic concepts. Latin suffixes take the range of values 1, 2, 3 and the summation convention operates on repeated suffixes. The coordinates x_i are rectangular cartesians, and differentiation with respect to a coordinate is indicated by a comma ($F_{,i}=\partial F/\partial x_i$).

By a *state* of an elastic body we understand a set of six stress components E_{ij} ($E_{ij}=E_{ji}$), given as functions of the coordinates throughout the body. For simplicity, we shall consider only functions which are continuous in the body and on its surface, and possess continuous first and second order derivatives in the body. It will be obvious that these conditions may be weakened, in the sense that these conditions hold throughout each of a finite number of parts into which the body is divided, with suitable continuity conditions across the surfaces which separate the parts.

The body is assumed to possess a *strain-energy function*

$$W = \tfrac{1}{2}c_{ijmn}E_{ij}E_{mn}, \tag{2.1}$$

* Received September 20, 1946.

where the constants c_{ijmn} satisfy

$$c_{ijmn} = c_{jimn} = c_{ijnm} = c_{mnij}, \tag{2.2}$$

and are such that the form W is positive-definite (i.e. it is positive unless all the stress components vanish).

We now introduce the *strain components* e_{ij} in an unconventional way. Instead of defining strain in terms of displacement, we define it by the generalized Hooke's law

$$e_{ij} = c_{ijmn}E_{mn}. \tag{2.3}$$

Thus (2.1) may be written

$$W = \tfrac{1}{2}e_{ij}E_{ij}. \tag{2.4}$$

It follows from (2.2) that for any two states, E_{ij} and E'_{ij}, we have the *reciprocity relation*

$$e_{ij}E'_{ij} = e'_{ij}E_{ij}. \tag{2.5}$$

The usual *equations of compatibility* read

$$e_{ij,mn} + e_{mn,ij} = e_{im,jn} + e_{jn,im}. \tag{2.6}$$

By (2.3), these can be translated into conditions on the stress components. In general, they will *not* be satisfied by an arbitrary state. But if they are satisfied, then the partial differential equations

$$u_{i,j} + u_{j,i} = 2e_{ij}, \tag{2.7}$$

have solutions u_i, unique to within an infinitesimal rigid body displacement,[1] and u_i is the *displacement* for the state E_{ij}. We are to remember that an arbitrary state has in general no corresponding displacement.

Should we wish to construct a state satisfying the equations of compatibility, the simplest plan is to choose a set of displacements, use (2.7) to obtain e_{ij}, and then solve (2.3) to obtain E_{ij}. The compatibility equations (2.6) will be automatically satisfied on account of (2.7).

Throughout this paper, we shall suppose that body forces are absent.* Thus the *equations of equilibrium* read

$$E_{ij,j} = 0. \tag{2.8}$$

In general these equations will not be satisfied by an arbitrary state.

We shall denote by n_i the unit vector normal to the surface of the body, pointing outward. Then the stress across the surface is

$$T_i = E_{ij}n_j. \tag{2.9}$$

The following notation for "inner products" will be found convenient. Since it will be obvious whether an integration throughout the body or an integration over its surface is implied, one type of notation serves for both types of inner product. We shall write

[1] I. S. Sokolnikoff, Mathematical Theory of Elasticity, McGraw-Hill, New York, 1946, p. 24.

* The extension to the case where body forces are present will be made in a later paper.

$$(e \cdot E') = \int e_{ij}E'_{ij}dv,$$
$$(u \cdot T') = \int u_i T'_i dS, \tag{2.10}$$

the former integral being taken throughout the body, and the latter over its surface. The interpretation of similar expressions will be obvious to the reader.

In attempting to solve a problem in elasticity, we seek a state which satisfies

(*i*) the equations of compatibility (2.6),

(*ii*) the equations of equilibrium (2.8),

(*iii*) the assigned boundary conditions, which will generally be T_i assigned, or u_i assigned, or T_i assigned in part and u_i assigned in part.

We approach the solution by considering states in which one or more of these conditions are relaxed.

3. Geometry in the function spaces of states. Since the power of the present method lies largely in the stimulation of geometrical intuition, it is well to pictorialize from the start. We shall develop some simple properties of the function space of states.

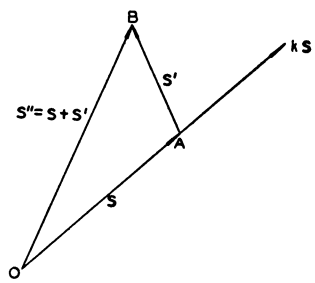

FIG. 1.

The unstressed state, $E_{ij} = 0$, is represented by the *origin* O (Fig. 1). Any other state is represented by a *point* such as A. Just as we describe the position of a point in ordinary space by a position-vector drawn from the origin, so we describe the point or state A by the *vector* $\overrightarrow{OA}$, or more compactly by a single letter (**S**) in heavy type. Thus the symbol **S** represents a state of stress (six functions of position throughout the body).

We *add* vectors in an obvious way. If we have two states,

S with stress components E_{ij}, and

S′ with stress components E'_{ij},

we define the sum $\mathbf{S}'' = \mathbf{S} + \mathbf{S}'$ to be the state with stress components

$$E''_{ij} = E_{ij} + E'_{ij}.$$

The definition of $\mathbf{S} - \mathbf{S}'$ is obvious.

To multiply a vector by a scalar constant k, we write $\mathbf{S}' = k\mathbf{S}$, and define **S**′ by the equations $E'_{ij} = kE_{ij}$. Note that E_{ij} are functions of position in the body, but k is a constant. The distributive laws are satisfied, so that

$$(k + k')\mathbf{S} = k\mathbf{S} + k'\mathbf{S}, \qquad k(\mathbf{S} + \mathbf{S}') = k\mathbf{S} + k\mathbf{S}'.$$

So far we have mentioned only vectors drawn out from the origin O, i.e. *bound vectors*. But the idea of a *free vector*, familiar in ordinary space, can be used with advantage. Apart from the geometrical representation, there is no real distinction between a free vector and a vector bound to O: each corresponds to a given set of stress components. But in the geometrical representation, a free vector **S** starts from *any* initial state A and proceeds to that state B for which the stress components exceed those of A by the stress components corresponding to **S**. In fact, a free vector may be

regarded as a transition from one state to another, and we make it a bound vector if we specify the initial state.

The algebraic procedures apply to free vectors. The addition of vectors follows the familiar parallelogram law (Fig. 1).

It should be amply clear from the context or from the appropriate diagram whether a free or bound vector is meant. In any case of doubt, a vector should be interpreted as a vector drawn from the origin O.

Fig. 1 will also draw the reader's attention to the geometrical relation between the vectors $\mathbf{S}$ and $k\mathbf{S}$. If k were negative, there would be a reversal of sense.

We define the *length* or *magnitude* S of a vector $\mathbf{S}$ by

$$S^2 = \int 2W dv = \int e_{ij}E_{ij}dv = (e\cdot E), \qquad S \geqq 0, \tag{3.1}$$

the integrals being taken throughout the body. Here e_{ij} is the strain corresponding to the stress E_{ij}, by (2.3). Since the strain-energy W is positive-definite, the length of any vector is real, and is zero only for the unstressed state, $E_{ij}=0$. Sometimes it will be convenient to denote the length of a vector $\mathbf{S}$ by $|\mathbf{S}|$. A vector $\mathbf{S}$ is a *unit vector* if $S=1$.

The distance between the extremities of two vectors, $\mathbf{S}$ and $\mathbf{S}'$, drawn from O, is defined to be the length of the vector $\mathbf{S}-\mathbf{S}'$. Thus this distance is given by

$$|\mathbf{S}-\mathbf{S}'|^2 = \int (e_{ij}-e'_{ij})(E_{ij}-E'_{ij})dv. \tag{3.2}$$

It vanishes if, and only if, $\mathbf{S}=\mathbf{S}'$, i.e. if the two states are identical.

The *scalar product* plays an important role. We define the scalar product of two vectors as the inner product of the corresponding states:

$$\mathbf{S}\cdot\mathbf{S}' = \int e_{ij}E'_{ij}dv = (e\cdot E'). \tag{3.3}$$

It is obvious, from the reciprocity relation (2.5), that the scalar product has the commutative property

$$\mathbf{S}\cdot\mathbf{S}' = \mathbf{S}'\cdot\mathbf{S}. \tag{3.4}$$

Two vectors are said to be *perpendicular* or *orthogonal* if their scalar product vanishes.

Let us explore the physical meaning of the scalar product. Let us suppose that $\mathbf{S}$ satisfies the equations of compatibility, so that it arises from a displacement, and that $\mathbf{S}'$ satisfies the equations of equilibrium. Then

$$\begin{aligned}\mathbf{S}\cdot\mathbf{S}' = (e\cdot E') &= \int e_{ij}E'_{ij}dv = \int u_{i,j}E'_{ij}dv \\ &= \int u_iE'_{ij}n_jdS - \int u_iE'_{ij,j}dv,\end{aligned} \tag{3.5}$$

by Green's theorem. The last integral vanishes, since $\mathbf{S}'$ satisfies the equations of equilibrium (2.8). Hence

$$\mathbf{S}\cdot\mathbf{S}' = \int u_iT'_i\,dS = (u\cdot T'); \tag{3.6}$$

in words, *the scalar product* $\mathbf{S}\cdot\mathbf{S}'$ *equals the work done in the displacement of* $\mathbf{S}$ *by the surface stress of* $\mathbf{S}'$. If the vectors are orthogonal, this work is zero.

As an example, consider a prismatic bar. Let the states $\mathbf{S}$ and $\mathbf{S}'$ correspond to simple tension and flexure, respectively. Then $\mathbf{S}$ is orthogonal to $\mathbf{S}'$.

It should be noted that the scalar product can be interpreted in terms of work only if one of the states satisfies the equations of equilibrium and the other the equations of compatibility.

We come now to a fundamental inequality. From the positive-definite character of strain-energy, it follows that if $\mathbf{S}$ and $\mathbf{S}'$ are any two arbitrary states, and k an arbitrary real number, then

$$\int (e'_{ij} - ke_{ij})(E'_{ij} - kE_{ij})dv \geqq 0, \tag{3.7}$$

the integral being taken throughout the body. This reduces to

$$S'^2 - 2k\mathbf{S}\cdot\mathbf{S}' + k^2S^2 \geqq 0, \tag{3.8}$$

and since this holds for all real values of k, we deduce the Schwarzian inequality

$$|\mathbf{S}\cdot\mathbf{S}'| \leqq SS'. \tag{3.9}$$

If the sign of equality holds in (3.9), then it is easy to see that the states $\mathbf{S}$ and $\mathbf{S}'$ must be connected by a relation $\mathbf{S}'=K\mathbf{S}$, where K is some real constant; in other words, the vectors $\mathbf{S}$ and $\mathbf{S}'$ have the same direction, or opposite directions.

We can now define the *angle* θ between two vectors $\mathbf{S}$ and $\mathbf{S}'$ by

$$\cos\theta = \frac{\mathbf{S}\cdot\mathbf{S}'}{SS'}, \qquad 0 \leqq \theta \leqq \pi. \tag{3.10}$$

By virtue of (3.9), the angle so defined is always real, and is of course equal to $\frac{1}{2}\pi$ when the vectors are orthogonal. For the angle between two unit vectors, $\mathbf{I}$ and $\mathbf{I}'$, we have

$$\cos\theta = \mathbf{I}\cdot\mathbf{I}'. \tag{3.11}$$

The whole of Euclidean geometry holds in any linear subspace of our function space based on a finite number of vectors. In particular, we have the theorem of Pythagoras, the theorem that any side of a triangle is less than the sum of the other two sides, and the theorem that the greater side of a triangle is opposite the greater angle. All this is well known, but the abstract and general form usually given to the theory of function space may easily obscure the intuitive simplicity of the approach.

