

SOME GENERAL RESULTS IN THE THEORY OF NOISE THROUGH NON-LINEAR DEVICES*

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1. Introduction. Because of the great prevalence of noise in almost all electronic processes, a study of the nature and properties of such noise seems desirable. The term "properties of noise" is used here to indicate such measurable quantities as the average and the mean-square voltages and currents, the mean power spectrum, the power or energy associated with the wave, and the power and the correlation function of the disturbance, or of part of it, when noise or a signal and noise is modified by passage through non-linear apparatus. From an analytical point of view, the theory of noise is intrinsically related to that of the Brownian motion so that the results in the discussion of the one may bear rather closely upon the other. There are two different but equivalent lines of attack on the over-all problem: in noise theory we are primarily interested in what happens to random noise waves (with or without an accompanying signal) when they are passed through non linear devices, such as second detectors or mixers in radio receivers, for example, or amplifiers in which cutoff and/or overloading is present. Here the *Fourier series method of Rice*,^{1,2} is the more natural approach and is the one followed in the present paper. In the study of Brownian motion and fluctuation phenomena in general, where the variations in the system are described by a diffusion process, the second method of *Fokker-Planck*, or the *diffusion equation method*, is used. We shall not consider this approach here; an excellent discussion has recently been given by Wang and Uhlenbeck,³ and less recently, an interesting treatment of somewhat similar subjects by Chandrasekhar.⁴ We mention only in passing that the two methods can be shown to yield identical results.³

Rice† and others⁵⁻⁹ have used the Fourier series method in the solution of special

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¹ S. O. Rice, *Mathematical analysis of random noise*, Bell Sys. T. J. **23**, 282 (1944).

² S. O. Rice, loc. cit. **24**, 46 (1945). Since this paper was written, Professor Brillouin has kindly called the author's attention to the interesting work of Dr. Blanc-Lapierre along somewhat similar lines, in particular, a thesis: *Sur certaines fonctions aléatoires stationnaires. Application à l'étude des fluctuations dues à la structure électronique de l'électricité*, and *Effet Schottky. Fluctuations dans les amplificateurs linéaires et dans les détecteurs*, Bull. de la Soc. fr. des Elec. (6) **5**, No. 53, Nov. 1945.

³ M. C. Wang and G. E. Uhlenbeck, *On the theory of the brownian motion II*, Rev. Mod. Phys. **17**, 323 (1945). This paper and reference 4 contain a considerable number of references to previous work along these lines.

⁴ S. Chandrasekhar, *Stochastic problems in physics and astronomy*, Rev. Mod. Phys. **15**, No. 1, 1, 1943.

† See part IV of ref. 2.

⁵ K. Fränz, Zeits. f. Hoch. u. Elek. **57**, 146 (1941).

⁶ W. R. Bennett, *The response of a linear rectifier to signal and noise*, J. Amer. Acous. Soc. **15**, 165 (1944).

⁷ D. O. North, *The modification of noise by certain non linear devices*, a synopsis of which was presented at the Jan. 1944 winter technical meeting of the I.R.E.

⁸ J. H. Van Vleck and D. Middleton, *A theoretical comparison of the visual, aural, and meter reception of pulsed signals in the presence of noise*, J. Appl. Phys. **17**, 940 (1946).

⁹ D. Middleton, *The response of biased, saturated linear and quadratic rectifiers to random noise*, Jour. A. Phys. **17**, 778 (1946).

problems involving the rectification of noise, or of a signal and noise. It is the purpose of this paper to generalize some of the results of previous work on this topic and to obtain original results for a number of unsolved problems in the analysis of noise through non-linear devices. Specifically, it is believed that the treatment of the following topics is new and of interest.

- (a) Passage of a *modulated* signal in the presence of noise through a general non-linear apparatus. The case of a sinusoidally modulated carrier (Sec. 2), is examined and attention is also given to the case of narrow-band noise, symmetrically distributed in frequency about the carrier (Sec. 3).
- (b) The biased ν th-law rectifier, for modulated and unmodulated carriers (Sec. 4). Limiting cases of large noise or signal voltages are also discussed. Reference 9 gives a detailed discussion of this problem for linear and quadratic rectifiers where noise alone is rectified.
- (c) The problem of a modulated signal and narrow-band noise, with a determination of the various probability densities associated with the envelope of the wave (Sec. 5). Section (c) and (d) offer alternative solutions to some of the problems discussed in (a) and (b).
- (d) The correlation function and mean power associated with the envelope of signal and noise. Attention is given to the low frequency output of the half-wave ν th-law device (Sec. 6).
- (e) The ν th-law, half-wave rectification of noise alone, the results of which are of interest in the measurement of noise by meters, spectrum analyzers, etc., and in the detection of pulse signals in the presence of noise^{8,9} (Sec. 7). This work is a generalization of reference 9 in that ν can take on any positive value, but is less general in that only half-wave detection is treated.
- (f) A general "small-signal" theory, in which the peak values of the incoming wave, whether noise or a signal and noise, are sufficiently small that overloading and cut-off do not occur. Rectification takes place because of the curvature of the dynamic characteristic of the device in question.

Not all the material in the present paper is original, it is realized, but in the discussion of (a)–(f) it has been necessary for clarity of treatment to bring together and extend, when necessary, a number of results previously derived in probability theory which are fundamental and hence unavoidable in the analysis of problems of this type. These results include the generalized, s -dimensional random-walk problem,⁴ from which in turn one may obtain the characteristic function, with the distinctive property of being the Fourier transform of the probability density, and finally the central-limit theorem,¹⁰ which in the limit of a very large number of events can be shown to yield the s -dimensional Gaussian distribution characteristic of all random processes fulfilling certain rather elastic conditions with regard to the separate distributions of the various events. Some of the details are available in Appendix I. Furthermore, although the concept of the correlation function and its relation to the mean power spectrum has been examined in varying detail elsewhere, we include a brief treatment in Appendix II, along with some of the more significant properties of the correlation function which are necessary in our work. Rigor has not been preserved at all costs, *vide* the use of the Dirac delta-function; the physical significance

¹⁰ H. Cramèr, *Random variables and probability distributions*, Cambridge Tract No. 36 (1937), Chapters VI and X. References 1 and 2 also contain references to this problem; see Secs. 2.9 and 2.10.

of the results is used to check any possible weakness in the rigor. Finally, in Appendix III some of the more unfamiliar special functions that appear in our results are briefly considered. It should be kept in mind that the present discussion applies only when the dynamic path of the non linear device is a one-valued function of the input disturbance. When the path is multi-valued, the theory breaks down. For example, when the plate-load of a rectifying tube is a pure resistance, or at worst, is primarily resistive, the dynamic path is one-valued, or nearly so. Figure 1 illustrates a typical tube characteristic under these conditions. However, when there is apprecia-

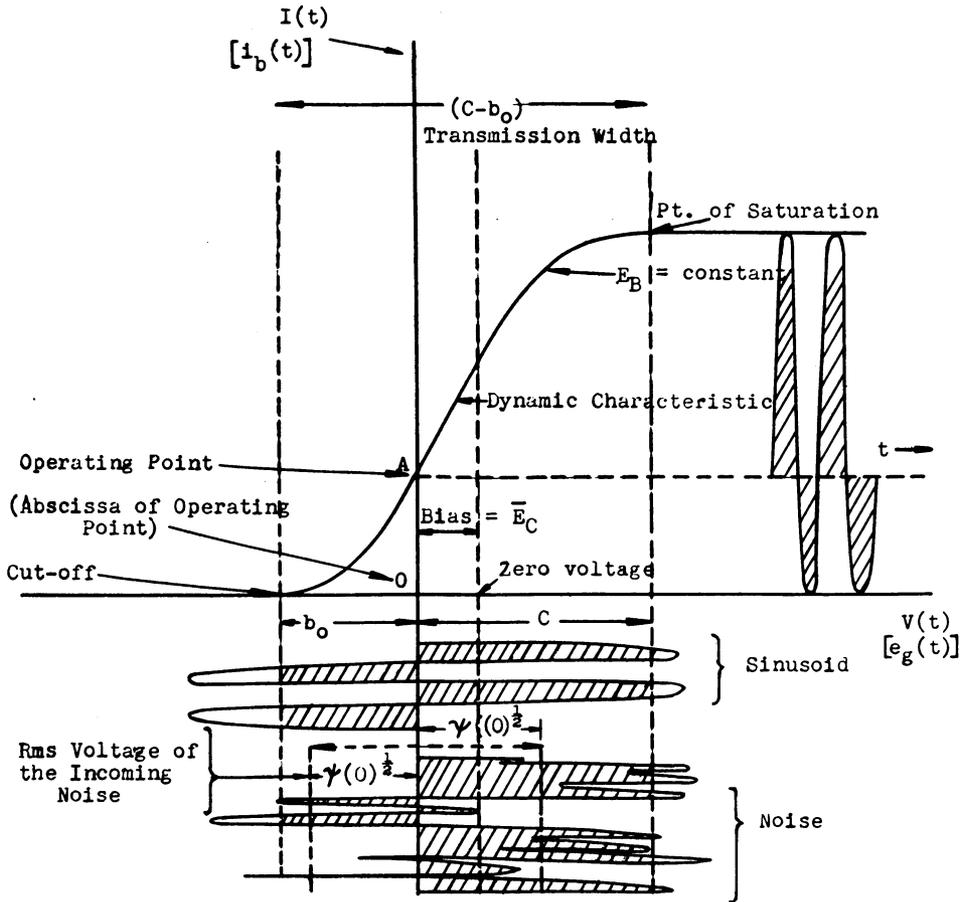


FIG. 1. Typical dynamic characteristic for resistive loads.

ble reactance X_p in comparison with the resistance R_p , the problem is not tractable with the present methods.