Scalar multiplication is distributive, as is easily seen, so that

$$\mathbf{S}_1\cdot(\mathbf{S}_2 + \mathbf{S}_3) = \mathbf{S}_1\cdot\mathbf{S}_2 + \mathbf{S}_1\cdot\mathbf{S}_3. \tag{3.12}$$

As an exercise, let us prove the theorem of Pythagoras. We have, for any two vectors, $\mathbf{S}$ and $\mathbf{S}'$,

$$\begin{aligned} |\mathbf{S} - \mathbf{S}'|^2 &= (\mathbf{S} - \mathbf{S}')\cdot(\mathbf{S} - \mathbf{S}') \\ &= S^2 + S'^2 - 2\mathbf{S}\cdot\mathbf{S}' \\ &= S^2 + S'^2 - 2SS'\cos\theta, \end{aligned} \tag{3.13}$$

where θ is the angle between $\mathbf{S}$ and $\mathbf{S}'$. Putting $\theta=\frac{1}{2}\pi$, we get the theorem of Pythagoras.

4. First approximation with surface stress given. Let us suppose that the surface stress T_i is assigned, satisfying the conditions of statical equilibrium, so that a solution $\mathbf{S}$ exists. We may refer to the solution $\mathbf{S}$ as the *natural* state, to distinguish it from the artificial states which we shall introduce.

Let $\mathbf{S}^*$ be a state which satisfies the equations of equilibrium and the boundary conditions, but not the equations of compatibility. (If the equations of compatibility were satisfied, we would have $\mathbf{S}^*=\mathbf{S}$, the natural state.) Let us evaluate the scalar product $\mathbf{S}\cdot\mathbf{S}^*$. We find

$$\begin{aligned}\mathbf{S}\cdot\mathbf{S}^* = (e\cdot E^*) &= \int e_{ij}E^*_{ij}dv = \int u_{i,j}E^*_{ij}dv \\ &= \int u_iT^*_i dS = \int u_iT_i dS = (e\cdot E) = S^2;\end{aligned} \tag{4.1}$$

in carrying this out, we have used the fact that $\mathbf{S}^*$ satisfies the equations of equilibrium, and also the fact that it satisfies the boundary conditions, so that $T_i^*=T_i$. Since $S^2=\mathbf{S}\cdot\mathbf{S}$, we may write (4.1) in the form

$$\mathbf{S}\cdot(\mathbf{S}-\mathbf{S}^*) = 0. \tag{4.2}$$

This tells us that the vectors $\mathbf{S}$ and $\mathbf{S}-\mathbf{S}^*$ are orthogonal; *this means that the extremity of* $\mathbf{S}$ *is located on a hypersphere* H *having the vector* $\mathbf{S}^*$ *for diameter*; in fact, (4.2) is the equation of this hypersphere, $\mathbf{S}$ being regarded as a current vector. The center of the hypersphere is at $\frac{1}{2}\mathbf{S}^*$, and its radius is $\frac{1}{2}S^*$ (Fig. 2). The equation (4.2) may also be written in the equivalent form

$$|\mathbf{S}-\tfrac{1}{2}\mathbf{S}^*| = \tfrac{1}{2}S^*, \tag{4.3}$$

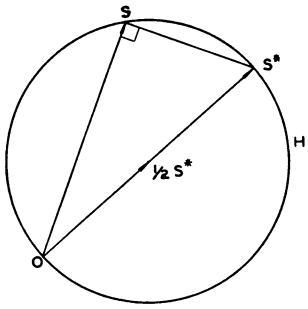

FIG. 2.

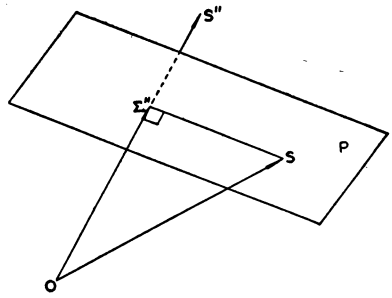

FIG. 3.

which shows up the center and the radius.

We now take another state $\mathbf{S}''$ which satisfies the equations of compatibility, but not in general either the equations of equilibrium or the boundary conditions. In fact, we can build such a vector $\mathbf{S}''$ by choosing an arbitrary set of displacements throughout the body. Let us investigate the scalar product $\mathbf{S}\cdot\mathbf{S}''$, $\mathbf{S}$ being the natural state as before. It is easy to see that

$$\mathbf{S}\cdot\mathbf{S}'' = (e''\cdot E) = (u''\cdot T). \tag{4.4}$$

Although $\mathbf{S}$ is unknown, we do know T_i from the assigned boundary conditions, and so the expression $(u'' \cdot T)$ is calculable. If θ is the angle between $\mathbf{S}$ and $\mathbf{S}''$, we have

$$S \cos \theta = \mathbf{S} \cdot \mathbf{S}''/S'' = (u'' \cdot T)/S''. \tag{4.5}$$

The right hand side is a definite calculable number, and $S \cos \theta$ is the orthogonal projection of $\mathbf{S}$ on $\mathbf{S}''$. It follows that the extremity of the natural vector $\mathbf{S}$ lies on a definite hyperplane P which is orthogonal to $\mathbf{S}''$ (Fig. 3).

Now

$$\mathbf{S}^* \cdot \mathbf{S}'' = (e'' \cdot E^*) = (u'' \cdot T^*) = (u'' \cdot T). \tag{4.6}$$

Comparison with (4.4) reveals a rather remarkable fact: *the extremity of the vector* $\mathbf{S}^*$ *lies in the hyperplane* P.

On account of (4.6), we can write (4.4) in the form

$$\mathbf{S} \cdot \mathbf{S}'' = \mathbf{S}^* \cdot \mathbf{S}''; \tag{4.7}$$

thinking of $\mathbf{S}$ as a current vector, we may regard this as the equation of the hyperplane P.

We have now located the extremity of the natural vector $\mathbf{S}$ on the hypersphere (4.2) and the hyperplane (4.7); we have therefore located it on the *hypercircle* Γ which is the intersection of the hypersphere and the hyperplane. The situation is therefore as shown schematically in Fig. 4. The hypercircle Γ passes through the extremity of $\mathbf{S}^*$ and also through $\mathbf{\Sigma}''$, the foot of the perpendicular dropped from O on P. We have

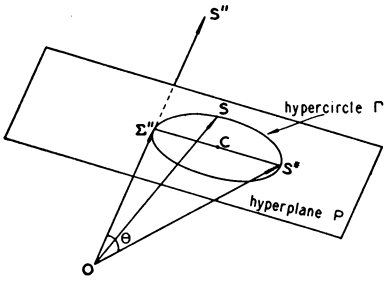

FIG. 4.

$$\mathbf{\Sigma}'' = \mathbf{S}'' \frac{\mathbf{S}^* \cdot \mathbf{S}''}{S''^2} \cdot \tag{4.8}$$

If $\mathbf{I}''$ is a unit vector codirectional with $\mathbf{S}''$ (and therefore also satisfies the equations of compatibility), we have

$$\mathbf{I}'' = \mathbf{S}''/S'',$$

and (4.8) can be written in the form

$$\mathbf{\Sigma}'' = \mathbf{I}''(\mathbf{S}^* \cdot \mathbf{I}''). \tag{4.9}$$

The points $\mathbf{\Sigma}''$ and $\mathbf{S}^*$ are at the ends of a diameter of the hypercircle Γ, because

$$(\mathbf{S} - \mathbf{\Sigma}'') \cdot (\mathbf{S} - \mathbf{S}^*) = 0$$

as follows from (4.2) and (4.7). The center $\mathbf{C}$ of Γ is therefore at

$$\mathbf{C} = \tfrac{1}{2}(\mathbf{S}^* + \mathbf{\Sigma}'') = \tfrac{1}{2}[\mathbf{S}^* + \mathbf{I}''(\mathbf{S}^* \cdot \mathbf{I}'')], \tag{4.10}$$

and the radius R of Γ is given by

$$R = \tfrac{1}{2}(S^{*2} - \mathbf{\Sigma}''^2)^{1/2} = \tfrac{1}{2}[S^{*2} - (\mathbf{S}^* \cdot \mathbf{I}'')^2]^{1/2}. \tag{4.11}$$

Of all points on Γ, the endpoint of $\mathbf{\Sigma}''$ is nearest to O and the endpoint of $\mathbf{S}^*$ most distant from O. Thus, $\Sigma''^2 \leqq S^2 \leqq S^{*2}$ or

$$(\mathbf{S}^* \cdot \mathbf{I}'')^2 \leqq S^2 \leqq S^{*2}. \tag{4.12}$$

This relation gives upper and lower bounds for the strain energy associated with the state $\mathbf{S}$.

We may also write (4.11) in the form

$$R = \tfrac{1}{2} S^* \sin \theta, \tag{4.13}$$

where θ is the angle between $\mathbf{S}^*$ and $\mathbf{S}''$.

Let us now consider the question of getting an approximation to the solution $\mathbf{S}$ of the given boundary value problem. Various definitions of a "good" approximation might be given. We might base a definition on the maximum deviation of displacement or stress from the correct value. We shall, however, use our geometrical picture as the basis of a test for goodness, and say that *an approximate solution* $\bar{\mathbf{S}}$ *is good if the distance (in function space) between* $\bar{\mathbf{S}}$ *and the true or natural solution* $\mathbf{S}$ *is small.* We shall define the *error* ϵ of any approximate solution by

$$\epsilon = |\bar{\mathbf{S}} - \mathbf{S}|.$$

Returning to the discussion of the hypercircle Γ, we ask whether our knowledge based on $\mathbf{S}^*$ and $\mathbf{S}''$ enables us to give an approximate solution. Suppose we accepted $\mathbf{S}^*$ as an approximate solution, how great an error would we commit? It is easy to set an upper bound to this error, because $\mathbf{S}$ lies on Γ, and the point on Γ most distant from $\mathbf{S}^*$ is the point $\mathbf{\Sigma}''$, which is diametrically opposed and at a distance $2R$. Thus we may state that *the error of* $\mathbf{S}^*$ *does not exceed* $2R$, where R is given by (4.11).

Similarly, if Σ'', as in (4.8) or (4.9), is taken as an approximate solution, the error does not exceed $2R$.

We can however do better. *If we take* $\mathbf{C}$, *the center of the hypercircle* Γ, *as approximate solution, the error is precisely* R. Thus, if we change from $\mathbf{S}^*$ to $\mathbf{C}$, we exchange an unknown error which *may* be as great as $2R$ for a *certain* error which is only R.

It should be pointed out that the approximate solution $\mathbf{C}$ will, in general, not satisfy the equations of compatibility, nor the equations of equilibrium, nor the boundary conditions.

The hypercircle Γ, formed for states $\mathbf{S}^*$ and $\mathbf{S}''$, gives us a yardstick by which we can assess any proposed approximate solution $\bar{\mathbf{S}}$. If D_1 is the least distance of $\bar{\mathbf{S}}$ from Γ and D_2 the greatest distance, then the error ϵ of $\bar{\mathbf{S}}$ satisfies

$$D_1 \leqq \epsilon \leqq D_2. \tag{4.14}$$

It is easy to prove by elementary trigonometry that D_1^2 and D_2^2 are given by

$$(\bar{\mathbf{S}} - \mathbf{C})^2 + R^2 \mp 2R\,[(\bar{\mathbf{S}} - \mathbf{C})^2 - (\mathbf{I}'' \cdot (\bar{\mathbf{S}} - \mathbf{C}))^2]^{1/2}. \tag{4.15}$$

5. First approximation with surface displacement given. In the preceding section we have considered the case where the surface stress T_i is given. We shall now take up the other type of problem in which the displacement u_i is given on the bounding surface of the body. The argument runs closely parallel to that of Sec. 4.