At this point it is well to mention what we mean by a random process, and with what class of such processes problems of the present kind deal. We consider $X(t)$ to be a random process when $X(t)$ does not depend in a definite way on the independent variable t , but instead may be specified by an aggregate of different functions $X(t)$, of which experimentally only probability distributions are observable. Here $X(t)$ may be the displacement of the electron beam on the screen of a cathode ray oscilloscope,

or, say, the velocity of a particle in Brownian motion. The quantity $X(t)$ may also represent a combination of two or more such stochastic variables, in which case we may speak of a two- or more-dimensional random process. In what follows, we shall consider formally that X and t are continuous variables; for our work, based as it is on the problem of the random walk (see Appendix I), this a plausible assumption. Now, as Wang and Uhlenbeck³ have shown, it is possible to describe a random process completely by means of a set of distributions determining the probability that X lie in the range $X, X+dX$, at time, t , that X_1 and X_2 fall in the intervals X_1, X_1+dX_1 and X_2, X_2+dX_2 , at times t_1 and t_2 , respectively, and so on. The classification is considerably simplified when the initial times of the observations do not enter; processes of this kind are *stationary*. Their statistical properties are independent of when the measurements are made, and depend only on the duration of the observations. There is an important class in this group known as *Markoff processes*, which are completely described by the second order density $W_2(X_1, X_2; t)$:

$$W_2(X_1, X_2; t) = \text{joint probability of obtaining } X_1 \text{ in the range } X_1, X_1+dX_1 \text{ and } X_2 \text{ in the range } X_2, X_2+dX_2 \text{ at a time } t \text{ later.} \quad (1.1)$$

Since the present paper deals exclusively with distributions that are stationary and Markoffian, or if not originally the latter, can be extended to a more complex Markoff process by the introduction of suitable additional random variables,¹¹ we find it helpful to mention the properties of W_2 . We have

$$W_2(X_1, X_2; t) = W_1(X_1)P_2(X_1 | X_2; t), \quad (1.2)$$

where following Wang and Uhlenbeck,³ $P_2(X | X_2; t)$ denotes the *conditional* probability that, given X_1 , we find X_2 in the interval X_2, X_2+dX_2 a time t later. Here $W_1(X_1)$ is simply the first order distribution, giving the probability of locating X_1 in the interval X_1, X_1+dX_1 ; the time does not enter because the process is assumed to be stationary. We have also the relations

$$P_2(X_1 | X_2; t) \geq 0; \quad \int dX_2 P_2(X_1 | X_2; t) = 1; \quad (1.3)$$

$$W_1(X_2) = \int W_1(X_1)P_2(X_1 | X_2; t)dX_1,$$

where the region of integration includes all possible values of the variables. The further condition

$$P_n(X_1, t_1; X_2, t_2; \dots; X_{n-1}, t_{n-1} | X_n; t_n) = P_2(X_{n-1}, t_{n-1} | X_n; t_n), \quad t_n > t_{n-1} > \dots > t_1 \quad (1.4)$$

shows that (1.1), or P_2 , completely determines the Markoff process, for once W_2 is known, we may obtain all distributions W_n , $n \geq 2$, from the above. It should be observed that $P_2 \rightarrow W_1(X_2)$ as $t \rightarrow \infty$, provided X_2 is purely random and does not contain periodic components. As has been previously pointed out,³ W_2 or P_2 cannot be selected arbitrarily, for it must satisfy the fundamental equation

¹¹ See, for example, sec. 3 (c), of ref. 3.

$$P_2(X_1 | X_2; t) = \int dX P_2(X_1 | X; t_0) P_2(X | X_2; t - t_0), \quad 0 \leq t_0 \leq t. \quad (1.5)$$

which is due to Smoluchowski. Equations (1.1)–(1.5) have been written for one- and two-dimensional systems; the extension to a greater number follows at once without modification of the concepts.

The successful solution of non-linear problems involving noise by the Fourier series method of Rice depends, then, on the following definitions, devices, and assumptions.

(1) It is *assumed* that the random process is stationary and Gaussian, i.e., X_1 , X_2 , etc., all obey a Gaussian distribution law. In the completely Gaussian case, all components of X_1 , X_2 , . . . , as well, are postulated to have a normal distribution of amplitudes. For our purposes, however, this is not necessary, for by the central limit theorem X_1 , X_2 , etc., will be Gaussian, irrespective of the distribution law of their components as long as the latter are sufficiently numerous, under conditions easily fulfilled physically in most instances. For convenience, we shall assume that our random processes are initially entirely random in the normal sense. In either case $W_2(X_1, X_2; t)$ is given by (A1.3).

(2) Our choice of input spectra will determine whether or not the system is Markoffian.^{12,13} If it is, then W_2 completely describes it; however in the present paper it is not usually critical that we be able to classify the process thoroughly; the chief concern is with the amplitude distribution and with the mean power spectrum, which may always be found when W_1 and W_2 are known, and here it is sufficient that the system be stationary.

(3) The mean power spectrum $W(f)$ may be obtained directly from the correlation function $R(t)$, defined by Eq. (A2.3), since by a well-known result^{14–16} they are each other's cosine Fourier transforms:

$$W(f) = 4 \int_0^\infty R(t) \cos \omega t dt; \quad \omega = 2\pi f; \quad R(t) = \int_0^\infty W(f) \cos \omega t df. \quad (1.6)$$

The correlation function, in turn, follows from the definition (A2.3) and with the help of W_2 is seen to be

$$R(t) = \int \int_{-\infty}^\infty X_1 X_2 W_2(X_1, X_2; t) dX_1 dX_2. \quad (1.7a)$$

If the correlation function for the random wave $X(t)$ *after* passage through a non linear device is desired (1.7a) becomes

$$R(t) = \int \int_{-\infty}^\infty g(X_1) g(X_2) W_2(X_1, X_2; t) dX_1 dX_2, \quad (1.7b)$$

¹² J. L. Dobb, Ann. Amer. Stat. **15**, 229 (1944).

¹³ Reference 3, Sec. 7, and note II of the appendix for details.

¹⁴ N. Wiener, Acta Math. **55**, 117 (1930).

¹⁵ A. Khintchine, Math. Ann. **109**, 604 (1934).

¹⁶ G. I. Taylor, Proc. Lond. Math. Soc. Sec. 2, **20**, 196 (1920), and Proc. Royal Soc. **164**, 476 (1938) gives applications to the theory of turbulence. Rice's interesting papers^{1,2} treat a large number of noise problems with the help of (1–6), while ref. 3 puts more emphasis on the Brownian motion.

where $g(X)$ is the dynamic characteristic of the apparatus in question, and X_1 is the incoming disturbance at some initial time and X_2 at time t later.

(4) The explicit evaluation of the integral (1.7b) is often very difficult in this form because discontinuities in the characteristic g appear as finite limits in the integration. Then it is convenient, almost mandatory, to introduce the *Fourier transform* $f(iz)$ of the dynamic path, so that the output wave is given in terms of the input by¹⁷

$$g(X) = \frac{1}{2\pi} \int_{\mathbf{C}} f(iz) e^{iXz} dz. \quad (1.8)$$

The contour \mathbf{C} extends from $-\infty$ to $+\infty$ along the real axis and is indented downward about a pole or branch point at the origin. The evaluation of (1.8) is effected when the contour is extended in an infinitely large semi-circle in a counterclockwise (positive) or clockwise (negative) sense, depending on whether the coefficient of iz in the exponent is positive or negative, respectively. When \mathbf{C} is traversed in a positive direction, the residue at $z=0$ yields the output as the desired function of the input, while for a negative circuit of the contour, $g(X)$ vanishes. Other contours are also possible, and combinations of such paths may be used for complicated characteristics. In this way, we are able to distinguish between the transmission and the cutoff states of the apparatus. It should be mentioned that when the contour representation (1.8) is employed, the results for the correlation function (1.7b) will be expressed in terms of *the characteristic function* associated with W_2 , namely Eq. (A1.1), in the general s -dimensional case, rather than by W_2 itself. This is not surprising, for it is easily shown that the characteristic function is the Fourier transform of the corresponding probability density. See Eqs. (2.13) to (2.16), in the next section.

(5) When a signal, modulated or unmodulated, is introduced along with the noise, the above concepts and artifices hold, with slight modification. For instance, (1.7a, 7b) now represent only the correlation function due to the noise components in the wave; a further average or averages are necessary to account for the signal and modulation components. Here the average over the phases of these components must be performed. In the next section this extension of the method is outlined and illustrated (see also references 2, 6, and 8).

2. Rectification of modulated signals in the presence of noise. Before determining what happens when a modulated wave is detected in the presence of noise, we must represent the incoming disturbance analytically. Let us consider first the signal component, which we shall denote by $V_s(t, t')$. Here the variable t refers to the time-variations of the carrier and t' to those of the modulation. Physically, t and t' both represent the same instant in time, our choice of symbols being merely one of mathematical convenience. For the type of signal of greatest interest here we may therefore write

$$V_s(t, t') = A_0(t') \cos \omega_0 t, \quad \omega_0 = 2\pi f_0, \quad A_0(t') \geq 0, \quad (2.1)$$

where f_0 is the carrier frequency and $A_0(t')$ is some function of the time, which is only properly called the modulation when it varies slowly compared with $\cos \omega_0 t$. In general the carrier and modulation are not commensurable and hence are uncorre-

¹⁷ W. R. Bennett and S. O. Rice, *Phil. Mag.* **18**, 422 (1934). For an extensive application of the contour representation, see reference 9.

lated; t and t' are consequently independent variables. On the other hand, there are cases when $A_0(t')$ and $\cos \omega_0 t$ bear a commensurable relation to each other, correlation exists, and t and t' are then functionally related. Such instances may occur, for example, when the carrier is over-modulated.

Now let us consider briefly the noise portion of the disturbance entering the non linear device. Several satisfactory representations are possible,¹⁸ of which we choose the Fourier series form

$$V_N(t) = \sum_{n=1}^N \{a_n \cos \omega_n t + b_n \sin \omega_n t\}, \quad \omega_n = 2\pi f_n = 2\pi n/T, \quad (2.2)$$

where the interval of expansion lies between 0 and T , and N is very much greater than unity. The quantities a_n and b_n are independent random variables, having the following properties:

$$\overline{a_n} = \overline{b_n} = 0; \quad \overline{a_n b_m} = 0; \quad \overline{a_n a_m} = \overline{b_n b_m} = w(f_n) \Delta f \delta_n^m = \{ |S(f)|^2 / T \} \delta_n^m, \quad (2.3)$$

this last from (A2.4). The bar indicates the statistical average over the various random quantities (see Appendix II); here δ_n^m is the familiar Kronecker delta, which has the values $\delta_n^m = 0$, $m \neq n$; $\delta_n^n = 1$. We assume the random process describing the noise to be stationary; this assumption can be verified experimentally. The time average for $T \rightarrow \infty$ and the ensemble average, in which the average of an (indefinitely) large number of finite intervals or "strips" corresponding to separate observations is taken, then yield the same results. Under these circumstances the distribution of the random variables a_n and b_n is unaltered. It is convenient to assume also that this distribution is Gaussian, with the standard deviation $[w(f_n) \Delta f]^{1/2}$, but as the central limit theorem shows, [see Appendix I (A1.1, A1.2, A1.3)], any other distribution with the same average and second moments [Eq. (2.3)] leads to identical expressions for the noise wave. Equations (2.3) may then be interpreted as follows: the first relation shows that $\overline{V_N(t)} = 0$, the second shows that a_n and b_n are independent, and the third apportions the mean power in the frequency range $f_n, f_n + \Delta f_n$ dissipated by a current flowing in a unit resistance when a potential difference $V_N(t)$ is maintained between the terminals. In the limit of a very long time, or what is the same thing, of a very large number of "strips" in the ensemble, $\Delta f (= 1/T)$ becomes infinitesimal, and the summation may be replaced by an integral from $f = 0$ to $f = \infty$.