First we choose a state $\mathbf{S}^*$ which satisfies the equations of compatibility and the boundary conditions. This is a much easier task than we had in Sec. 4, because

all we have to do is to take a system of displacements which assume assigned values on the bounding surface, and satisfy the general conditions regarding continuity of derivatives.

Next we choose a state $\mathbf{S}''$ which satisfies the equations of equilibrium, but not in general the equations of compatibility, so that satisfaction of the boundary conditions is meaningless, since displacement does not exist. Let $\mathbf{I}''$ be the corresponding unit vector.

We have

$$\mathbf{S}\cdot\mathbf{S}^* = (e^*\cdot E) = (u^*\cdot T) = (u\cdot T) = S^2. \tag{5.1}$$

Accordingly, the endpoint of $\mathbf{S}$ lies on the hypersphere

$$\mathbf{S}\cdot(\mathbf{S} - \mathbf{S}^*) = 0. \tag{5.2}$$

Also,

$$\mathbf{S}\cdot\mathbf{S}'' = (e\cdot E'') = (u\cdot T'') \tag{5.3}$$

which is calculable, and

$$\mathbf{S}^*\cdot\mathbf{S}'' = (e^*\cdot E'') = (u^*\cdot T'') = (u\cdot T''). \tag{5.4}$$

Thus,

$$\mathbf{S}\cdot\mathbf{S}'' = \mathbf{S}^*\cdot\mathbf{S}''. \tag{5.5}$$

Let us compare the equations (5.2) and (5.5) with (4.2) and (4.7), respectively. They are formally the same equations, although $\mathbf{S}^*$ and $\mathbf{S}''$ now have new meanings. Hence, just as we obtained (4.10), (4.11) and (4.12), we are now led to the result: *when the surface displacement is given, the natural vector* $\mathbf{S}$ *has its endpoint on a hypercircle with the center at*

$$\mathbf{C} = \tfrac{1}{2}[\mathbf{S}^* + \mathbf{I}''(\mathbf{S}^*\cdot\mathbf{I}'')], \tag{5.6}$$

and the radius R given by

$$R = \tfrac{1}{2}[S^{*2} - (\mathbf{S}^*\cdot\mathbf{I}'')^2]^{1/2}; \tag{5.7}$$

the strain energy associated with the state $\mathbf{S}$ *is bounded in accordance with*

$$(\mathbf{S}^*\cdot\mathbf{I}'')^2 \leqq S^2 \leqq S^{*2}. \tag{5.8}$$

6. Boundary conditions. Two types of boundary conditions may be regarded as fundamental: (i) stress assigned, (ii) displacement assigned. As we have seen in Secs. 4 and 5, there is remarkable duality between these two types of boundary condition. This duality extends to more general types of boundary condition, which will now be described before we proceed to generalize the ideas developed in the preceding sections.

We consider boundary conditions which involve the following elements at points of the bounding surface:

normal stress $T_{(n)}$; tangential stress $T_{(t)}$;

normal displacement $u_{(n)}$; tangential displacement $u_{(t)}$.

The normal elements are of course scalars and the tangential elements vectors in the tangent plane.

We shall classify points on the bounding surface according to the conditions assigned at them. Thus, a point will be said to be of the class $[u_{(n)}, T_{(t)}]$ if the normal displacement and the tangential stress are assigned at the point. Cases of vanishing elements are important, and for them we shall use the following type of notation: $[u_{(n)}=0, T_{(t)}]$ means that the normal displacement and the tangential stress are assigned, and that the normal displacement is zero.

Two types of boundary condition will now be defined. These may be regarded as generalizations of the simple types of boundary conditions in which stress is assigned all over the surface, or displacement is assigned all over the surface.

STRESS BOUNDARY CONDITIONS (or briefly SBC): The bounding surface may be divided into regions such that all the points in any region belong to one of the following classes:

$$[T_{(n)}, T_{(t)}], \qquad [T_{(n)}, u_{(t)} = 0], \qquad [T_{(t)}, u_{(n)} = 0], \qquad [u_{(n)} = 0, u_{(t)} = 0]. \tag{6.1}$$

Obviously, if the complete surface stress T_i is assigned all over the surface, all points belong to the first class, and so this is a particular case of SBC. Note that in SBC any displacement elements which may be assigned are zero.

DISPLACEMENT BOUNDARY CONDITIONS (or briefly DBC): The bounding surface may be divided into regions such that all the points in any region belong to one of the following classes:

$$[u_{(n)}, u_{(t)}], \qquad [u_{(n)}, T_{(t)} = 0], \qquad [u_{(t)}, T_{(n)} = 0], \qquad [T_{(n)} = 0, T_{(t)} = 0]. \tag{6.2}$$

Note that the case where displacement is assigned all over the surface is a particular case of DBC, and that in DBC any stress elements which may be assigned are zero.

As an illustration, consider a horizontal beam carrying a load on its smooth upper surface and supported on smooth unyielding supports. The points of the surface may be classified as follows:

upper surface under load $[T_{(n)}, T_{(t)}=0]$
surface in contact with supports $[T_{(t)}=0, u_{(n)}=0]$
rest of surface $[T_{(n)}=0, T_{(t)}=0]$

The boundary conditions are evidently SBC, only the first and third classes of (6.1) occurring.

As a second illustration, consider the classical Saint Venant torsion problem for a beam of arbitrary section. The classification is as follows:

ends of beam $[u_{(t)}, T_{(n)}=0]$
sides of beam $[T_{(n)}=0, T_{(t)}=0]$

Here we have DBC, only the third and fourth classes of (6.2) occurring.

7. Associated and complementary differential equations and states. When we have to deal with SBC, we have reason to regard the equations of equilibrium, rather than the equations of compatibility, as particularly associated with the boundary conditions. On the other hand, when we have to deal with DBC, we regard the equations of compatibility as particularly associated with the boundary conditions. The real reason for these associations lies in the theory which follows, but we may note the following facts which will help in remembering which differential equations are associated with which boundary conditions.

In the simplest type of SBC ($E_{ij}n_j$ assigned) and in the equations of equilibrium ($E_{ij,j}=0$), the stress components are involved in a very simple way. On the other

hand, the assignment of DBC would be meaningless unless the equations of compatibility were satisfied, because displacement would not exist.

We shall therefore speak of *associated* and *complementary* differential equations in accordance with the following table:

TABLE I.

Type of boundary condition	Associated differential equations	Complementary differential equations
Stress (SBC)	Equilibrium (2.8)	Compatibility (2.6)
Displacement (DBC)	Compatibility (2.6)	Equilibrium (2.8)

This terminology will permit us to discuss both types of boundary condition with a single argument, which bifurcates only for details.

We note that the solution of an elastic problem involves the satisfaction of

(1) the associated differential equations,

(2) the complementary differential equations,

(3) the boundary conditions.

The following table sets forth schematically notation and terms to be used:

TABLE II.

Symbol for State	Name	Differential Equations Satisfied	Stress Boundary Conditions (SBC)		Displacement Boundary Conditions (DBC)	
			Diff. Equations Satisfied	Boundary Condns. Satisfied	Diff. Equations Satisfied	Boundary Condns. Satisfied
S	Natural state	Associated and Complementary	Equilibrium and Compatibility	$[T_{(n)}, T_{(t)}]$ $[T_{(n)}, u_{(t)}=0]$ $[T_{(t)}, u_{(n)}=0]$ $[u_{(n)}=0, u_{(t)}=0]$	Compatibility and Equilibrium	$[u_{(n)}, u_{(t)}]$ $[u_{(n)}, T_{(t)}=0]$ $[u_{(t)}, T_{(n)}=0]$ $[T_{(n)}=0, T_{(t)}=0]$
S^*	Completely associated state	Associated	Equilibrium	$T^*_{(n)}=T_{(n)}$ where $T_{(n)}$ assigned for S; $T^*_{(t)}=T_{(t)}$ where $T_{(t)}$ assigned for S.	Compatibility	$u^*_{(n)}=u_{(n)}$ where $u_{(n)}$ assigned for S; $u^*_{(t)}=u_{(t)}$ where $u_{(t)}$ assigned for S.
S'_p $(p=1, \cdots, m)$	Homogeneous associated states	Associated	Equilibrium	$T'_{(n)}=0$ where $T_{(n)}$ assigned for S; $T'_{(t)}=0$ where $T_{(t)}$ assigned for S.	Compatibility	$u'_{(n)}=0$ where $u_{(n)}$ assigned for S; $u'_{(t)}=0$ where $u_{(t)}$ assigned for S.
S''_q $(q=1, \cdots, n)$	Complementary states	Complementary	Compatibility	$u''_{(n)}=0$ where $u_{(n)}=0$ for S; $u''_{(t)}=0$ where $u_{(t)}=0$ for S.	Equilibrium	$T''_{(n)}=0$ where $T_{(n)}=0$ for S; $T''_{(t)}=0$ where $T_{(t)}=0$ for S.
I'_p $(p=1, \cdots, m)$	Orthonormal homogeneous associated states	← As for S'_p →				
I''_q $(q=1, \cdots, n)$	Orthonormal complementary states	← As for S''_q →				

We note that **S**, the *natural state*, is the solution of the elastic problem with which we are concerned. It is to be regarded as unknown. All the other states are artificial states which we build up with a view to getting an approximation to **S**; they are to be regarded as known.

In any particular problem, the boundary conditions will be either SBC or DBC. Hence, in any particular problem, we shall have occasion to consult the two columns under SBC, or the two columns under DBC, according to the type of boundary condition.

The table is self-explanatory, but it may assist the reader if we interpret it in the case where the stress T_i is given all over the surface. Here we have SBC, and the following interpretation:

$\mathbf{S}^*$ is any state which satisfies the equations of equilibrium (2.8), and has at each point of the surface the assigned value of T_i.

$\mathbf{S}_p'$ form a set of states, each satisfying the equations of equilibrium and homogeneous boundary conditions $T_i=0$.

$\mathbf{S}_q''$ form a set of states, each satisfying the equations of compatibility, but not subjected to any boundary conditions.

$\mathbf{I}_p'$ form an orthonormal set of states, each being a linear combination of the states $\mathbf{S}_p'$; each therefore satisfies the equations of equilibrium and makes $T_i=0$ on the surface.

$\mathbf{I}_q''$ form an orthonormal set of states, each being a linear combination of the states $\mathbf{S}_q''$; each therefore satisfies the equations of compatibility.

The procedure for forming an orthonormal set by linear combination of linearly independent states is a well known procedure. The general conditions for orthonormality read

$$\begin{aligned} \mathbf{I}_p'\cdot\mathbf{I}_r' &= \delta_{pr} \qquad (p, r = 1, \cdots, m) \\ \mathbf{I}_q''\cdot\mathbf{I}_s'' &= \delta_{qs} \qquad (q, s = 1, \cdots, n). \end{aligned} \tag{7.1}$$

These conditions imply that each vector is of unit magnitude, and that all vectors in each set are mutually orthogonal.

We shall always assume that the states $\mathbf{S}_p'$ are chosen linearly independent; otherwise the number of orthonormal vectors $\mathbf{I}_p'$ would be less than the number of vectors $\mathbf{S}_p'$. The same remarks apply to the states $\mathbf{S}_q''$.

The following statements are easily verified:

I. *Any linear combination of homogeneous associated states is itself a homogeneous associated state.*

II. *If any linear combination of homogeneous associated states is added to a completely associated state, the resulting state*

$$\mathbf{S}^* + \sum_{p=1}^{m} a_p'\mathbf{S}_p' \tag{7.2}$$

is a completely associated state.