With the aid of Appendix I, we proceed to determine $W_1(X)$ and $W_2(X_1, X_2; t)$, cf. (1.1), where we identify V_N at some time t_0 with X_1 and some time t later with X_2 . Referring to Eq. (A1.2), we find from (2.2) and (2.3) that on letting $X_{(j=n)} = a_n \cos \omega_n t + b_n \sin \omega_n t$,

$$\overline{X} = \nu = \sum_{n=1}^N (\overline{a_n} \cos \omega_n t + \overline{b_n} \sin \omega_n t) = 0; \quad (2.4)$$

$$\mu_{11} = \overline{X^2} = \sum_{n=1}^N (\overline{a_n^2} \cos^2 \omega_n t + \overline{b_n^2} \sin^2 \omega_n t) = \sum_{n=1}^N (\overline{a_n^2} + \overline{b_n^2}) / 2 \rightarrow \int_0^\infty w(f) df \equiv \psi(0),$$

with $\psi(0) = \overline{X^2}$ the mean power in the wave. Clearly here the matrix \mathbf{u} associated with W_2 , (see A1.3), has the single element $\psi(0)$ and $|\mu| = \psi^{1/2}$, $\mu^{11} = 1$, so that the first order distribution for $V_N = X$ is, from (A1.3),

¹⁸ See reference 1, Sec. 2.8, and reference 6.

$$W_1(X) = [2\pi\psi(0)]^{-1/2} \exp[-X^2/2\psi(0)]. \quad (2.5)$$

In a similar fashion the joint probability density $W_2(X_1, X_2; t)$ follows; we have

$$\begin{aligned} \bar{X}_1 &= \bar{X}_2 = \nu_1 = \nu_2 = 0, \quad \text{and} \\ \mu_{11} &= \sum_{n=1}^N (\bar{a}_n^2 \cos^2 \omega_n t_0 + \bar{b}_n^2 \sin^2 \omega_n t_0) = \bar{X}_1^2 = \psi(0) \\ \mu_{22} &= \sum_{n=1}^N (\bar{a}_n^2 \cos^2 \omega_n(t_0 + t) + \bar{b}_n^2 \sin^2 \omega_n(t_0 + t)) = \bar{X}_1^2 = \psi(0), \end{aligned} \quad (2.6)$$

from (A1.2) and (A1.4), as a consequence of the conditions (2.3). For the off-diagonal moments one may write

$$\begin{aligned} \mu_{12} = \mu_{21} &= \overline{X_1 X_2} = \sum_{n=1}^N [\bar{a}_n^2 \cos \omega_n t_0 \cos \omega_n(t_0 + t) + \bar{b}_n^2 \sin \omega_n t_0 \sin \omega_n(t_0 + t)] \\ &= \sum_{n=1}^N (\bar{a}_n^2 \cos^2 \omega_n t_0 + \bar{b}_n^2 \sin^2 \omega_n t_0) \cos \omega_n t \\ &\rightarrow \int_0^\infty w(f) \cos \omega t df \equiv \psi(t). \end{aligned} \quad (2.7)$$

The quantity $\psi(t)$ is the correlation function of the input noise, and by (1.6),¹⁹ the Fourier transform of the input spectrum $w(f)$. For the two-dimensional distribution \mathbf{u} becomes

$$\begin{aligned} &= \begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{bmatrix} = \begin{bmatrix} \psi(0) & \psi(t) \\ \psi(t) & \psi(0) \end{bmatrix}; \quad |\mu| = \psi^2(1 - r^2(t)), \quad \text{and} \\ &\mu^{11} = \mu^{22} = \psi(0), \quad \mu^{12} = \mu^{21} = -\psi(t); \end{aligned} \quad (2.8)$$

$r(t)$ is the normalized correlation function of the incoming disturbance, Eq. (A2.7). With the aid of (A1.3) the joint distribution is easily observed to take the familiar form²⁰

$$W_2(X_1, X_2; t) = \frac{1}{2\pi\psi(1 - r^2)^{1/2}} e^{-(X_1^2 + X_2^2 - 2rX_1X_2)/2\psi(1 - r^2)}. \quad (2.9)$$

The incoming wave is simply the sum of (2.1) and (2.2), with the various properties of the two components outlined above:

$$V_{in} = X + V_s(t', t). \quad (2.10)$$

Accordingly, if $g(V_{in})$ is the output of the device [see (1.7b)] the correlation function for the noise components is

$$R_N(t_0, t_0'; t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(X_1 + V_s[t_0, t_0']) g(X_2 + V_s[t_0 + t, t_0' + t]) W_2(X_1, X_2; t) dX_1 dX_2. \quad (2.11)$$

¹⁹ See Appendix II for details.

²⁰ Henceforth we abbreviate $\psi(0)$ by ψ .

The complete correlation function, which includes the average over the phases of the signal as well as over the noise, is

$$R(t) = T_0'^{-1} \int_0^{T_0'} dt_0' \int_0^{T_0} T_0^{-1} R_N(t_0, t_0'; t) dt_0, \tag{2.12}$$

where T_0 and T_0' are the respective periods of the carrier and the modulation. The double integration in (2.12) reduces to a single operation in case t_0 and t_0' are functionally related, i.e., when carrier and modulation have a constant phase difference. Since W_2 is given explicitly [Eq. (2.9)] from our assumptions regarding the character of the noise, (2.11) and (2.12) represent the formal solution to our problem, in as much as the mean power spectrum follows immediately from (1.6). However, as these equations in their present form can be handled only for a few special cases, chiefly in the instance of "small-signal" rectification, where the distortion of the dynamic path arising from cut-off and saturation is gradual, we make use of the device of contour integration, (1.8), for the large and important class of problems in which cut-off and saturation take place abruptly. Then the outputs at times t_0 and t_0+t are respectively

$$g(V_1) = g(V_s(t_0, t_0') + X_1) = \frac{1}{2\pi} \int_{\mathbf{C}} f(iz) e^{iz(X_1 + V_s(t_0, t_0'))} dz, \tag{2.13a}$$

$$g(V_2) = g(V_s[t_0 + t, t_0' + t] + X_2) = \frac{1}{2\pi} \int_{\mathbf{C}'} f(i\xi) e^{i\xi(X_2 + V_s[t_0+t, t_0'+t])} d\xi, \tag{2.13b}$$

where \mathbf{C}' is a contour similar to \mathbf{C} but in the ξ -plane. Equations (2.11) and (2.12) become finally

$$\begin{aligned} R(t) &= \frac{1}{4\pi^2} \int_{\mathbf{C}} f(iz) dz \int_{\mathbf{C}'} f(i\xi) d\xi \left\{ \iint_{-\infty}^{\infty} W_2(X_1, X_2; t) e^{izX_1 + i\xi X_2} dX_1 dX_2 \right\} \\ &\times \left\{ \frac{1}{T_0 T_0'} \int_0^{T_0} \int_0^{T_0'} dt_0' dt_0 e^{izV_s(t_0, t_0') + i\xi V_s[t_0+t, t_0'+t]} \right\} \\ &= \frac{1}{4\pi^2} \int_{\mathbf{C}} f(iz) dz \int_{\mathbf{C}'} f(i\xi) F_N(z, \xi, t) F_S(z, \xi; t) d\xi, \end{aligned} \tag{2.14}$$

in which $F_S(z, \xi; t)$ is by definition the characteristic function of the signal, given here by

$$F_S(z, \xi; t) = \frac{1}{T_0 T_0'} \int_0^{T_0'} dt_0' \int_0^{T_0} dt_0 e^{izV_s(t_0, t_0') + i\xi V_s(t_0+t, t_0'+t)}, \tag{2.15}$$

and $F_N(z, \xi; t)$ is similarly seen to be the characteristic function for the noise, and may be written

$$\begin{aligned} F_N(z, \xi; t) &= \iint_{-\infty}^{\infty} W_2(X_1, X_2; t) e^{izX_1 + i\xi X_2} dX_1 dX_2 \\ &= \exp \left[-\frac{1}{2} \psi(z^2 + \xi^2) - \psi(t) z \xi \right]. \end{aligned} \tag{2.16}$$

This last result is obtained from (A1.1) with the help of (2.6)–(2.9). From (2.14), it

appears that the characteristic function for a modulated signal and noise is merely the product of the separate characteristic functions, for physically the signal and noise components are uncorrelated, and hence their distributions must be independent.

For the modulated carrier Eq. (2.1) may be substituted directly in (2.15). The more usual case of no carrier-modulation correlation gives us

$$\begin{aligned}
 F_s(z, \xi; t) &= \frac{1}{T'_0} \int_0^{T'_0} dt'_0 \left\{ \frac{1}{T_0} \int_0^{T_0} e^{izA_0(t'_0)\cos\omega_0 t_0 + i\xi A_0(t'_0+t)\cos\omega_0(t_0+t)} dt_0 \right\} \\
 &= \frac{1}{T'_0} \int_0^{T'_0} dt'_0 \left\{ \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \epsilon_m \epsilon_p i^{m+p} J_m(A_0[t'_0]z) J_p(A_0[t'_0+t]\xi) \right. \\
 &\quad \left. \times \frac{1}{T_0} \int_0^{T_0} dt_0 \cos \omega_0 m t_0 \cos p \omega_0 (t_0 + t) \right\}, \tag{2.17}
 \end{aligned}$$

where we have used the familiar expansion of $\exp(i\alpha \cos \beta t)$ in a Fourier series,²¹ and where $\epsilon_0 = 1$, $\epsilon_m = 2$, $m \geq 1$. The trigonometric integral has the value $(\cos m\omega_0 t) \delta_m^p / \epsilon_m^p$, so that (2.17) may be written finally

$$F_s(z, \xi; t) = \sum_{m=0}^{\infty} (-1)^m \epsilon_m \cos m\omega_0 t \left\{ \frac{1}{T'_0} \int_0^{T'_0} dt'_0 J_m(A_0[t'_0]z) J_m(A_0[t'_0+t]\xi) \right\}. \tag{2.18}$$

When (2.16) and (2.18) are substituted in the expression for the complete correlation function $R(t)$, we may expand $\exp[-\psi(t)z\xi]$, obtaining

$$R(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} \epsilon_m \psi(t)^n \cos m\omega_0 t}{n!} [H_{mn}(t'_0) H_{mn}(t'_0 + t)]_{av.}, \tag{2.19}$$

where

$$\left. \begin{aligned}
 H_{mn}(t'_0) &= \frac{1}{2\pi} \int_{\mathbf{C}} z^n f(iz) J_m(zA_0[t'_0]) e^{-\psi z^2/2} dz, \\
 H_{mn}(t'_0 + t) &= \frac{1}{2\pi} \int_{\mathbf{C}'} \xi^n f(i\xi) J_m(\xi A_0[t'_0 + t]) e^{-\psi \xi^2/2} d\xi.
 \end{aligned} \right\} \tag{2.20}$$

The average indicated above in (2.19) is the time average over the phases of the modulation. The mean power spectrum of the output wave follows from (1.6) and (2.19), and is

$$W(f) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} \frac{\epsilon_m}{n!} C_{mn}(f), \tag{2.21}$$

where $C_{mn}(f)$ is given by

$$C_{mn}(f) = 4 \int_0^{\infty} \psi(t)^n \cos m\omega_0 t \cos \omega t [H_{mn}(t'_0) H_{mn}(t'_0 + t)]_{av.} dt, \quad \omega = 2\pi f. \tag{2.22}$$

The quantity $[H_{mn}(t'_0) H_{mn}(t'_0 + t)]_{av.}$ is, in fact, a kind of correlation function, by formal comparison with (A2.1), except that here no statistical averages appear because $A_0(t)$ is periodic. We can also write formally

²¹ Watson, *Theory of Bessel Functions*, Cambridge Univ. Press, 1944, sec. 2.22.

$$\begin{aligned}
 & [H_{mn}(t'_0)H_{mn}(t'_0 + t)]_{av.} \\
 &= \frac{1}{4\pi^2} \int_C z^n f(iz) e^{-\psi z^2/2} dz \int_C \xi^n f(i\xi) e^{-\psi \xi^2/2} [J_m(zA_0[t'_0])J_m(\xi A_0[t'_0 + t])]_{av.} d\xi \\
 &= \sum_{k=0}^{\infty} \epsilon_k h_{kmn}^2 \cos k\omega_A t, \tag{2.23}
 \end{aligned}$$

where $f_A (= \omega_A/2\pi)$ is the fundamental frequency of $A_0(t)$ and h_{kmn}^2 is a mean-square amplitude, which depends on ψ , the amplitude of $A_0(t)$, and on the dynamic characteristic of the apparatus in question.²² Equation (2.23) is the result of developing the Bessel functions in series, or the expressions integrated first over z and ξ , and arranging the result as a series in $\cos k\omega_A t$. The explicit evaluation of h_{kmn} is in general a difficult task. In some cases the work is simplified by taking the time average first, in others, by integrating over z and ξ initially. There seems to be no simple way of handling the integration problem presented by (2.23), but usually the average over the phases of the modulation must be performed last. Some illustrative examples are given later in this section.