III. *Any linear combination of complementary states*

$$\sum_{q=1}^{n} a_q''\mathbf{S}_q'' \tag{7.3}$$

is itself a complementary state.

8. The associated and complementary hyperplanes. If, in (7.2), we assign arbitrary values to the constants a_p', we get a linear subspace or hyperplane of m dimensions. We shall call this the *associated hyperplane*, and denote it by L_m^*. Similarly, we define the *complementary hyperplane* by (7.3), and denote it by L_n''. Fig. 5 shows these hyperplanes for the case $m=1$, $n=1$, when, of course, they are straight lines in function space.

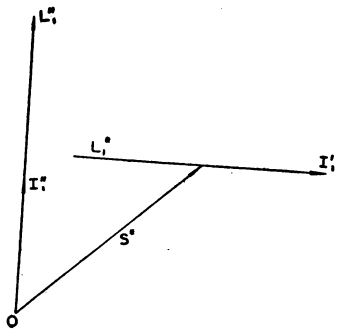

FIG. 5.

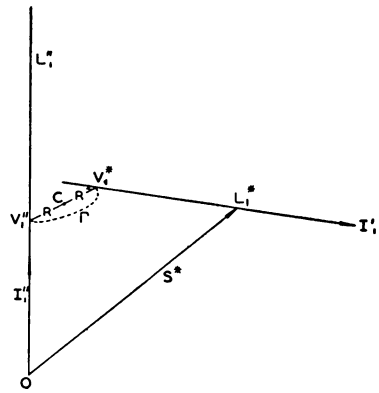

FIG. 6.

The associated hyperplane and the complementary hyperplane play important roles. If they possess a point of intersection, then the state corresponding to that intersection is the natural state **S**; for this state satisfies the associated differential equations, the complementary differential equations, and all the boundary conditions. The object of our approximations is to draw these hyperplanes together, and to locate their closest points, so that we may get a good approximation to **S**.

We shall now prove the following:

IV. *The associated hyperplane L_m^* and the complementary hyperplane L_n'' are mutually orthogonal.*

The hyperplanes are orthogonal if every vector lying wholly in one hyperplane is orthogonal to every vector lying wholly in the other. The general vector lying wholly in L_m^* is

$$\mathbf{S}' = \sum_{p=1}^{m} a_p' \mathbf{I}_p',$$

and the general vector lying wholly in L_n'' is

$$\mathbf{S}'' = \sum_{q=1}^{n} a_q'' \mathbf{I}_q''.$$

The necessary and sufficient condition for the orthogonality of the hyperplanes is

$$\mathbf{S}'.\mathbf{S}'' = 0. \tag{8.1}$$

To prove that this relation holds, we must consider SBC and DBC separately. For

SBC we have, since $\mathbf{S}'$ satisfies the equations of equilibrium and $\mathbf{S}''$ the equations of compatibility,

$$\begin{aligned}\mathbf{S}'.\mathbf{S}'' = (e''.E') &= \int e''_{ij}E'_{ij}dv = \int u''_{i,j}E'_{ij}dv \\ &= \int u''_i T'_i dS - \int u''_i E'_{ij,j}dv \\ &= \int u''_{(n)}T'_{(n)}dS + \int u''_{(t)}T'_{(t)}dS,\end{aligned} \tag{8.2}$$

the last integrand being the scalar product of vectors of displacement and stress lying in the tangent plane of the bounding surface. Let us now use the columns under SBC in Table II, and consider the integrals in the last line of (8.2) taken over the several regions into which the surface is divided. We see that

$$\begin{array}{lll} \text{on} & [T_{(n)}, T_{(t)}] & \text{we have} \quad T'_{(n)} = 0,\ T'_{(t)} = 0, \\ \text{on} & [T_{(n)}, u_{(t)} = 0] & \text{we have} \quad T'_{(n)} = 0,\ u''_{(t)} = 0, \\ \text{on} & [T_{(t)}, u_{(n)} = 0] & \text{we have} \quad T'_{(t)} = 0,\ u''_{(n)} = 0, \\ \text{on} & [u_{(n)} = 0, u_{(t)} = 0] & \text{we have} \quad u''_{(n)} = 0,\ u''_{(t)} = 0. \end{array}$$

Hence the integrals in the last line of (8.2) vanish, and (8.1) is proved; this establishes the orthogonality of the two hyperplanes. The proof for DBC follows the same lines.

From (8.2) we have

$$\mathbf{I}'_p \cdot \mathbf{I}''_q = 0 \qquad (p = 1, 2, \cdots, m;\ q = 1, 2, \cdots, n). \tag{8.3}$$

9. Some basic relations. We shall now prove a number of results.

V. *For the natural state* $\mathbf{S}$ *and any completely associated state* $\mathbf{S}^*$, *the following relation holds*:

$$\mathbf{S}\cdot(\mathbf{S} - \mathbf{S}^*) = 0. \tag{9.1}$$

Hence the extremity of $\mathbf{S}$ *lies on a hypersphere having the vector* $\mathbf{S}^*$ *for diameter*; *the center is at* $\frac{1}{2}\mathbf{S}^*$ *and the radius is* $\frac{1}{2}S^*$.

To prove this, we separate the cases SBC and DBC. For SBC we have

$$\begin{aligned}\mathbf{S}\cdot\mathbf{S} = (u.T) &= \int u_{(n)}T_{(n)}dS + \int u_{(t)}T_{(t)}dS, \\ \mathbf{S}\cdot\mathbf{S}^* = (u.T^*) &= \int u_{(n)}T^*_{(n)}dS + \int u_{(t)}T^*_{(t)}dS.\end{aligned} \tag{9.2}$$

By Table II,

$$\begin{array}{lll} \text{on} & [T_{(n)}, T_{(t)}] & \text{we have} \quad T^*_{(n)} = T_{(n)},\ T^*_{(t)} = T_{(t)} \\ \text{on} & [T_{(n)}, u_{(t)} = 0] & \text{we have} \quad T^*_{(n)} = T_{(n)},\ u_{(t)} = 0, \\ \text{on} & [T_{(t)}, u_{(n)} = 0] & \text{we have} \quad T^*_{(t)} = T_{(t)},\ u_{(n)} = 0, \\ \text{on} & [u_{(n)} = 0, u_{(t)} = 0] & \text{we have} \quad u_{(n)} = 0,\ u_{(t)} = 0. \end{array}$$

On using these values in (9.2), we see that $\mathbf{S}\cdot\mathbf{S}=\mathbf{S}\cdot\mathbf{S}^*$, and so (9.1) is proved for SBC. The proof for DBC is similar. Since (9.1) may be written

$$(\mathbf{S} - \tfrac{1}{2}\mathbf{S}^*)^2 = \tfrac{1}{4}S^{*2}, \tag{9.3}$$

the truth of the statement about the hypersphere is obvious.

VI. *The natural state* $\mathbf{S}$ *is orthogonal to any homogeneous associated state* $\mathbf{S}'$:

$$\mathbf{S}\cdot\mathbf{S}' = 0. \tag{9.4}$$

Hence $\mathbf{S}$ *is orthogonal to the associated hyperplane* L_m^* *and*

$$\mathbf{S}\cdot\mathbf{I}_p' = 0 \qquad (p = 1, 2, \cdots, m). \tag{9.5}$$

Since any homogeneous associated state may be expressed as the difference between two completely associated states (cf. Table II), (9.4) follows if we write down relations of the form (9.1) for the two completely associated states and subtract one from the other.

VII. *The difference between the natural state* $\mathbf{S}$ *and any completely associated state* $\mathbf{S}^*$ *is orthogonal to any complementary state* $\mathbf{S}''$:

$$(\mathbf{S} - \mathbf{S}^*)\cdot\mathbf{S}'' = 0. \tag{9.6}$$

Hence this difference is orthogonal to the complementary hyperplane L_n'' *and*

$$\mathbf{S}\cdot\mathbf{I}_q'' = \mathbf{S}^*\cdot\mathbf{I}_q'' \qquad (q = 1, 2, \cdots, n). \tag{9.7}$$

For SBC we have

$$\begin{aligned} \mathbf{S}\cdot\mathbf{S}'' &= (e''\cdot E) = (u''\cdot T) = \int u_{(n)}'' T_{(n)} dS + \int u_{(t)}'' T_{(t)} dS, \\ \mathbf{S}^*\cdot\mathbf{S}'' &= (e''\cdot E^*) = (u''\cdot T^*) = \int u_{(n)}'' T_{(n)}^* dS + \int u_{(t)}'' T_{(t)}^* dS. \end{aligned} \tag{9.8}$$

On reference to Table II, it is easy to see that these expressions are equal, and so (9.6) is established for SBC. The proof for DBC is similar.

VIII. *The difference between the natural state* $\mathbf{S}$ *and any completely associated state* $\mathbf{S}^*$ *is orthogonal to the difference between* $\mathbf{S}$ *and any complementary state* $\mathbf{S}''$:

$$(\mathbf{S} - \mathbf{S}^*)\cdot(\mathbf{S} - \mathbf{S}'') = 0. \tag{9.9}$$

Thus the natural state lies on every hypersphere which has for diameter the line joining a point of the associated hyperplane to a point of the complementary hyperplane.

Equation (9.9) is an immediate consequence of (9.1) and (9.6). The statement about the hypersphere is verified by writing (9.9) in the form

$$[\mathbf{S} - \tfrac{1}{2}(\mathbf{S}^* + \mathbf{S}'')]^2 = \tfrac{1}{4}(\mathbf{S}^* - \mathbf{S}'')^2. \tag{9.10}$$

10. The hypercircle. In (9.1) we located the natural state $\mathbf{S}$ on a hypersphere, and in (9.5), (9.7) we located it on $m+n$ hyperplanes. This means that the extremity of the natural vector $\mathbf{S}$ lies on a hypercircle Γ, which is the intersection of the hypersphere and the hyperplanes.

The center $\mathbf{C}$ of Γ may be found by starting from the center of the hypersphere

($\frac{1}{2}\mathbf{S}^*$), and proceeding through suitable distances in directions normal to the hyperplanes until we arrive at a point on all the hyperplanes. Thus, we write

$$\mathbf{C} = \tfrac{1}{2}\mathbf{S}^* + \sum_{p=1}^{m} a_p'\mathbf{I}_p' + \sum_{q=1}^{n} a_q''\mathbf{I}_q'', \tag{10.1}$$

where the coefficients are to be determined from the conditions

$$\mathbf{C}\cdot\mathbf{I}_p' = 0 \quad (p = 1, 2, \cdots, m), \qquad \mathbf{C}\cdot\mathbf{I}_q'' = \mathbf{S}^*\cdot\mathbf{I}_q'' \quad (q = 1, 2, \cdots, n). \tag{10.2}$$

By (7.1) and (8.3), we find

$$a_p' = -\tfrac{1}{2}\mathbf{S}^*\cdot\mathbf{I}_p', \qquad a_q'' = \tfrac{1}{2}\mathbf{S}^*\cdot\mathbf{I}_q'', \tag{10.3}$$

and so the center $\mathbf{C}$ of the hypercircle Γ is at

$$\mathbf{C} = \frac{1}{2}\left[\mathbf{S}^* - \sum_{p=1}^{m} \mathbf{I}_p'(\mathbf{S}^*\cdot\mathbf{I}_p') + \sum_{q=1}^{n} \mathbf{I}_q''(\mathbf{S}^*\cdot\mathbf{I}_q'')\right]. \tag{10.4}$$

We note that the magnitude of $\mathbf{C}$ is given by

$$C^2 = \frac{1}{4}\left[S^{*2} - \sum_{p=1}^{m} (\mathbf{S}^*\cdot\mathbf{I}_p')^2 + 3\sum_{q=1}^{n} (\mathbf{S}^*\cdot\mathbf{I}_q'')^2\right]. \tag{10.5}$$

We find the radius R of Γ from

$$\begin{aligned} R^2 &= (\mathbf{S} - \mathbf{C})^2 \\ &= \mathbf{S}\cdot(\mathbf{S} - \mathbf{S}^*) + \mathbf{S}\cdot(\mathbf{S}^* - 2\mathbf{C}) + C^2. \end{aligned} \tag{10.6}$$

The first term vanishes by (9.1), and we easily find from (10.4) and (10.5)

$$R^2 = \frac{1}{4}\left[S^{*2} - \sum_{p=1}^{m} (\mathbf{S}^*\cdot\mathbf{I}_p')^2 - \sum_{q=1}^{n} (\mathbf{S}^*\cdot\mathbf{I}_q'')^2\right]. \tag{10.7}$$

To sum up:

IX. *The natural state* $\mathbf{S}$ *is located on a hypercircle* Γ *with center* $\mathbf{C}$ *as given by* (10.4) *and radius* R *as given by* (10.7). *Using* $\mathbf{X}$ *as a current position vector, we may write the equations of* Γ *in the form*

$$\begin{gathered} \mathbf{X}\cdot(\mathbf{X} - \mathbf{S}^*) = 0, \qquad \mathbf{X}\cdot\mathbf{I}_p' = 0 \quad (p = 1, 2, \cdots, m), \\ \mathbf{X}\cdot\mathbf{I}_q'' = \mathbf{S}^*\cdot\mathbf{I}_q'' \quad (q = 1, 2, \cdots, n). \end{gathered} \tag{10.8}$$

Let us now approach the hypercircle in a different way.