From (2.23) we observe that (2.22) becomes

$$C_{mn}(f) = \sum_{k=0}^{\infty} \epsilon_k h_{kmn}^2 c_{kmn}(f), \tag{2.24a}$$

where

$$c_{kmn}(f) = 4 \int_0^{\infty} \psi(t)^n \cos m\omega_0 t \cos k\omega_A t \cos \omega t dt; \tag{2.24b}$$

the mean output spectrum (2.21) finally takes the form

$$W(f) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{m+n} \frac{\epsilon_m \epsilon_k}{n!} h_{kmn}^2 c_{kmn}(f). \tag{2.25}$$

The effect of the rectifier or similar non-linear device is to “mix” or cross-modulate the noise and signal components with one another so that the (unfiltered) output contains the following three classes of modulation product: (a) *noise* \times *noise*, which gives rise to noise, no longer random with Gaussian properties, (b) *noise* \times *signal*, which results from the beating of the signal components with the noise wave and which in turn also yields non-Gaussian noise, and finally, (c) *signal* \times *signal*, produced by the cross-modulation of the various signal components: this last is entirely periodic and free from noise. A d-c component is often present, but not when the dynamic characteristic is symmetrical, for then the average of the output, as well as the input, vanishes. Further subdivision is possible when there is modulation, as we may then distinguish between modulation products generated by the separate (incommensurable) components of the carrier and modulation, but essentially the three main classifications (a)–(c) still hold.

The different contributions to (a)–(c) may be distinguished in (2.25), since for (a), exclusive of the d-c, we may write, with $m = k = 0, n \geq 1$,

²² It is also possible to express the average in terms of the moments $A_{01}^m A_{02}^p$, on expanding the Bessel functions with the help of the relation on page 148 of reference 21.

$$W(f)_{\text{noise} \times \text{noise}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} h_{00n}^2 c_{00n}(f), \tag{2.26a}$$

and for (b) we have $m \geq 1, n \geq 1, k \geq 1$, giving

$$W(f)_{\text{noise} \times \text{signal}} = 4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{m+n}}{n!} h_{kmn}^2 c_{kmn}(f). \tag{2.26b}$$

Type (c) occurs only for $m \geq 1, k \geq 0, n = 0$ or $m \geq 0, k \geq 1, n = 0$:

$$W(f)_{\text{signal} \times \text{signal}} = \sum_{\substack{m=1 \\ =0}}^{\infty} \sum_{\substack{k=0 \\ =1}}^{\infty} \epsilon_k (-1)^m h_{km0}^2 \{ \delta(f - mf_0 - kf_A) + \delta(f - mf_0 + kf_A) \}, \tag{2.26c}$$

where we have used the delta function to indicate the "discrete" nature of these components, located in frequency at $mf_0 \pm kf_A$. The spectral distributions (a) and (b), on the other hand, representing noise, are continuous. There remains only the d-c, which is specified by $m = n = k = 0$:

$$W_{d-c} = h_{000}^2 4 \int_0^{\infty} df \left(\int_0^{\infty} \cos \omega t dt \right) = h_{000}^2 2 \int_0^{\infty} \delta(f - 0) df = h_{000}^2. \tag{2.26d}$$

The parts of the correlation function which contribute to (a)-(c) are easily found from (2.19), (2.21), and (1.6).

We turn now to a number of interesting special cases:

I. Unmodulated carrier. This is the first and simplest case involving a signal; A_0 is a constant quantity representing the amplitude of the carrier wave. It is immediately clear that the only non-vanishing terms require $k = 0$ ($m, n \geq 0$), so that the correlation function reduces to the form given by Eqs. (4.9-7) of reference 2, while $h_{mn} (\equiv h_{mn0})$ is expressed by Eq. (4.9-6). For *noise alone* entering the non linear device, the expressions (2.19)-(2.26d) are still simpler, reducing to the forms given by Eqs. (3.1)-(3.6) of reference 9.

*II. Carrier modulated by sine wave.*²³ For this important case we have for the signal (2.1)

$$V_s = A_0(1 + \lambda \cos \omega_A t) \cos \omega_0 t, \quad A_0(t'_0) = A_0(1 + \lambda \cos \omega_A t'_0), \quad 0 \leq \lambda \leq 1, \tag{2.27}$$

where λ is the modulation index and A_0 is the (peak) amplitude of the carrier component. For the moment no restriction is set on f_A and f_0 , save that they be incommensurable, which is another way of stating that $A_0(t)$ and $\cos \omega_0 t$ are uncorrelated. Of course, for $A_0(t)$ to be spoken of as a modulation in the usual sense, the frequency f_A must be much less than f_0 .

We require the value of $[H_{mn}(t'_0)H_{mn}(t'_0 + t)]_{\text{av.}}$. Expansion of the Bessel functions in (2.20) yields various moments of the modulation, viz. $[A_0^{2m}(t'_0)A_0^{2n}(t'_0 + t)]_{\text{av.}}$. The average over t'_0 follows in the present instance, cf. (2.27), with the aid of the relation

²³ For some earlier work on this problem, see J. R. Ragazzini, Proc. I.R.E. 30, 227, 1942. See also sections 4.1, 4.2, 4.10 of ref. 2. A theoretical discussion of this problem in the case of half-wave rectification has been given by the author in a paper submitted to the I.R.E. A companion paper by Fubini and Johnson verifies the results experimentally.

$$I_{a_1, a_2}(\phi) = \frac{1}{2\pi} \int_0^{2\pi} (1 + \lambda \cos \theta)^{a_1} (1 + \lambda \cos [\theta + \phi])^{a_2} d\theta, \quad \phi = \omega_A t,$$

$$= \sum_{j=1}^{a_2} 2^{1-2j} \lambda^{2j} {}_{a_2} C_j (-1)^j \sum_{i=0}^{j/2, (j-1)/2} {}_j C_i {}_j \Delta_i [-a_1]_{j-2i} \frac{(2/\lambda)^{2i}}{(j-2i)!} \quad (2.28)$$

$$\times \cos(j-2i) \phi {}_2F_1\left[\frac{1}{2}(j-2i-a_1), \frac{1}{2}(j-2i-a_1+1); 1+j-2i; \lambda^2\right], \quad (2.29)$$

where a_1 and a_2 are integers, the C 's are the usual binomial coefficients, and ${}_j \Delta_i = \frac{1}{2}$ when $j = 2i$, and ${}_j \Delta_i = 1, j \neq 2i$. The limits on the second summation apply accordingly as j is even or odd, and $[\alpha]_\beta = \alpha(\alpha+1) \cdots [\alpha+\beta-1], [\alpha]_0 = 1$. The expression (2.29) follows from the development of $(1 + \lambda \cos [\theta + \phi])^{a_2}$ in a series in $\cos \theta$ and the result²⁴

$$\frac{1}{2\pi} \int_0^{2\pi} (1 + \lambda \cos \theta)^l \cos k\theta d\theta$$

$$= \frac{(-l)_k (-\lambda)^k}{2^k k!} {}_2F_1\left[\frac{1}{2}(k-l), \frac{1}{2}(k-l+1); k+1; \lambda^2\right]. \quad (2.30)$$

From this it is also easily seen that when $\phi = 0$

$$I_{a_1, a_2}(0) = {}_2F_1\left[-\frac{1}{2}(a_1 + a_2), \frac{1}{2}(-a_1 - a_2 + 1); 1; \lambda^2\right]. \quad (2.31)$$

Equation (2.31) is helpful if one desires to determine the mean power in the wave. A short table of $I_{a_1, a_2}(\phi)$ is given below.

TABLE I

	$a_1=0$	1	2	3	4	5
$a_1=0$	1	1	$1+\lambda^2/2$	$1+3\lambda^2/2$	$1+3\lambda^2/2+3\lambda^4/8$	$1+5\lambda^2+15\lambda^4/8$
1	1	$1+(\lambda^2/2) \cos \phi$	$1+\lambda^2(\frac{1}{2}+\cos \phi)$	$1+(3\lambda^2/2)(1+\cos \phi)+ (3\lambda^4/8) \cos \phi$	$1+\lambda^2(3+2 \cos \phi)+ (3\lambda^4/8)(1+4 \cos \phi)$	
2	$1+\lambda^2/2$	$1+\lambda^2(\frac{1}{2}+\cos \phi)$	$1+\lambda^2(1+2 \cos \phi)+ (\lambda^4/8)(2+\cos 2\phi)$	$1+\lambda^2(2+3 \cos \phi)+ (3\lambda^4/8)(2+2 \cos \phi+\cos 2\phi)$		
3	$1+3\lambda^2/2$	$1+(3\lambda^2/2)(1+\cos \phi)+ (3\lambda^4/8) \cos \phi$	$1+\lambda^2(2+3 \cos \phi)+ (3\lambda^4/8)(2+2 \cos \phi+\cos 2\phi)$			
4	$1+3\lambda^2/2+3\lambda^4/8$	$1+\lambda^2(3+2 \cos \phi)+ (3\lambda^4/8)(1+4 \cos \phi)$				
5	$1+5\lambda^2+15\lambda^4/8$					

Once the correlation function has been found it is comparatively easy to determine the mean power in the output by setting $t=0$ in the appropriate expressions for $R(t)$. As shown by (A2.6) and as discussed more fully in Appendix II, this is equivalent to integrating over all spectral components, "discrete" or continuous. It is also evident that the mean total power W_r in the output wave is independent of spectral shape of the input distribution provided all significant components are included in our calculation. Thus from (2.19) and (2.23) we may write, setting $t=0$,

²⁴ Reference 2, Eq. (4.2.17).

$$\begin{aligned}
 W_\tau = R(0) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{n!} \epsilon_m \psi(0)^n [H_{mn}(t'_0)^2]_{\text{av.}} \\
 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{m+n} \epsilon_m \epsilon_k \psi(0)^n}{n!} h_{kmn}^2
 \end{aligned} \tag{2.32}$$

for a general, modulated carrier rectified in the presence of random noise. The power in the continuum W_τ is readily obtained from (2.32) and (2.26d), viz., $W_\tau = W_\tau - W_{d-c} = W_\tau - h_{000}^2$, or by omitting from the sum the term in $k=m=n=0$ in (2.32). In a similar fashion, to determine the portion of the total power output attributable to modulation products formed by (a), *noise* \times *noise*, (b), *noise* \times *signal*, and (c), *signal* \times *signal*, we have only to consider in (2.32) those terms for which $m=k=0, n \geq 1, m \geq 1, n \geq 1, k \geq 1$, and $m \geq 1, k \geq 0, n=0$ or $m \geq 0, k \geq 1, n=0$, respectively.