Points on the associated hyperplane L_m^* satisfy some of the conditions imposed on $\mathbf{S}$, and points on the complementary hyperplane L_n'' satisfy others. If the two hyperplanes had a common point, then (as remarked earlier) that common point would be the natural state $\mathbf{S}$. Unless we are particularly lucky in choosing the states $\mathbf{S}^*$, $\mathbf{S}_p'$, $\mathbf{S}_q''$, this common point will not occur, and then it seems appropriate to seek those points, $\mathbf{V}_m^*$ on L_m^* and $\mathbf{V}_n''$ on L_n'' such that the distance $|\mathbf{V}_m^* - \mathbf{V}_n''|$ is as small as possible. We may call these two points the *vertices* of the hyperplanes.

The general point on L_m^* is

$$\mathbf{U}^* = \mathbf{S}^* + \sum_{p=1}^{m} b_p'\mathbf{I}_p', \tag{10.9}$$

and the general point on L_n'' is

$$\mathbf{U}'' = \sum_{q=1}^{n} b_q''\mathbf{I}_q''. \tag{10.10}$$

The square of the distance between these two points is

$$(\mathbf{U}^* - \mathbf{U}'')^2 = \left(\mathbf{S}^* + \sum_{p=1}^{m} b_p'\mathbf{I}_p' - \sum_{q=1}^{n} b_q''\mathbf{I}_q''\right)^2. \tag{10.11}$$

It is easily seen that to minimize this, we must choose

$$b_p' = -\mathbf{S}^*\cdot\mathbf{I}_p', \qquad b_q'' = \mathbf{S}^*\cdot\mathbf{I}_q''. \tag{10.12}$$

Substitution in (10.9) and (10.10) gives for the vertices of the associated and complementary hyperplanes respectively

$$\begin{aligned} \mathbf{V}_m^* &= \mathbf{S}^* - \sum_{p=1}^{m} \mathbf{I}_p'(\mathbf{S}^*\cdot\mathbf{I}_p') \\ \mathbf{V}_n'' &= \sum_{q=1}^{n} \mathbf{I}_q''(\mathbf{S}^*\cdot\mathbf{I}_q''). \end{aligned} \tag{10.13}$$

We see from (10.4) that

$$\mathbf{C} = \tfrac{1}{2}(\mathbf{V}_m^* + \mathbf{V}_n''), \tag{10.14}$$

and from (10.7)

$$R^2 = \tfrac{1}{4}(\mathbf{V}_m^* - \mathbf{V}_n'')^2. \tag{10.15}$$

Thus the points $\mathbf{V}_m^*$ and $\mathbf{V}_n''$ are the extremities of a diameter of the hypercircle Γ.

Any chord of Γ may be represented by a vector $\mathbf{Y}$; it follows from (10.8) that every such chord satisfies $\mathbf{Y}\cdot\mathbf{I}_q''=0$ $(q=1, 2, \cdots n)$, and hence $\mathbf{Y}\cdot\mathbf{V}_n''=0$.

We may sum up as follows:

X. *The vertices of the associated and complementary hyperplanes, as given in* (10.13), *are the extremities of a diameter of the hypercircle* Γ, *described in* IX. *The position vector* $\mathbf{V}_n''$ *of the vertex of the complementary hyperplane is orthogonal to every chord of* Γ (Fig. 6).

The above geometrical statement suggests the inequalities

$$V_n'' \leqq S \leqq V_m^*, \tag{10.16}$$

and a formal proof is easy. Any point $\mathbf{X}$ on Γ satisfies (cf. (10.8))

$$(\mathbf{X} - \mathbf{V}_m^*)\cdot(\mathbf{X} - \mathbf{V}_n'') = 0, \qquad \mathbf{X}\cdot\mathbf{V}_n'' = \mathbf{S}^*\cdot\mathbf{V}_n'', \tag{10.17}$$

and if we seek maxima and minima of X^2 subject to these conditions, it is easy to see that we must have

$$\mathbf{X} = a\mathbf{V}_m^* + b\mathbf{V}_n'', \tag{10.18}$$

where a and b are undetermined multipliers. When we substitute from this in (10.17), the second equation gives b as a linear expression in a, and then the first equation gives a quadratic for a. Thus there are just two solutions, and it is easy to verify from

(10.17) that they are $(a=1,\ b=0)$ and $(a=0,\ b=1)$, i.e. $\mathbf{X}=\mathbf{V}_m^*$ and $\mathbf{X}=\mathbf{V}_n''$. Moreover, by (10.13),

$$V_m^{*2} = S^{*2} - \sum_{p=1}^{m} (\mathbf{S}^*\cdot\mathbf{I}_p')^2,$$
$$V_n''^2 = \sum_{q=1}^{n} (\mathbf{S}^*\cdot\mathbf{I}_q'')^2, \tag{10.19}$$

and so, by (10.7),

$$V_m^{*2} - V_n''^2 = 4R^2 \geqq 0, \qquad V_m^* \geqq V_n''. \tag{10.20}$$

Thus V_m^* is the maximum of X and V_n'' the minimum, when $\mathbf{X}$ is any vector satisfying (10.17). Thus (10.16) is proved. To sum up:

XI. *The vertex of the associated hyperplane is further from the origin than the vertex of the complementary hyperplane, and the distance of the natural state from the origin is intermediate between them, so that*

$$V_n''^2 = \sum_{q=1}^{n} (\mathbf{S}^*\cdot\mathbf{I}_q'')^2 \leqq S^2 \leqq S^{*2} - \sum_{p=1}^{m} (\mathbf{S}^*\cdot\mathbf{I}_p'')^2 = V_m^{*2}. \tag{10.21}$$

These inequalities place lower and upper bounds on the strain energy of the natural state $\mathbf{S}$.

11. Approximations. If $\bar{\mathbf{S}}$ (a set of six stress components expressed as functions of position throughout the body) is suggested as an approximation to the natural state $\mathbf{S}$, the error, as defined in (4.14), is

$$\epsilon = |\bar{\mathbf{S}} - \mathbf{S}|, \tag{11.1}$$

and the squared error is

$$\epsilon^2 = (\bar{\mathbf{S}} - \mathbf{S})^2. \tag{11.2}$$

But we cannot in general calculate this error, because we do not know $\mathbf{S}$.

The hypercircle Γ comes to our assistance. The following is obvious:

XII. *If the center* $\mathbf{C}$ *of the hypercircle* Γ *as given by* (10.4), *namely*

$$\mathbf{C} = \frac{1}{2}\left[\mathbf{S}^* - \sum_{p=1}^{m} \mathbf{I}_p'(\mathbf{S}^*\cdot\mathbf{I}_p') + \sum_{q=1}^{n} \mathbf{I}_q''(\mathbf{S}^*\cdot\mathbf{I}_q'')\right] \tag{11.3}$$

is taken as an approximation to the natural state $\mathbf{S}$, *the error* ϵ *is precisely the radius* R *of* Γ, *so that by* (10.7)

$$4\epsilon^2 = 4R^2 = S^{*2} - \sum_{p=1}^{m} (\mathbf{S}^*\cdot\mathbf{I}_p')^2 - \sum_{q=1}^{n} (\mathbf{S}^*\cdot\mathbf{I}_q'')^2. \tag{11.4}$$

Since the distance between any two points on Γ cannot exceed the length of a diameter, we have the following result:

XIII. *If any state on the hypercircle* Γ *is taken as an approximation to* $\mathbf{S}$, *the error satisfies*

$$\epsilon \leqq 2R, \tag{11.5}$$

where R *is as in* (10.7). *The same inequality is satisfied if we take as an approximation*

either the vertex of the associated hyperplane,

$$\mathbf{V}_m^* = \mathbf{S}^* - \sum_{p=1}^{m} \mathbf{I}_p'(\mathbf{S}^* \cdot \mathbf{I}_p'), \tag{11.6}$$

or the vertex of the complementary hyperplane

$$\mathbf{V}_n'' = \sum_{q=1}^{n} \mathbf{I}_q''(\mathbf{S}^* \cdot \mathbf{I}_q''). \tag{11.7}$$

We note that since $\mathbf{C}$ does not satisfy the equations of compatibility, there is no corresponding displacement. Hence, if we are looking for an approximation for the displacement throughout the body, $\mathbf{V}_n''$ is the best approximation to take for SBC, and $\mathbf{V}_m^*$ the best for DBC.

We may add that if we take as an approximation

$$\bar{\mathbf{S}} = a\mathbf{V}_m^* + b\mathbf{V}_n'' \qquad (a + b = 1,\ a \geqq 0,\ b \geqq 0) \tag{11.8}$$

then ϵ satisfies

$$\begin{aligned} 2Rb \leqq \epsilon \leqq 2Ra \quad &\text{if} \quad b \leqq a, \\ 2Ra \leqq \epsilon \leqq 2Rb \quad &\text{if} \quad a \leqq b. \end{aligned} \tag{11.9}$$

Finally, we shall show how to use the hypercircle Γ to obtain lower and upper bounds for the error of any suggested approximation $\bar{\mathbf{S}}$. It is a question of finding the least and greatest distances of $\bar{\mathbf{S}}$ from the hypercircle with the equations (10.8). That means that we are to find the minimum and maximum values of

$$(\mathbf{X} - \bar{\mathbf{S}})^2$$

when the vector $\mathbf{X}$ is subject to the conditions (10.8).

It is clear that we must have

$$\mathbf{X} - \bar{\mathbf{S}} = a(2\mathbf{X} - \mathbf{S}^*) + \sum_{p=1}^{m} c_p'\mathbf{I}_p' + \sum_{q=1}^{n} c_q''\mathbf{I}_q'', \tag{11.10}$$

where the coefficients are undetermined multipliers. Equivalently,

$$(1 - 2a)\mathbf{X} = \bar{\mathbf{S}} - a\mathbf{S}^* + \sum_{p=1}^{m} c_p'\mathbf{I}_p' + \sum_{q=1}^{n} c_q''\mathbf{I}_q''. \tag{11.11}$$

Hence, by (10.8),

$$\begin{aligned} (1 - 2a)\mathbf{X} \cdot \mathbf{I}_p' &= c_p' + \bar{\mathbf{S}} \cdot \mathbf{I}_p' - a\mathbf{S}^* \cdot \mathbf{I}_p' = 0, \\ (1 - 2a)\mathbf{X} \cdot \mathbf{I}_q'' &= c_q'' + \bar{\mathbf{S}} \cdot \mathbf{I}_q'' - a\mathbf{S}^* \cdot \mathbf{I}_q'' = (1 - 2a)\mathbf{S}^* \cdot \mathbf{I}_q''. \end{aligned} \tag{11.12}$$

If we now substitute in (11.11) the values of c_p', c_q'' given by these equations, we get

$$(1 - 2a)\mathbf{X} = \bar{\mathbf{V}} - 2a\mathbf{C} \tag{11.13}$$

where $\mathbf{C}$ is the center of Γ, as in (11.3), and $\bar{\mathbf{V}}$ is defined as

$$\bar{\mathbf{V}} = \bar{\mathbf{S}} - \sum_{p=1}^{m} \mathbf{I}_p'(\bar{\mathbf{S}} \cdot \mathbf{I}_p') - \sum_{q=1}^{n} \mathbf{I}_q''[(\bar{\mathbf{S}} - \mathbf{S}^*) \cdot \mathbf{I}_q'']. \tag{11.14}$$

We may interpret (11.13) as follows: The points on Γ at minimax distances from $\mathbf{S}$ lie on the diameter of Γ which passes through $\bar{\mathbf{V}}$, and these points divide the line joining $\mathbf{C}$ to $\bar{\mathbf{V}}$ in the ratios $a:-\frac{1}{2}$. We have still to determine a.

The constant a is found by the first of (10.8). On substitution from (11.13), it is found that a satisfies the quadratic equation

$$4a^2\mathbf{C}\cdot(\mathbf{C}-\mathbf{S}^*)+2a[\bar{\mathbf{V}}\cdot(\mathbf{S}^*-\mathbf{C})+\mathbf{C}\,(\mathbf{S}^*-\bar{\mathbf{V}})]+\bar{\mathbf{V}}\cdot(\bar{\mathbf{V}}-\mathbf{S}^*)=0. \tag{11.15}$$

We may state the result as follows:

XIV. *The square error of any state* $\bar{\mathbf{S}}$ *lies between*

$$(\mathbf{X}_1-\bar{\mathbf{S}})^2 \quad \textit{and} \quad (\mathbf{X}_2-\bar{\mathbf{S}})^2,$$

where $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are the vectors given by* (11.13) *when the roots of* (11.15) *are substituted for* a.

Less precise but simpler bounds on the error of $\bar{\mathbf{S}}$ may be established by using, instead of the hypercircle Γ, the hypersphere with same center $\mathbf{C}$ and the same radius R. We first find whether $\bar{\mathbf{S}}$ lies outside or inside this hypersphere by computing

$$(\bar{\mathbf{S}}-\mathbf{C})^2-R^2. \tag{11.16}$$

If this is positive, $\bar{\mathbf{S}}$ is outside, if negative, inside. If $\bar{\mathbf{S}}$ is outside, we have

$$|\bar{\mathbf{S}}-\mathbf{C}|-R\leqq\epsilon\leqq|\bar{\mathbf{S}}-\mathbf{C}|+R. \tag{11.17}$$

If $\bar{\mathbf{S}}$ is inside, we have

$$R-|\bar{\mathbf{S}}-\mathbf{C}|\leqq\epsilon\leqq R+|\bar{\mathbf{S}}-\mathbf{C}|. \tag{11.18}$$

12. Example. As an illustration of the preceding relations, we consider the torsion of a prism of square cross section. We choose the centroid of one end section as the origin of coordinates and let the x_3 axis coincide with the axis of the prism. To avoid carrying along unessential constants, we consider the prism bounded by the planes $x_1=\pm1$, $x_2=\pm1$, $x_3=0$, $x_3=1$ and assume that the shear modulus G and the angle of twist ϑ equal unity. On the end sections of the prism we have $T_3=0$, and the tangential displacements u_α $(\alpha=1,2)$ are given, viz.

$$u_\alpha=0 \qquad \text{on} \quad x_3=0,$$
$$u_\alpha=-\epsilon_{\alpha\beta}x_\beta \quad \text{on} \quad x_3=1,$$

where $\epsilon_{11}=\epsilon_{22}=0$, $\epsilon_{12}=-\epsilon_{21}=1$. On the lateral faces of the prism $T_i=0$ $(i=1,2,3)$. We thus have a case of DBC.

A suitable completely associated state is defined by

$$u_\alpha=-\epsilon_{\alpha\beta}x_\beta x_3, \qquad u_3=0, \tag{12.1}$$

or, with the shear modulus $G=1$,

$$F_{\alpha3}=-\epsilon_{\alpha\beta}x_\beta, \qquad E_{\alpha\beta}=E_{33}=0. \tag{S*}$$

We have

$$S^{*2}=\int_{-1}^{1}\int_{-1}^{1}E_{\alpha3}E_{\alpha3}dx_1dx_2=\int_{-1}^{1}\int_{-1}^{1}(x_1^2+x_2^2)dx_1dx_2=\frac{8}{3}\cdot \tag{12.2}$$

A homogeneous associated state is defined by

$$u_\alpha = 0, \qquad u_3 = \varphi(x_1, x_2). \tag{12.3}$$

The corresponding vector $\mathbf{S}'$ is

$$E_{\alpha 3} = \varphi_{,\alpha} \qquad E_{\alpha\beta} = E_{33} = 0, \tag{S'}$$

and we have

$$S'^2 = \int_{-1}^{1}\int_{-1}^{1} \varphi_{,\alpha}\varphi_{,\alpha} dx_1 dx_2. \tag{12.4}$$

As is easily seen, the "warping function" φ must assume values of equal magnitude but opposite signs in any two points which are symmetrically situated with respect to any one of the lines $x_1=0$, $x_2=0$, $x_1=x_2$, $x_1=-x_2$. A function of this type is

$$\varphi_p = x_1^{2p-1} x_2^{2p-1}(x_1^2 - x_2^2), \tag{12.5}$$

leading to the vector

$$\begin{aligned} E_{13} &= x_1^{2p-2} x_2^{2p-1}[(2p+1)x_1^2 + (2p-1)x_2^2], \\ E_{23} &= x_1^{2p-1} x_2^{2p-2}[(2p-1)x_1^2 + (2p+1)x_2^2]. \end{aligned} \tag{$\mathbf{S}_p'$}$$

We find

$$\mathbf{S}_p' \cdot \mathbf{S}_r' = \frac{2^5\{4[(p+r)^2 + 2pr - (p+r)] - 7\}}{[2(p+r)-3][2(p+r)-1][2(p+r)+1][2(p+r)+3]} \tag{12.6}$$

and

$$\mathbf{S}^* \cdot \mathbf{S}_p' = -\frac{2^4}{(2p+1)(2p+3)}. \tag{12.7}$$

A complementary state can be derived from a stress function $\psi(x_1, x_2)$ which vanishes along the contour of the cross section. The corresponding vector $\mathbf{S}''$ is

$$E_{\alpha 3} = \epsilon_{\alpha\beta}\psi_{,\beta}, \qquad E_{\alpha\beta} = E_{33} = 0, \tag{S''}$$

and we have

$$S''^2 = \int_{-1}^{1}\int_{-1}^{1} \psi_{,\alpha}\psi_{,\alpha} dx_1 dx_2. \tag{12.8}$$

As is easily seen, the stress function must be symmetric with respect to x_1 and x_2 and even in both these variables. Such a function which vanishes on the boundary of the cross section is

$$\psi_q = (x_1^{2q} - 1)(x_2^{2q} - 1), \tag{12.9}$$

leading to the vector

$$\begin{aligned} E_{13} &= 2q(x_1^{2q} - 1)x_2^{2q-1}, \\ E_{23} &= -2qx_1^{2q-1}(x_2^{2q} - 1). \end{aligned} \tag{$\mathbf{S}_q''$}$$

We find

$$\mathbf{S}_q''\cdot\mathbf{S}_s''=\frac{2^8q^2s^2(q+s+1)}{(2q+1)(2s+1)[2(q+s)-1][2(q+s)+1]} \tag{12.10}$$

and

$$\mathbf{S}^*\cdot\mathbf{S}_q''=\frac{2^5q^2}{(2q+1)^2}\cdot \tag{12.11}$$

We now proceed to determine the vertices $\mathbf{V}_m^*$ and $\mathbf{V}_m''$ in accordance with (10.13). We have

$$\mathbf{I}_1'(\mathbf{S}^*\cdot\mathbf{I}_1')=\mathbf{S}_1'(\mathbf{S}^*\cdot\mathbf{S}_1')/S_1'^2, \tag{12.12}$$

where

$$\mathbf{S}^*\cdot\mathbf{S}_1'=-\frac{2^4}{3\cdot 5},\qquad S_1'^2=\frac{2^5\cdot 3}{5\cdot 7} \tag{12.13}$$

by Eqs. (12.7) and (12.6). Thus,

$$\mathbf{I}_1'(\mathbf{S}^*\cdot\mathbf{I}_1')=-\frac{7}{2\cdot 3^2}\mathbf{S}_1',$$

and

$$\mathbf{V}_1^*=\mathbf{S}^*-\mathbf{I}_1'(\mathbf{S}^*\cdot\mathbf{I}_1')=\mathbf{S}^*+\frac{7}{2\cdot 3^2}\mathbf{S}_1'. \tag{$\mathbf{V}_1^*$}$$

In a similar manner, we find

$$\mathbf{V}_1''=\mathbf{I}_1''(\mathbf{S}^*\cdot\mathbf{I}_1'')=\frac{5}{2^3}\mathbf{S}_1''. \tag{$\mathbf{V}_1''$}$$

To obtain $\mathbf{V}_2^*$ and $\mathbf{V}_2''$, we must first orthogonalize $\mathbf{S}_2'$ and $\mathbf{S}_2''$ with respect to $\mathbf{S}_1'$ and $\mathbf{S}_1''$, respectively. As is easily seen the vector

$$\tilde{\mathbf{S}}_2'=\mathbf{S}_2'-\mathbf{S}_1'(\mathbf{S}_1'\cdot\mathbf{S}_2')/S_1'^2 \tag{12.14}$$

is orthogonal to $\mathbf{S}_1'$. Using (12.6), we find

$$\tilde{\mathbf{S}}_2'=\mathbf{S}_2'-\frac{11}{3^3}\mathbf{S}_1' \tag{$\tilde{\mathbf{S}}_2'$}$$

and

$$\tilde{S}_2'^2=S_2'^2-(\mathbf{S}_1'\cdot\mathbf{S}_2')^2/S_1'^2=\frac{2^{12}}{3^5\cdot 7\cdot 11}\cdot \tag{12.15}$$

In a similar manner, a vector orthogonal to $\mathbf{S}_1''$ can be derived from $\mathbf{S}_2''$. One finds

$$\tilde{\mathbf{S}}_2''=\mathbf{S}_2''-\mathbf{S}_1''(\mathbf{S}_1''\cdot\mathbf{S}_2'')/S_1''^2=\mathbf{S}_2''-\frac{2^4\cdot 3}{5\cdot 7}\tilde{\mathbf{S}}_1'' \tag{$\tilde{\mathbf{S}}_2''$}$$

and

$$\tilde{S}_2''^2 = S_2''^2 - (\mathbf{S}_1''\cdot\mathbf{S}_2'')^2/S_1''^2 = \frac{2^{12}\cdot 31}{3^2\cdot 5^3\cdot 7^2}\cdot \tag{12.16}$$