One may also calculate the mean power in the output without knowledge of the correlation function, for if $W_1(V_{in})$ is the first order probability density for the input wave (2.10), then by definition the mean total power dissipated by the rectified disturbance in a unit resistance is

$$W_\tau = \overline{g(V_{in})^2} = \int_{-\infty}^{\infty} g(V_{in})^2 W_1(V_{in}) dV_{in}, \tag{2.33a}$$

with a corresponding expression for the d-c component:

$$W_{d-c} = [\overline{g(V_{in})}]^2 = \left\{ \int_{-\infty}^{\infty} g(V_{in}) W_1(V_{in}) dV_{in} \right\}^2. \tag{2.33b}$$

The quantity $W_1(V_{in})$ has been derived independently by Rice²⁵ and Bennett²⁶ for the case of the simple sine wave and noise. For noise alone it is given by (2.5); the distribution is more complicated when a signal is present. The drawback to (2.33a) as a method of determining the power output, however, lies in the difficulty of performing the integrations when g is discontinuous, for instance in the half-wave linear rectifier, where $g > 0, V_{in} > 0; g = 0, V < 0$. Equation (2.33a) is useful only when the characteristic g exhibits no abrupt variations, cf., "small-signal" detection, in which the incoming disturbance has a mean square amplitude small compared with the curvature of the dynamic path (see Sec. 8), or when noise alone is subject to rectification, see Ref. 9. In such cases (2.33a) is to be preferred because it yields results in terms of a finite number of (tabulated) functions, whereas the corresponding expression for W_τ obtained from $R(0)$ is given as an infinite series which does not converge too rapidly. On the other hand, use of the correlation function has the advantage that from it can be predicted the order and nature of particular modulation products or groups of products, i.e., whether the contribution arises from *noise* \times *noise* or *noise* \times *signal*, etc. This, as we shall see in the next section, is an important property of the correlation function, especially useful when the incoming wave is spectrally narrow compared with its central frequency, for then the output is composed of separate harmonic "zones," any one of which may in principle be isolated with a suitable filter and examined.

3. Narrow band waves. So far, in discussing spectra and correlation functions no restrictions have been placed on the spectral nature of the input wave, which is com-

²⁵ Reference 2, Eq. (3.10.6).

²⁶ Reference 6, Eq. (3.6).

posed, in its general form, of periodic or of a set of periodic components and of a noise term. Moreover, in determining the mean power spectrum of the output disturbance, no additional modification of the wave has been assumed other than that attributable to the non linear $i_p - e_g$ (output current vs. applied voltage) path of the rectifying device. In somewhat different language this means that the circuits of our apparatus contain only filters whose amplitude response is uniform (for all frequencies) and whose phase shifts vary in linear fashion with the frequency. This means that we restrict ourselves to non linear elements whose instantaneous output does not affect the time dependence of the incoming wave, i.e., to resistive loads, no feedback, or at worst to plate (and grid) circuits whose reactive components are small compared with their resistive ones—all of which is required for essentially one-valued dynamic paths (Fig. 1)—then we may introduce the effect of filters into the analysis, observing that now they simply modify the *frequency* spectrum. In the case of filtering the input the spectral shape is altered, but the quality of the wave is not altered; it is still a Gaussian random process. On the other hand, if the output is filtered, not only may the original, i.e., the unfiltered output, be changed spectrally, but also the quality of

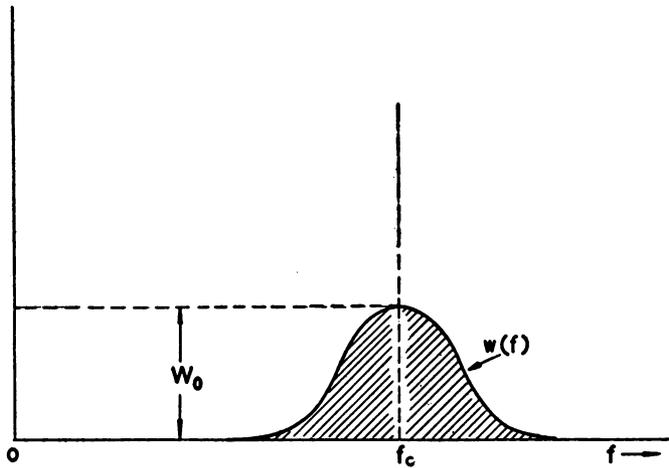


FIG. 2. Narrow-band input noise (power) spectrum, centered about f_c .

the wave is affected, in as much as the output is no longer Gaussian, because of the intervening non linear operation. The distributions of the output amplitude, however, remain unchanged by a filter, *provided that the pass-band of this filter is sufficiently wide compared with the spectrum of the transmitted wave*. If it is not, then there is a tendency of the filter to restore randomness to the distribution by filtering out the higher frequencies characteristic of the distortions due to curvature of the dynamic path and to the "ceiling" produced by cut-off, saturation, or both.* But we can still determine the mean power and mean power spectrum, although the distributions W_1, W_2, \dots, W_n are at present beyond our powers, by observing that

$$W(f)_{\text{filter in output}} = W(f)_{\text{inf. filter in output}} \cdot |G(f)|^2,$$

where $|G(f)|$ is the modulus of the output filter response.

* *Note added in proof:* The difficult problem of determining the distribution of the amplitudes after rectification and arbitrary filtering has been solved by M. Kac and A. J. F. Siegert, *J. Appl. Phys.*, **18**, 383 (1947) for a quadratic detector.

When the spectral width, measured between points at which the mean power spectrum is but a few percent of its maximum value, is small compared to the central or "resonant" frequency, we term the wave narrow-band. A precise definition, of course, requires a knowledge of the spectral shape, whether it has one or more maxima, where we measure width, etc., but here we shall assume that the spectrum is a simple, symmetrical distribution about a single maximum, as shown in Fig. 2. Now an important property of narrow band noise waves is that their correlation functions may be expressed as the product of the correlation functions of a slowly and of a rapidly varying term, corresponding to an envelope or "modulation," and to a "carrier," having the central or resonant frequency f_c of the disturbance. (This is shown to hold for symmetrical noise spectra in Appendix II (A2.8); for a signal the same demonstration may be made, except now, of course, we have a series of discrete components distributed about a true carrier, rather than a continuum about a central frequency.) The correlation function and the spectrum of the output follow directly from (2.14) and (1.6) as before, but instead of expanding $\exp[-\psi(t)z\xi]$ at once to obtain the form (2.19), we may use this property of the input correlation, viz:

$$\psi(t) = \psi(t)r_c(t) \cos \omega_c t \quad (\text{A2.8})$$

to show that the outgoing wave is composed of an infinite number of spectral bands centered about harmonics of the central or of the carrier frequency, or of their modulation products. Generally speaking, these bands will overlap to varying degrees, if f_0 and f_c are different; if $f_0 = f_c$, which is the important case in practice, the modulation products involving f_0 and f_c will coincide exactly. The different behaviors are apparent when we attempt to examine a given spectral region, say one centered about the carrier f_0 . In the former instance ($f_0 \neq f_c$) our filter* will exclude part of the spectral zone due to the one and include some of the zone associated with the other, while for the latter ($f_c = f_0$), our filter passes contributions symmetrically disposed about f_0 . For the harmonic regions of higher spectral order, i.e., $lf_0, lf_c, l \geq 2$, the former effect becomes more pronounced, since now the maxima of the distributions due to the noise and signal are separated by approximately $l|f_0 - f_c|$ cycles. Further there is a "smearing-out" of the spectra, because all orders of *noise* \times *noise*, *noise* \times *signal*, etc., components are generated. This accounts for the broadening and filling-in of the various output spectra; smearing of the carrier and modulation is also observed. When $l=0$, we obtain the low-frequency output, or envelope, distorted or not depending on the nature of the tube's dynamic path. This spectral region is usually of principal interest in reception, while in transmission the zone $l=1$ is of chief concern, where now our rectifier assumes the rôle of mixer (though with a different characteristic, of course). Figure 3 illustrates this discussion.

Now with the help of (A2.8) and the expansion

$$e^{-\psi r_0(t) \cos \omega_0 t} = \sum_{p=0}^{\infty} \epsilon_p (-1)^p I_p(z\xi\psi r_0(t)) \cos p\omega_c t \quad (\text{3.1})$$

we may express the correlation function (2.14) as

* For this purpose all our filters are assumed, to be ideal, i.e., to have a uniform response over a frequency range wide enough to include (almost) all the components in a given zone. We say almost, because the non linear device produces some frequencies which will lie outside the finite pass-band (see the end of Appendix II).

$$R(t) = \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \epsilon_m \epsilon_p (-1)^{m+p} \cos p\omega_c t \cos m\omega_0 t \left\{ \frac{1}{4\pi^2} \int_C f(iz) J_m(zA_0[t_0']) e^{-\psi z^2/2} dz \right. \\ \left. \times \int_{C'} f(i\xi) J_m(\xi A_0[t_0' + t]) e^{-\psi \xi^2/2} d\xi \cdot I_p(z\xi r_0) \right\}_{av}. \tag{3.2}$$

Equation (3.2) is general for all f_0 and f_c . However, in the case of principal interest carrier and central frequency coincide, so that $f_0 = f_c$, and (3.2) reduces to a series of

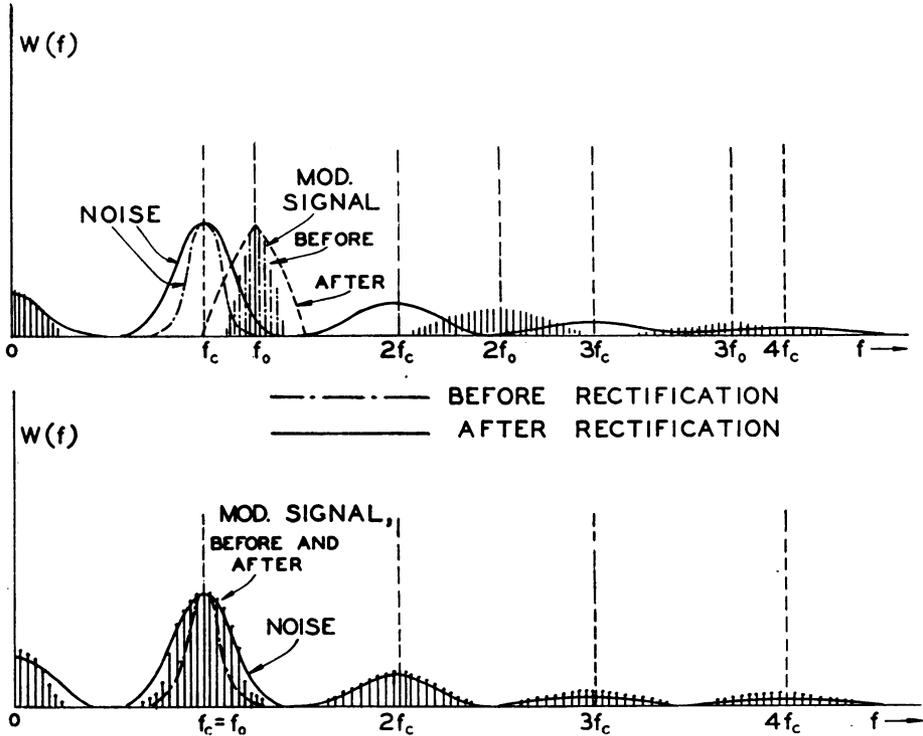


FIG. 3. Narrow-band noise and modulated signal spectra, before and after rectification. The upper of the two applies when $f_c \neq f_0$, where the spectral width is much less than either f_c or f_0 . The lower illustrates the spectra when $f_c = f_0$. The normalization is arbitrary.

distinct harmonic regions distributed about $lf_0, l=0, 1, 2, 3 \dots$, the harmonics of the carrier. Thus, from the fact that replacing $\sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \epsilon_m \epsilon_p ()$ by $\sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} ()$, since $J_m(\alpha) J_m(\beta) = J_{-m}(\alpha) J_{-m}(\beta)$ and $I_p(\gamma) = I_{-p}(\gamma)$, and collecting all terms of $\cos p\omega_0 t \cos m\omega_0 t$ which yield $\cos l\omega_0 t$, we observe that the correlation function for the l th band is

$$R_l(t) = \frac{(-1)^l \cos l\omega_0 t}{4\pi^2} \sum_{m=0}^{\infty} \epsilon_{|m-l|} \left\{ \int_C f(iz) J_m(A_0[t_0'] z) e^{-\psi z^2/2} dz \right. \\ \left. \times \int_{C'} f(i\xi) J_m(A_0[t_0' + t] \xi) I_{|m-l|}(\psi \cdot z\xi r_0) e^{-\psi \xi^2/2} d\xi \right\}_{av}. \\ = (-1)^l \cos l\omega_0 t \sum_{m=0}^{\infty} \frac{\epsilon_{|m-l|} \psi^{m-l}}{2^{|m-l|}} \sum_{q=0}^{\infty} \frac{\psi^{2q} r_0^{2q+|m-l|}}{2^{2q} q! (q + |m-l|)!} \\ \times [H_{m, |m-l|+2q}(t_0') \cdot H_{n, |m-l|+2q}(t_0' + t)]_{av}. \tag{3.3}$$

on expanding the Bessel function $I_{|m-l|}$. The complete correlation function is, of course, $\sum_{l=0}^{\infty} R_l(t)$. By (A2.6) and (2.23) the power in any zone is

$$R_l(0) = (-1)^l \sum_{m=0}^{\infty} \frac{\psi^{|m-l|} \epsilon_{|m-l|}}{2^{|m-l|}} \sum_{q=0}^{\infty} \frac{\psi^{2q} 2^{-2q}}{q!(q+|m-l|)!} \sum_{k=0}^{\infty} \epsilon_k h_{k,m,|m-l+2q}^2, \tag{3.4}$$

since $r_0(0) = 1$. Here also we see an illustration of the theorem stated previously, cf. (2.32) and Appendix II, namely, that the power in a spectral region, here an harmonic zone rather than the entire output spectrum, is independent of the spectral shape of the input noise and also of the signal spectrum. This naturally simplifies the calculation of the power, once $R_l(t)$ has been found.

4. The biased ν th-law rectifier. In the preceding section certain general aspects of the theory of modulated signals and noise passed through non linear devices have been outlined. Specific results in each instance depend on the evaluation of integrals of the form (2.20). Here attention is directed to the important case of the biased ν th-law detector, whose dynamic characteristic $g(V)$ is given by

$$\begin{aligned} I = g(V) &= \beta(V - b_0)^\nu, & V > b_0, & \nu > 0 \\ &= 0, & V < b_0, & \end{aligned} \tag{4.1}$$

and β has suitable dimensions for the output to be a current when the incoming wave is a voltage; b_0 represents the cut-off voltage, measured from the operating point in the manner of Fig. 1. The Fourier transform of g is readily shown to be

$$f(iz) = \int_0^\infty e^{-(iz)V} g(V) dV = \frac{\beta \Gamma(\nu + 1) e^{-izb_0}}{(iz)^{\nu+1}}. \tag{4.2}$$

With the help of (4.2) the integrals (2.20) may now be determined, in a variety of ways leading to different, but equivalent results, the form of either depending on whether $\exp(-ib_0z)$ or $J_m(A_0z)$ is expanded in a series, followed in each instance by termwise integration. The former approach yields a series in $b \equiv b_0/\psi^{1/2}$, where the coefficients of the various powers of b are themselves series in the parameter $p(t) \equiv [A_0(t)]^2/2\psi$, while the latter is expressible as a series in $p(t)$, the coefficient of whose general term likewise is an expansion in b . The two developments, while valid for all values of the parameters b , $p(t)$, and ψ , vary in their usefulness for different ranges of b , $p(t)$, and ψ .

Let us consider first the case where $\exp(-izb_0)$ is given as a series. The general integral (2.20) becomes accordingly, with the aid of (4.2),

$$H_{mn}(t) = \frac{\beta \Gamma(\nu + 1)}{2\pi} \int_{\mathbf{C}} z^n e^{-izb_0} J_m(zA_0(t)) e^{-\psi z^2/2} dz / (iz)^{\nu+1} \tag{4.3}$$

$$\begin{aligned} &= \frac{\beta \Gamma(\nu + 1)}{2\pi} i^{-\nu-1} \sum_{k=0}^{\infty} \frac{(-i)^k b_0^k}{k!} \int_{\mathbf{C}} z^{k+n+\nu-1} J_m(zA_0(t)) e^{-\psi z^2/2} dz \\ &= \frac{\beta \Gamma(\nu + 1) e^{-(m+n)\pi i/2}}{2m!} \left(\frac{\psi}{2}\right)^{(\nu-n)/2} p(t)^{m/2} \\ &\quad \cdot \sum_{k=0}^{\infty} \frac{(-1)^k 2^{k/2} b^k F_1[(k+m+n-\nu)/2; m+1; -p(t)]}{k! \Gamma[(2+\nu-k-m-n)/2]}, \end{aligned} \tag{4.4}$$

this last from Appendix III, Eq. (A3.15), where also are mentioned some of the prop-

erties of the confluent hypergeometric function ${}_1F_1$. For an unmodulated carrier $p(t)$ is merely a constant $p = A_0^2/2\psi$. Observe also that when $(2 + \nu - k - m - n)$ is zero or an even *negative* integer the modulation products contributed by such terms vanish, in virtue of the poles of $\Gamma[(2 + \nu - k - m - n)/2]$. Further, when ν is integral, only odd values of k appear, $m + n$ even, and only even values of k when $m + n$ is odd (except for $2 + \nu > k + m + n > 0$). Equation (4.4) is best suited for computation when $b = b_0/\psi^{1/2}$ is of the order of unity or less, even for a wide range of values of $p(t)$. Values of $b > 1$ yield too slow a convergence, and the alternative development is then needed.

The expansion of the Bessel function in (4.3) gives us

$$H_{mn}(t) = \frac{\beta\Gamma(\nu + 1)}{2\pi} i^{-\nu-1} \sum_{k=0}^{\infty} \frac{(-1)^k A_0(t)^{2k+m}}{k!(k+m)!2^{2k+m}} \int_C z^{2k+m+n-\nu-1} e^{-izb_0-\psi z^2/2} dz,$$

and from Eq. (A3.17) we obtain finally

$$H_{mn}(t) = \frac{\beta\Gamma(\nu + 1)}{2} e^{-\pi i(m+n)/2} \left(\frac{\psi}{2}\right)^{(\nu-n)/2} p(t)^{m/2} \sum_{k=0}^{\infty} \alpha_{mnk} p(t)^k, \tag{4.5}$$

where

$$\alpha_{mnk} = \frac{1}{k!(k+m)!} \left\{ \frac{{}_1F_1[(2k+m+n-\nu)/2; 1/2; -b^2/2]}{\Gamma[(2+\nu-2k-m-n)/2]} - \sqrt{2} b \frac{{}_1F_1[(2k+m+n-\nu-1)/2; 3/2; -b^2/2]}{\Gamma[(1+\nu-2k-m-n)/2]} \right\}. \tag{4.6}$$

Here large values of b are clearly more easily handled, especially when the asymptotic series for ${}_1F_1$, cf. (A3.3), can be used. Also equations (4.5) and (4.6) in general offer a more satisfactory form from which to determine the time average over the phases of the modulation, viz:

$$\begin{aligned} & [H_{mn}(t'_0)H_{mn}(t'_0 + t)]_{av.} \\ &= \frac{\beta^2\Gamma(\nu + 1)^2(-1)^{m+n}}{4} \left(\frac{\psi}{2}\right)^{\nu-n} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \alpha_{mnk_1}\alpha_{mnk_2} [p(t'_0)^{m/2+k_1}p(t'_0 + t)^{m/2+k_2}]_{av.}, \end{aligned} \tag{4.7}$$

when the spectrum of the output is desired, cf. Eqs. (2.21), (2.22), since a double series is sufficient, whereas if (4.4) were used, a fourfold infinite series would result for the average. A similar superiority of (4.7) over (4.4) is noted in the limiting case of $p(t) \ll 1$. Observe also from (A3.9) that for integral values of $\nu (> 0)$ the various α_{mnk} reduce to the functions defined in Appendix III (A3.4), and are thus expressible as Hermitian polynomials.

A detailed discussion of the above in the case of the biased linear rectifier or mixer is reserved for a later paper.

Limiting cases are of interest: We consider first:

Case I: $b_0 \rightarrow 0, A_0(t), \psi \neq 0$ and finite. Equations (4.4) and (4.7) reduce, equivalently, to

$$\begin{aligned} H_{mn}(t)_{b_0 \rightarrow 0} &= \frac{\beta\Gamma(\nu + 1)}{2m!} e^{-(m+n)\pi i/2} \left(\frac{\psi}{2}\right)^{(\nu-m)/2} p(t)^{m/2} \left\{ \frac{{}_1F_1[(n+m-\nu)/2; m+1; -p(t)]}{\Gamma[(2+\nu-m-n)/2]} \right. \\ &\quad \left. - b\sqrt{2} \frac{{}_1F_1[(1+m+n-\nu)/2; m+1; -p(t)]}{\Gamma[(1+\nu-m-n)/2]} + \dots \right\}. \end{aligned} \tag{4.8}$$

From (4.8) the expressions for half-wave rectification, $b = 0$, follow immediately.