We now determine

$$\mathbf{I}_2'(\mathbf{S}^*\cdot\mathbf{I}_2') = \tilde{\mathbf{S}}_2'(\mathbf{S}^*\cdot\tilde{\mathbf{S}}_2')/\tilde{S}_2'^2.$$

With the use of Eqs. (12.6), (12.7) and (12.15), we find

$$\mathbf{I}_2'(\mathbf{S}^*\cdot\mathbf{I}_2') = -\frac{3\cdot 11}{2^6\cdot 5}\tilde{\mathbf{S}}_2'. \tag{12.17}$$

Similarly,

$$\mathbf{I}_2''(\mathbf{S}^*\cdot\mathbf{I}_2'') = \frac{3\cdot 5\cdot 7}{2^5\cdot 31}\tilde{\mathbf{S}}_2''. \tag{12.18}$$

Thus, by Eqs. (10.13),

$$\mathbf{V}_2^* = \mathbf{V}_1^* + \frac{3.11}{2^6\cdot 5}\tilde{\mathbf{S}}_2' \tag{$\mathbf{V}_2^*$}$$

and

$$\mathbf{V}_2'' = \mathbf{V}_1'' + \frac{3\cdot 5\cdot 7}{2^5\cdot 31}\tilde{\mathbf{S}}_2''. \tag{$\mathbf{V}_2''$}$$

In connection with the evaluation of (10.21), we need the values of $(\mathbf{S}^*\cdot\mathbf{I}_p')^2$ and $(\mathbf{S}^*\cdot\mathbf{I}_q'')^2$. One finds

$$(\mathbf{S}^*\cdot\mathbf{I}_1')^2 = (\mathbf{S}^*\cdot\mathbf{S}_1')^2/S_1'^2 = \frac{2^3\cdot 7}{3^3\cdot 5} > 0.4148$$

$$(\mathbf{S}^*\cdot\mathbf{I}_2')^2 = \frac{11}{3^3\cdot 5^2\cdot 7} > 0.0023,$$

$$(\mathbf{S}^*\cdot\mathbf{I}_1'')^2 = \frac{2^2\cdot 5}{3^2} > 2.2222,$$

$$(\mathbf{S}^*\cdot\mathbf{I}_2'')^2 = \frac{2^2}{5\cdot 31} > 0.0258.$$

Now, S^2 here equals the product of the torque and the angle of twist. Since the latter has been taken as unity, S^2 represents the torque per unit angle of twist or the torsional stiffness. Equation (10.21) thus furnishes the following bounds for the torsional stiffness:*

a) for $m=n=1$: $2.2222 < S^2 < \frac{8}{3} - 0.4148 < 2.2519$.

b) for $m=n=2$: $2.2480 < S^2 < 2.2496$;

the exact value, to four decimal places, is $S^2 = 2.2492$.

* C. Weber [Z. angew. Math. Mech. **11**, 244–245 (1931)] used the variational principles of elasticity to obtain lower and upper bounds for the torsional stiffness of a square hollow prism. His basic formula corresponds essentially to the form which (10.21) assumes in the case of torsion for $m=n=1$.

Next, let us consider the maximum shearing stress, i.e. the stress component E_{23} at the point $x_1=1$, $x_2=0$. For the various states considered above the value of this stress component is given in the following table:

State	$\mathbf{S}^*$	$\mathbf{S}_1'$	$\mathbf{S}_2'$	$\mathbf{S}_1''$	$\mathbf{S}_2''$	$\tilde{\mathbf{S}}_2'$	$\tilde{\mathbf{S}}_2''$
$E_{23}(1,0)$	1	1	0	2	4	$-11/3^3$	$2^2\cdot 11/(5\cdot 7)$

Accordingly, the stress values corresponding to the vertices of the associated and complementary hyperplanes are:

$$1+\frac{7}{2\cdot 3^2}=\frac{25}{18}=1.389 \quad \text{for} \quad \mathbf{V}_1^*,$$

$$\frac{25}{18}-\frac{3\cdot 11}{2^6\cdot 5}\cdot\frac{11}{3^3}=\frac{431}{320}=1.347 \quad \text{for} \quad \mathbf{V}_2^*,$$

$$\frac{5}{2^3}\cdot 2=\frac{5}{4}=1.250 \quad \text{for} \quad \mathbf{V}_1'', \quad \text{and}$$

$$\frac{5}{4}+\frac{3\cdot 5\cdot 7}{2^5\cdot 31}\cdot\frac{2^2\cdot 11}{5\cdot 7}=\frac{343}{248}=1.383 \quad \text{for} \quad \mathbf{V}_2''.$$

The stress values corresponding to

$$\mathbf{C}_1=\tfrac{1}{2}(\mathbf{V}_1^*+\mathbf{V}_1'')$$

and

$$\mathbf{C}_2=\tfrac{1}{2}(\mathbf{V}_2^*+\mathbf{V}_2'')$$

are therefore 1.319 and 1.365, respectively; the exact value, to three decimal places, is 1.352.

13. Comparison with other work. S. Bergman* has given an exact solution of the general boundary value problem for an isotropic elastic body, the surface stress or the surface displacement being assigned. In the vector notation of the present paper, his solution may be written

$$\mathbf{S}=\sum_{p=1}^{\infty}\mathbf{Z}_p(\mathbf{Z}_p\cdot\mathbf{S}),$$

where $\mathbf{Z}_p$ form an infinite orthonormal sequence of states, each satisfying the equations of equilibrium and compatibility. These states are determined explicitly. The scalar product $\mathbf{Z}_p\cdot\mathbf{S}$ can be computed when either stress or displacement is given all over the bounding surface. A point of Bergman's function space is a set of three functions (components of displacement), whereas a point of our function space is a set of six functions (components of stress), since we relax the equations of compatibility. But a more important difference between Bergman's work and the present theory lies in the fact that he was primarily concerned with an exact solution, whereas we are interested in comparatively simple approximate solutions in which the error can be easily computed.

It is interesting to discuss, along the lines of the present paper, the results given in

* Mathematische Annalen, **98**, 248–263 (1928).

Courant-Hilbert, "Methoden der mathematischen Physik," vol. 1, J. Springer, Berlin, 1931, pp. 228–230. We shall refer to this work as CH. We shall show how their results can be obtained without recourse to the calculus of variations. Then we shall strengthen these results, and express them as a single inequality. Finally, for the boundary conditions considered in CH, we shall show that the natural state lies on a certain hypersphere.

The boundary conditions used by CH are different from those of the present paper. In their simpler type of boundary condition, they divide the boundary into two parts, Γ_1 and Γ_2, with T_i assigned on Γ_1, and u_i assigned on Γ_2. Body forces are included in CH, but we shall here omit them.

Unless either Γ_1 or Γ_2 disappears, these boundary conditions are neither SBC nor DBC. We shall therefore abandon our previous notation, and use the following symbols:

$\mathbf{S}$ = natural state.

$\mathbf{S}'$ = a state satisfying the equations of equilibrium and making $T_i' = T_i$ on Γ_1.

$\mathbf{S}''$ = a state satisfying the equations of compatibility and making $u_i'' = u_i$ on Γ_2.

Now

$$\int (e_{ij} - e_{ij}')(E_{ij} - E_{ij}')dv \geqq 0, \tag{13.1}$$

or

$$S^2 + S'^2 - 2(e \cdot E') \geqq 0, \tag{13.2}$$

or, in an obvious extension of the notation of (2.10),

$$S^2 + S'^2 \geqq 2(u \cdot T')_1 + 2(u \cdot T')_2, \tag{13.3}$$

where the subscripts 1 and 2 refer to integration over Γ_1 and Γ_2 respectively. Also

$$S^2 = (u \cdot T)_1 + (u \cdot T)_2. \tag{13.4}$$

Subtracting twice (13.4) from (13.3) and using the fact that $T_i' = T_i$ on Γ_1, we get

$$S'^2 - S^2 \geqq 2(u \cdot T')_2 - 2(u \cdot T)_2, \tag{13.5}$$

or

$$\tfrac{1}{2}S^2 - (u \cdot T)_2 \leqq \tfrac{1}{2}S'^2 - (u \cdot T')_2. \tag{13.6}$$

This is the same as the principle II of CH. By a similar argument, we obtain

$$\tfrac{1}{2}S^2 - (u \cdot T)_1 \leqq \tfrac{1}{2}S''^2 - (u'' \cdot T)_1, \tag{13.7}$$

which is the same as the principle I of CH.

If we add (13.6) and (13.7), and use (13.4), we get

$$\tfrac{1}{2}S'^2 + \tfrac{1}{2}S''^2 - (u \cdot T')_2 - (u'' \cdot T)_1 \geqq 0, \tag{13.8}$$

a result which is of no particular interest.

Now let us strengthen the preceding results, using (3.9), which yields

$$\mathbf{S} \cdot \mathbf{S}' \leqq SS', \qquad \mathbf{S} \cdot \mathbf{S}'' \leqq SS''. \tag{13.9}$$

The first of these gives, instead of (13.3), the stronger inequality

$$SS' \geqq (u \cdot T')_1 + (u \cdot T')_2 = (u \cdot T)_1 + (u \cdot T')_2. \tag{13.10}$$

Subtracting (13.4), we get

$$SS' - S^2 \geqq (u\cdot T')_2 - (u\cdot T)_2, \tag{13.11}$$

or, instead of (13.6),

$$S^2 - (u\cdot T)_2 \leqq SS' - (u\cdot T')_2. \tag{13.12}$$

Similarly, instead of (13.7), we get

$$S^2 - (u\cdot T)_1 \leqq SS'' - (u''\cdot T)_1. \tag{13.13}$$

When we added (13.6) and (13.7) we got the trivial inequality (13.8), which tells us nothing about the natural state. But if we add (13.12) and (13.13), and use (13.4), we get

$$S^2 \leqq S(S' + S'') - (u\cdot T')_2 - (u''\cdot T)_1, \tag{13.14}$$

or

$$[S - \tfrac{1}{2}(S' + S'')]^2 \leqq \tfrac{1}{4}(S' + S'')^2 - (u\cdot T')_2 - (u''\cdot T)_1. \tag{13.15}$$

This single inequality bounds the strain energy of the natural state below and above, for all quantities except S are calculable, and we have

$$\tfrac{1}{2}(S' + S'') - R \leqq S \leqq \tfrac{1}{2}(S' + S'') + R, \tag{13.16}$$

where $R>0$ and

$$R^2 = \tfrac{1}{4}(S' + S'')^2 - (u\cdot T')_2 - (u''\cdot T)_1. \tag{13.17}$$

But we can do better than this. We have

$$\begin{aligned} S^2 &= (u\cdot T)_1 + (u\cdot T)_2, \\ \mathbf{S}\cdot\mathbf{S}' &= (u\cdot T')_1 + (u\cdot T')_2, \\ \mathbf{S}\cdot\mathbf{S}'' &= (u''\cdot T)_1 + (u''\cdot T)_2. \end{aligned} \tag{13.18}$$

Since $(u\cdot T)_1=(u\cdot T')_1$, $(u\cdot T)_2=(u''\cdot T)_2$, we get

$$S^2 - \mathbf{S}\cdot(\mathbf{S}' + \mathbf{S}'') = -(u\cdot T')_2 - (u''\cdot T)_1, \tag{13.19}$$

or

$$[\mathbf{S} - \tfrac{1}{2}(\mathbf{S}' + \mathbf{S}'')]^2 = R^2, \tag{13.20}$$

where $R^2=\frac{1}{4}(\mathbf{S}'+\mathbf{S}'')^2-(u\cdot T')_2-(u''\cdot T)_1$. *Thus the natural state lies on a hypersphere with center at* $\frac{1}{2}(\mathbf{S}'+\mathbf{S}'')$ *and radius* R.

As remarked in a footnote to CH, their results apply to more general boundary conditions. These more general boundary conditions include our SBC and DBC as special cases, and may be described as follows: The boundary surface may be divided into four regions:

$$\begin{aligned} &\Gamma_{11}: \quad T_{(n)},\ T_{(t)} \text{ assigned.} \\ &\Gamma_{12}: \quad T_{(n)},\ u_{(t)} \text{ assigned.} \\ &\Gamma_{21}: \quad u_{(n)},\ T_{(t)} \text{ assigned.} \\ &\Gamma_{22}: \quad u_{(n)},\ u_{(t)} \text{ assigned.} \end{aligned} \tag{13.21}$$

We define

$\mathbf{S}$ = natural state.

$\mathbf{S}'$ = state satisfying equations of equilibrium and all boundary conditions on $T_{(n)}$ and $T_{(t)}$.

$\mathbf{S}''$ = state satisfying equations of compatibility and all boundary conditions on $u_{(n)}$ and $u_{(t)}$.

Then

$$S^2 = (u_{(n)}T_{(n)})_{11} + (u_{(t)}T_{(t)})_{11} + (u_{(n)}T_{(n)})_{12} + (u_{(t)}T_{(t)})_{12} + (u_{(n)}T_{(n)})_{21} + (u_{(t)}T_{(t)})_{21} + (u_{(n)}T_{(n)})_{22} + (u_{(t)}T_{(t)})_{22}, \tag{13.22}$$

$$\mathbf{S}\cdot\mathbf{S}' = \text{same expression as in (13.22) but with } T \text{ changed to } T', \tag{13.23}$$

$$\mathbf{S}\cdot\mathbf{S}'' = \text{same expression as in (13.22) but with } u \text{ changed to } u''. \tag{13.24}$$

Now we have

$$\begin{aligned}
(u_{(n)}T'_{(n)})_{11} &= (u_{(n)}T_{(n)})_{11}, & (u_{(t)}T'_{(t)})_{11} &= (u_{(t)}T_{(t)})_{11}\\
(u_{(n)}T'_{(n)})_{12} &= (u_{(n)}T_{(n)})_{12}, & (u_{(t)}T'_{(t)})_{21} &= (u_{(t)}T_{(t)})_{21}\\
(u''_{(n)}T_{(n)})_{21} &= (u_{(n)}T_{(n)})_{21}, & (u''_{(t)}T_{(t)})_{12} &= (u_{(t)}T_{(t)})_{12}\\
(u''_{(n)}T_{(n)})_{22} &= (u_{(n)}T_{(n)})_{22}, & (u''_{(t)}T_{(t)})_{22} &= (u_{(t)}T_{(t)})_{22}.
\end{aligned}$$

So we obtain from (13.22), (13.23), (13.24),

$$\begin{aligned}
S^2 - \mathbf{S}\cdot(\mathbf{S}' + \mathbf{S}'') = &- (u''\cdot T'')_{11} - (u\cdot T')_{22} - (u_{(n)}T'_{(n)})_{21} - (u_{(t)}T'_{(t)})_{12}\\
&- (u''_{(n)}T_{(n)})_{12} - (u''_{(t)}T_{(t)})_{21}.
\end{aligned} \tag{13.26}$$

The right hand side is computable. The result may be written

$$[\mathbf{S} - \tfrac{1}{2}(\mathbf{S}' + \mathbf{S}'')]^2 = R^2, \tag{13.27}$$

where

$$\begin{aligned}
R^2 = &\tfrac{1}{4}(\mathbf{S}' + \mathbf{S}'')^2 - (u''\cdot T'')_{11} - (u\cdot T')_{22} - (u_{(n)}T'_{(n)})_{21}\\
&- (u_{(t)}T'_{(t)})_{12} - (u_{(n)}T''_{(n)})_{12} - (u_{(t)}T''_{(t)})_{22}.
\end{aligned} \tag{13.28}$$

We may sum up as follows:

XIV. *With the boundary conditions as described in* (13.21), *the natural state lies on a hypersphere with center at*

$$\mathbf{C} = \tfrac{1}{2}(\mathbf{S}' + \mathbf{S}'') \tag{13.29}$$

and radius R as given by (13.28), *where* $\mathbf{S}'$ *is any state satisfying the equations of equilibrium and all boundary conditions on stress, and* $\mathbf{S}''$ *is any state satisfying the equations of compatibility and all boundary conditions on displacement.*

Added November 18, 1946: The authors are indebted to Professor A. Weinstein for showing a connection between the present method and the method given by E. Trefftz.* Trefftz dealt with the Dirichlet problem in a plane (to determine a harmonic function with prescribed boundary values). The Ritz method gives an upper bound to the Dirichlet integral of the solution u, namely, the Dirichlet integral of a function v satisfying the boundary conditions. Trefftz supplied a lower bound by

* Proc. 2nd International Congress for Applied Mechanics, Zurich, 1926, pp. 131–137.

taking a linear combination w of harmonic functions, and choosing the coefficients so that the Dirichlet integral of the difference $(u-w)$ is a minimum. It should be pointed out that, although the torsion problem is reducible to the Dirichlet problem, the torsional rigidity is not given by the Dirichlet integral of a harmonic function. Hence, in applying his method to the estimation of torsional rigidity, Trefftz had to proceed in an indirect way.

Let us now consider the Dirichlet problem in the notation of the present paper. Let a vector in function space correspond to a function with continuous first derivatives in a plane region R, bounded by a curve B. Let $\mathbf{S}^*$ be a *completely associated vector*; this means a function u^* satisfying the boundary conditions. Let $\mathbf{S}''$ be a *complementary vector;* this means a *harmonic function* u''. Let $\mathbf{S}$ be the solution-function u. Let the scalar product of two vectors be defined by the Dirichlet integral

$$\mathbf{S}_1\cdot\mathbf{S}_2 = \iint_R\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y}\right)dxdy.$$

Then

$$\mathbf{S}^*\cdot\mathbf{S}'' = \iint_R\left(\frac{\partial u^*}{\partial x}\frac{\partial u''}{\partial x}+\frac{\partial u^*}{\partial y}\frac{\partial u''}{\partial y}\right)dxdy = \int_B u^*\frac{\partial u''}{\partial n}ds,$$

since u'' is harmonic. Since $\mathbf{S}$ is itself a completely associated vector, we may write $\mathbf{S}$ for $\mathbf{S}^*$ and u for u^*. But $u^*=u$ on B, and so we have

$$\mathbf{S}^*\cdot\mathbf{S}'' = \mathbf{S}\cdot\mathbf{S}'' \quad\text{or}\quad \mathbf{S}''\cdot(\mathbf{S}^*-\mathbf{S}) = 0. \tag{1}$$

But $\mathbf{S}$ is also a complementary vector, and so we may write $\mathbf{S}$ instead of $\mathbf{S}''$. This gives

$$\mathbf{S}^*\cdot\mathbf{S} = \mathbf{S}^2 \quad\text{or}\quad \mathbf{S}\cdot(\mathbf{S}^*-\mathbf{S}) = 0. \tag{2}$$

This last equation is the Ritz equation (4) of Trefftz' paper.

The equation (1) is not that of Trefftz. However, it tells us that $\mathbf{S}^*-\mathbf{S}$ is perpendicular to $\mathbf{S}''$. From elementary solid geometry it follows that, if we drop perpendiculars from the extremities of $\mathbf{S}^*$ and $\mathbf{S}$ on the line of $\mathbf{S}''$, these perpendiculars have a common foot, say with position vector $\mathbf{T}''=k\mathbf{S}''$. We have then

$$\mathbf{T}''\cdot(\mathbf{S}-\mathbf{T}'') = 0. \tag{3}$$

This is identical with Trefftz' equation (10), in which the right side is actually zero. Obviously, $\mathbf{T}''$ is that vector of form $k\mathbf{S}''$ which minimizes $(\mathbf{S}-\mathbf{T}'')^2$; thus $\mathbf{T}''$ corresponds to that harmonic function of form ku'' which minimizes the Dirichlet integral of $u-ku''$, where u is the solution.

To apply the above ideas to the torsion problem, we recall that the torsional rigidity (for $G=1$) is

$$\iint_R\left(x^2+y^2-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)dxdy,$$

where u is a harmonic function, taking the value $\frac{1}{2}(x^2+y^2)$ on the boundary. If we write $\mathbf{S}_0$ for the vector $\frac{1}{2}(x^2+y^2)$, then the torsional rigidity is

$$\mathbf{S}_0^2 - \mathbf{S}_0\cdot\mathbf{S}. \tag{4}$$

By virtue of (1) and (2) we can build up the theory of the hypercircle, as we did in the general elastic problem. We can introduce orthonormal homogeneous associated vectors $\mathbf{I}_p'$ $(p=1, \cdots, m)$ with $u'=0$ on B, and orthonormal complementary vectors $\mathbf{I}_q''$ $(q=1, \cdots, n)$, each harmonic. The center $\mathbf{C}$ of the hypercircle is then given by (10.4) and its radius by (10.7). We have

$$\begin{aligned} (\mathbf{S}-\mathbf{C})\cdot\mathbf{I}_p' &= 0 \qquad (p = 1, \cdots, m) \\ (\mathbf{S}-\mathbf{C})\cdot\mathbf{I}_q'' &= 0 \qquad (q = 1, \cdots, n). \end{aligned} \tag{5}$$

Thus we may write

$$\mathbf{S} = \mathbf{C} + R\mathbf{J}, \tag{6}$$

where $\mathbf{J}$ is a unit vector satisfying

$$\begin{aligned} \mathbf{J}\cdot\mathbf{I}_p' &= 0 \qquad (p = 1, \cdots, m) \\ \mathbf{J}\cdot\mathbf{I}_q'' &= 0 \qquad (q = 1, \cdots, n). \end{aligned} \tag{7}$$

Thus

$$\mathbf{S}_0\cdot\mathbf{S} = \mathbf{S}_0\cdot\mathbf{C} + R\mathbf{S}_0\cdot\mathbf{J}. \tag{8}$$

The maximum and minimum of this expression, as $\mathbf{S}$ ranges over the hypercircle, are to be found from

$$\mathbf{S}_0 = \sum_{p=1}^{m} c_p'\mathbf{I}_p' + \sum_{q=1}^{n} c_q''\mathbf{I}_q'' + c\mathbf{J}, \tag{9}$$

where the c's are undetermined scalars. Since $\mathbf{I}_p'\cdot\mathbf{I}_q''=0$, we easily find

$$\begin{aligned} c_p' &= \mathbf{S}_0\cdot\mathbf{I}_p' \qquad (p = 1, \cdots, m) \\ c_q'' &= \mathbf{S}_0\cdot\mathbf{I}_q'' \qquad (q = 1, \cdots, n) \\ c^2 &= \mathbf{S}_0^2 - \sum_{p=1}^{m} (\mathbf{S}_0\cdot\mathbf{I}_p')^2 - \sum_{q=1}^{n} (\mathbf{S}_0\cdot\mathbf{I}_q'')^2 \\ \mathbf{S}_0\cdot\mathbf{J} &= c. \end{aligned} \tag{10}$$

Hence by (8)

$$\mathbf{S}_0\cdot\mathbf{C} - RA \leqq \mathbf{S}_0\mathbf{S} \leqq \mathbf{S}_0\cdot\mathbf{C} + RA, \tag{11}$$

where

$$A = \left[\mathbf{S}_0^2 - \sum_{p=1}^{m} (\mathbf{S}_0\cdot\mathbf{I}_p')^2 - \sum_{q=1}^{n} (\mathbf{S}_0\cdot\mathbf{I}_q'')\right]^{1/2}.$$

So we have upper and lower bounds for the torsional rigidity, as given by (4).