Case II: $b_0 \rightarrow \pm \infty$, $A_0(t)$, ψ finite. Here it is convenient to use the second form (4.5) of $H_{mn}(t)$ and apply the asymptotic series (A3.3) for ${}_1F_1$. It then may be shown that

$$H_{mn}(t)_{b_0 \rightarrow \infty} \cong 0. \tag{4.9}$$

since $\alpha_{mnk} \rightarrow 0$ for all terms in the asymptotic development. This is to be expected physically, since with an indefinitely large cut-off voltage the finite signal amplitudes and even the possibly infinite noise peaks are not transmitted, the latter because they arise a vanishingly small fraction of the time, since the mean power in the input signal and noise waves is finite. On the other hand, when $b_0 \rightarrow -\infty$, we find from (A3.3) and (4.5) that

$$H_{mn}(t)_{b_0 \rightarrow -\infty} \cong \frac{\beta \Gamma(\nu + 1) e^{-(m+n)\pi i/2} \sqrt{\pi} 2^{n-\nu} |b_0|^{-m-n+\nu} A_0(t)^m}{m! \Gamma[(1 + \nu - m - n)/2] \Gamma[(2 + \nu - m - n)/2]} \times \left\{ 1 + \frac{\psi}{2b_0^2} \left[1 - \frac{A_0^2(t)}{2\psi(m+1)} \right] (m+n-\nu+1)(m+n-\nu) + \dots \right\}, \tag{4.10}$$

which shows that for $\nu > m+n$ the output contains increasingly large terms, while for $\nu < m+n$ all such contributions become negligibly small. As an example, consider the linear rectifier, $\nu = 1$. Here only the terms $(m, n) = (0, 0), (0, 1), (1, 0)$ are significant:

$$H_{00} = \beta |b_0|; \quad H_{01} = -i\beta; \quad H_{10}(t) = -i\beta A_0(t)/2; \quad H_{mn}(t) \rightarrow 0, \quad m+n > 1. \tag{4.10a}$$

The first relation becomes infinite, since infinite (negative) cut-off voltage must be supplied when $b_0 \rightarrow -\infty$; the second result, when substituted in (2.21) and (2.22), gives the mean power spectrum of the output noise, which is easily seen to be the same as the input spectrum except for the constant factor β , and finally, the third expression is observed to be the mean power spectrum of the input signal, undistorted because now the linear rectifier is essentially a linear amplifier. In general (4.10) may be said to hold for a small-signal ν th-law detector (see Sec. 7).

Case III: $A_0(t) \rightarrow 0$, b_0 , ψ finite. Here the modulation or carrier contribution is allowed to become small. Then the principal term in $H_{mn}(t)$ obtains when $m=0$, and correction terms when $m=0, k=1$, $m=1, k=0$, and $m=2, k=0$. Either (4.4) or (4.5) reduces to the following

$$\begin{aligned} H_{0n}(t)_{p(t) \rightarrow 0} &= \frac{\beta \Gamma(\nu + 1) e^{-\pi n i/2} \left(\frac{\psi}{2}\right)^{(\nu-n)/2}}{2} \left\{ \alpha_{0n0} + \alpha_{0n1} p(t) + \dots \right\}, \\ H_{1n}(t) &= - \frac{i\beta \Gamma(\nu + 1) e^{-\pi n i/2} \left(\frac{\psi}{2}\right)^{(\nu-n)/2}}{2} \left\{ p(t)^{1/2} \alpha_{1n0} + \dots \right\}, \\ H_{2n}(t) &= - \frac{\beta \Gamma(\nu + 1) e^{-\pi n i/2} \left(\frac{\psi}{2}\right)^{(\nu-n)/2}}{2} \left\{ p(t) \alpha_{2n0} + \dots \right\}. \end{aligned} \tag{4.11}$$

For noise alone we have

$$H_{0n}(t) = h_{0n} = \frac{\beta \Gamma(\nu + 1) e^{-\pi n i/2} \left(\frac{\psi}{2}\right)^{(\nu-n)/2}}{2} \left\{ \frac{{}_1F_1[(n - \nu)/2; 1/2; -b^2/2]}{\Gamma[(2 + \nu - n)/2]} - \right.$$

$$- b\sqrt{2} \frac{{}_1F_1[(n - \nu + 1)/2; 3/2; - b^2/2]}{\Gamma[(1 + \nu - n)/2]} \Big\}, \quad (4.12)$$

and in the special cases $\nu = 1$ and $\nu = 2$ we obtain with the aid of (A3.9) the equations of Reference 9 when there is no saturation.

Case IV: $A_0(t) \rightarrow \infty, b_0, \psi$ finite. The signal is taken to be very large, and again the asymptotic relation (A3.3) is in order, this time applied to (4.4) as $p(t) \rightarrow \infty$. The result is

$$H_{mn}(t)_{A_0(t) \rightarrow \infty} \cong \frac{\beta \Gamma(\nu + 1) (-1)^{(m+n)/2} 2^{n-\nu-1} A_0(t)^{\nu-n}}{2\Gamma[(2 + \nu - n + m)/2]} \left\{ \frac{1}{\Gamma[(2 + \nu - n - m)/2]} - \frac{2b_0\Gamma[(2 + \nu - n + m)/2]}{A_0(t)\Gamma[(1 + \nu - n + m)/2](1 + \nu - n - m)/2} + \dots \right\}, \quad (4.13)$$

and including terms in $A_0(t)^{-1}$ we may write

$$H_{m0}(t) \cong \beta \Gamma(\nu + 1) 2^{-\nu-1} (-i)^m A_0(t)^\nu \left\{ \frac{1}{\Gamma[(2 + \nu + m)/2]\Gamma[(2 + \nu - m)/2]} - \frac{2b_0\Gamma(2 + \nu + m)}{A_0(t)\Gamma[(1 + \nu + m)/2]\Gamma[(1 + \nu + m/2)]} + \dots \right\},$$

$$H_{m1}(t) \cong i\beta \Gamma(\nu + 1) 2^{-\nu} (-i)^m A_0(t)^\nu \times \left\{ \frac{1}{\Gamma[(1 + \nu + m)/2]\Gamma[(1 + \nu - m)/2]A_0(t)} + \dots \right\}, \text{ etc.} \quad (4.14)$$

This shows the complete suppression of the noise when $A_0(t) \gg \psi^{1/2}$. Further, since the cut-off and r-m-s noise voltages are comparable in that they are far smaller than the signal, they, too, are negligible in their effect, and we have essentially half-wave rectification of a modulated signal alone. In the special instance of the low-frequency output of the half-wave linear detector, we have from (4.14), $m = 0, H_{00}(t) \cong \beta A_0(t)/\pi$, in agreement with Eq. (40) of reference 8.

Case V: $\psi \rightarrow \infty, b_0, A_0(t)$ finite. This corresponds to the case in which the noise overwhelms the signal and much exceeds the cut-off voltage. Equations (4.4) or (4.5) reduce to

$$H_{mn}(t)_{\psi \rightarrow \infty} \cong \frac{\beta \Gamma(\nu + 1)}{2m!} (-1)^{(m+n)/2} \psi^{(\nu-n-m)/2} 2^{(n-\nu-m)/2} A_0(t)^m \left\{ \frac{{}_1F_1[(n + m - \nu)/2; m + 1; - p(t)]}{\Gamma[(2 + \nu - m - n)/2]} - \frac{b_0\sqrt{2} {}_1F_1[(n + m + 1 - \nu)/2; m + 1; - p(t)]}{\psi^{1/2} \Gamma[(1 + \nu - m - n)/2]} + \frac{b_0^2}{\psi} \frac{{}_1F_1[(m + n + 2 - \nu)/2; m + 1; - p(t)]}{\Gamma[(\nu - m - n)/2]} + \dots \right\}, \quad (4.15)$$

and the significant terms are

$$\begin{aligned}
 H_{00}(t) &\cong \frac{\beta\Gamma(\nu + 1)\psi^{\nu/2}2^{-\nu/2}}{2\Gamma[(2 + \nu)/2]} \left\{ 1 + \frac{b_0\sqrt{2}\Gamma[(2 + \nu)/2]}{\psi^{1/2}\Gamma[(1 + \nu)/2]} \right. \\
 &\quad \left. + \frac{1}{\psi} \left[\frac{\nu A_0^2(t)}{4} + \frac{b_0^2\Gamma[(2 + \nu)/2]}{\Gamma(\nu/2)} \right] + \dots \right\}, \\
 H_{01}(t) &\cong - \frac{i\beta\Gamma(\nu + 1)\psi^{(\nu-1)/2}2^{-(\nu+1)/2}}{\Gamma[(\nu + 1)/2]} \left\{ 1 - \frac{b_0\sqrt{2}}{\psi^{1/2}} \frac{\Gamma[(1 + \nu)/2]}{\Gamma(\nu/2)} + \dots \right\}, \\
 H_{10}(t) &\cong - \frac{i\beta\Gamma(\nu + 1)\psi^{\nu/2}2^{-(\nu+3)/2}}{\Gamma[(1 + \nu)/2]} \left\{ \psi^{-1/2} - \frac{b_0\sqrt{2}}{\psi} \frac{\Gamma[(1 + \nu)/2]}{\Gamma(\nu/2)} + \dots \right\}, \\
 &\vdots \\
 H_{02}(t) &\cong - \beta\Gamma(\nu + 1)\psi^{(\nu/2)-1}2^{-\nu/2}/\Gamma(\nu/2); \\
 H_{20}(t) &\cong - \beta\Gamma(\nu + 1)\psi^{\nu/2} \cdot 2^{-(\nu/2)-3}A_0(t)^2/\psi\Gamma(\nu/2), \text{ etc.}
 \end{aligned} \tag{4.16}$$

Observe that the terms containing pure-signal or cross-term contributions, $n = 0$, or $m, n \neq 0$, vanish at least as $\psi^{-1/2}$, and only noise is left. Thus when the input noise voltage becomes sufficiently great in comparison with the signal, the latter is suppressed. In particular, for the half-wave linear and quadratic rectifier, the low-frequency amplitude of the output ($m = n = 0$) becomes from (4.16)

$$\begin{aligned}
 H_{00}(t) - (d - c) &= \frac{\beta\Gamma(\nu + 1)\psi^{\nu/2}}{2^{(\nu/2)+3}\Gamma[(\nu + 2)/2]} \left(\frac{\nu A_0^2}{\psi} \right) = \frac{\beta\psi^{1/2}}{2\sqrt{2}\pi} [A_0^2(t)/2\psi]_{\nu=1, b_0=0, p(t) \ll 1} \\
 &= \beta[A_0^2(t)/2]_{\nu=2, b_0=0},
 \end{aligned} \tag{4.17}$$

the first of which is Eq. (39) of reference 8, and the second, Eq. (29).

Case VI: $\psi \rightarrow 0, b_0, A_0(t)$ finite. Here it is convenient to distinguish two cases in the limit, one where $|b_0| > A_0(t) \geq 0$ and the other when $A_0(t) > |b_0| \geq 0$. For the first we use (4.5) and (4.6) as $b \rightarrow \infty$, along with (A3.3) to obtain the not unexpected result that

$$H_{nn}(t)_{\psi \rightarrow 0} \cong 0, \quad b_0 > A_0(t) \geq 0, \tag{4.18}$$

since the cut-off voltage exceeds the input signal (envelope), so that the wave is not passed. However, when $|b_0| > A_0(t), b_0 < 0$, we have a different situation, where now the incoming disturbance, which is essentially pure signal (hence $n = 0$ in the limit $\psi = 0$), is transmitted without distortion due to cut-off or saturation effects, albeit with distortion due to the non linear nature of the dynamic path, $\nu \neq 1$. The expression for the amplitude is accordingly

$$H_{m0}(t) = \frac{\beta\Gamma(\nu + 1)i^{-\nu-1}}{2\pi} \int_{\mathbf{c}} e^{-izb_0} J_m(A_0(t)z) dz/dz^{\nu+1}. \tag{4.19}$$

When $b_0 < 0, |b_0| > A_0(t)$ we may use (A3.8) in conjunction with (A3.19) to get finally

$$H_{m0}(t) = \frac{\beta\Gamma(\nu + 1)e^{-\pi mi/2}\sqrt{\pi} |b_0|^2}{2^\nu\Gamma[(2 + \nu - m)/2]\Gamma[(1 + \nu - m)/2]m!} \left[\frac{A_0(t)}{|b_0|} \right]^m {}_2F_1(m - \nu, m - \nu + 1; m + 1; A_0^2(t)/b_0^2), \quad b_0 < 0, \quad |b_0| > A_0(t), \quad \psi = 0. \quad (4.20)$$

On the other hand, when $|b_0| < A_0(t)$ we obtain from (4.4), including a correction term,

$$H_{mn}(t)_{\psi \rightarrow 0} = \beta\Gamma(\nu + 1)e^{-(m+n)\pi i/2} 2^{n-\nu-1} A_0(t)^{\nu-n} \cdot \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k 2^k b_0^k \{1 + \psi(m + n + k - \nu)(n - m + k - \nu)/2A_0(t)^2\}}{k! A_0(t)^k \Gamma[(2 + \nu - k - m - n)/2]\Gamma[(2 + \nu - k + m - n)/2]} + \dots \right\}, \quad (4.21)$$

and when $\psi = 0$ we may sum (4.21), $n = 0$, or use (A3.19) to get

$$H_{m0}(t)_{\psi=0} = \frac{\beta\Gamma(\nu + 1)}{2^{\nu+1}} e^{-m\pi i/2} A_0(t)^\nu \left\{ \frac{{}_2F_1[(m - \nu)/2, (-m - \nu)/2; \frac{1}{2}; b_0^2/A_0(t)^2]}{\Gamma[(2 + \nu + m)/2]\Gamma[(2 + \nu - m)/2]} - \frac{2b_0}{A_0(t)} \frac{{}_2F_1[(m - \nu + 1)/2, (-m - \nu + 1)/2; \frac{3}{2}; b_0^2/A_0(t)^2]}{\Gamma[(1 + \nu + m)/2]\Gamma[(1 + \nu - m)/2]} \right\}, \quad (4.22)$$

$0 \leq |b_0| < A_0(t).$

For half-wave detection we have the simple result

$$H_{m0}(t)_{b_0=0, \psi=0} = \frac{\beta\Gamma(\nu + 1)}{2^{\nu+1}} \frac{e^{-m\pi i/2} A_0(t)^\nu}{\Gamma[(2 + \nu + m)/2]\Gamma[(2 + \nu - m)/2]}. \quad (4.23)$$

Power and spectra may be obtained in the usual manner (see Sec. 2).

5. Probability density of the envelope and phase of a modulated signal and noise.

In handling the problem of passage of noise or a signal and noise through a non linear device, it is sometimes convenient first to determine the various first and second order probability densities associated with the incoming wave, and then with their help derive the expressions for the mean power output and the correlation function associated with the transmitted disturbance, after it has been modified by a rectifier or similar non linear apparatus, cf. Eq. (1.7b), for example. Now in particular when the noise and signal are narrow-band, *vide* Sec. 3, this method, alternative to the use of the characteristic function [Eqs. (2.10)–(2.16)] suggests itself. The purpose of this section and of section 6 following is to obtain explicit expressions for the probability density of the envelope and phase of the general type of modulated carrier in the presence of narrow-band noise. Only in such circumstances may one properly speak of an envelope or of phase—i.e., when the part of the incoming wave due to noise and modulation is essentially slowly varying in comparison with the carrier frequency f_0 and the central frequency f_c of the noise band. After rectification it is usually this envelope, the low-frequency part of the disturbance, that is observed.

The input is

$$V(t) = A_0(t) \cos \omega_0 t + V(t)_N$$

$$= (A \cos \omega_d t + V_c) \cos \omega_c t + (V_s - A \sin \omega_d t) \sin \omega_c t, \quad (5.1)$$

where f_d is the difference frequency $f_d = f_c - f_0$ and $A = A_0(t)$, from (2.1) and (2.2). Here V_c is the component of the noise "in phase" with $\cos \omega_c t$ and V_s is the component "in phase" with $\sin \omega_c t$, e.g.:

$$V_c = \sum_{n=1}^{\infty} (a_n \cos \omega_n' t + b_n \sin \omega_n' t),$$

$$V_s = \sum_{n=1}^{\infty} (-a_n \sin \omega_n' t + b_n \cos \omega_n' t); \quad \omega_n' = \omega_n - \omega_c,$$
(5.2)

respectively. By the envelope is meant

$$E(t) = [(A \cos \omega_d t + V_c)^2 + (V_s - A \sin \omega_d t)^2]^{1/2},$$
(5.3)

which is a slowly varying function of the time provided $f_d \ll f_0, f_c$ i.e., the centers of the noise and signal spectra are not too far apart in frequency. The quantities A, V_c, V_s are, of course, relatively low-frequency disturbances.

Our first task is to obtain the joint probability density of E_1 and E_2 , where E_1 is the envelope at time t_0 and E_2 its value at some later time $t_0 + t$. We derive the distribution of $V_{c1}, V_{c2}, V_{s1}, V_{s2}$, and V_{s2} initially in rectangular coordinates. Letting

$$X_1 = V_{c1} = \sum_{n=1}^{\infty} (a_n \cos \omega_n' t_0 + b_n \sin \omega_n' t_0);$$

$$X_3 = V_{c2} = \sum_{n=1}^{\infty} [a_n \cos \omega_n' (t_0 + t) + b_n \sin \omega_n' (t_0 + t)]$$

$$X_2 = V_{s1} = \sum_{n=1}^{\infty} (-a_n \sin \omega_n' t_0 + b_n \cos \omega_n' t_0);$$

$$X_4 = V_{s2} = \sum_{n=1}^{\infty} [-a_n \sin \omega_n' (t_0 + t) + b_n \cos \omega_n' (t_0 + t)],$$
(5.4)

with the help of (2.3), (2.4), and the generalized Gaussian distribution (A.3), we observe that

$$\overline{X_1^2} = \overline{X_2^2} = \overline{X_3^2} = \overline{X_4^2} = \int_0^{\infty} w(f) df \equiv \psi(0) = \mu_{11} = \mu_{22} = \mu_{33} = \mu_{44},$$
(5.5)

$w(f)$ as before being the mean power spectrum of the noise; for $\overline{X_1 X_3}$ and $\overline{X_2 X_4}$ we have

$$\overline{X_1 X_3} = \overline{X_2 X_4} = \int_0^{\infty} w(f) \cos (\omega - \omega_c) t df \equiv \psi_0(t) = r_0(t) \psi = \mu_{13} = \mu_{24};$$
(5.6)

(see Appendix II). It is also evident that $\overline{X_1 X_2} = \overline{X_3 X_4} = 0$, as there is no correlation between the "in-" and "out-of-phase" components of the noise. There remains finally

$$\overline{X_2 X_3} = -\overline{X_1 X_4} = \int_0^{\infty} w(f) \sin (\omega - \omega_c) t df \equiv \lambda_0(t) \psi = \mu_{23} = -\mu_{14}.$$
(5.7)

Accordingly, if we let $Y_1 = A \cos \omega_d t + V_c, Y_2 = -A \sin \omega_d t + V_s$, etc., the various Y_1, Y_2 ,

Y_3, Y_4 represent random variables distributed about the averages $A_0(t'_0) \cos \omega_d t_0, -A_0(t'_0) \sin \omega_d t_0, A_0(t'_0 + t) \cos \omega_d(t_0 + t),$ and $-A_0(t'_0 + t) \sin \omega_d(t_0 + t),$ where no correlation between the modulation and the carrier, as well as the noise, is assumed. Thus $A_1 = A_0(t'_0)$ represents the envelope modulation at a time $t'_0,$ not analytically related to $t_0,$ and $A_2 = A_0(t'_0 + t)$ is the same modulation a time t later. Equations (5.4) become

$$\begin{aligned} X_1 &= Y_1 - A_1 \cos \omega_d t_0; & X_2 &= Y_2 + A_1 \sin \omega_d t_0; \\ X_3 &= Y_3 - A_2 \cos \omega_d(t_0 + t); & X_4 &= Y_4 + A_2 \sin \omega_d(t_0 + t). \end{aligned} \tag{5.8}$$

Equation (A1.3), where $s=4,$ gives us the joint distribution $W_2(X_1 X_2; X_3 X_4; A_1 A_2; t),$ for which the fundamental matrix is

$$\boldsymbol{\mu} = \|\mu_{ki}\| = \begin{bmatrix} \psi & 0 & r_0\psi & -\lambda_0\psi \\ 0 & \psi & \lambda_0\psi & r_0\psi \\ r_0\psi & \lambda_0\psi & \psi & 0 \\ -\lambda_0\psi & r_0\psi & 0 & \psi \end{bmatrix}. \tag{5.9}$$

From (5.5)–(5.7) and from (5.9) the determinant $|\boldsymbol{\mu}|$ and the cofactors μ^{ki} follow at once. Since (5.1) may be written $V(t) = Y_1 \cos \omega_c t + Y_2 \sin \omega_c t,$ the envelopes of the wave at the two times t'_0 and $t'_0 + t$ are respectively

$$E_1 = (Y_1^2 + Y_2^2)^{1/2} \quad \text{and} \quad E_2 = (Y_3^2 + Y_4^2)^{1/2}, \tag{5.10}$$

and following the example of Rice² and others, we transform to polar coordinates with the help of

$$Y_1 = E_1 \cos \theta_1; \quad Y_2 = E_1 \sin \theta_1; \quad Y_3 = E_2 \cos \theta_2; \quad Y_4 = E_2 \sin \theta_2, \tag{5.11}$$

for which the Jacobian is easily shown to be $E_1 E_2.$

The probability density W_2 becomes finally

$$\begin{aligned} &W_2(E_1 E_2; \theta_1 \theta_2; A_1 A_2; t) \\ &= \frac{E_1 E_2}{4\pi^2 \psi (1 - \lambda_0^2 - r_0^2)} \exp \left[- \frac{1}{2\psi (1 - \lambda_0^2 - r_0^2)} \{ E_1^2 + E_2^2 + A_1^2 + A_2^2 \right. \\ &\quad - 2r_0 E_1 E_2 \cos (\theta_2 - \theta_1) - 2\lambda_0 E_1 E_2 \sin (\theta_2 - \theta_1) \\ &\quad - 2r_0 A_1 A_2 \cos \omega_d t + 2\lambda_0 A_1 A_2 \sin \omega_d t \\ &\quad + 2E_2 \sin \theta_2 [A_2 \sin \omega_d t_2 - r_0 A_1 \sin \omega_d t_1] \\ &\quad + 2E_1 \sin \theta_1 [A_1 \sin \omega_d t_1 - r_0 A_2 \sin \omega_d t_2] \\ &\quad + 2\lambda_0 [E_2 A_1 \sin \theta_2 \cos \omega_d t_1 - E_1 A_2 \sin \theta_1 \cos \omega_d t_2] \\ &\quad + 2\lambda_0 [E_2 A_1 \cos \theta_2 \sin \omega_d t_1 - E_1 A_2 \cos \theta_1 \sin \omega_d t_2] \\ &\quad - 2E_1 \cos \theta_1 [A_1 \cos \omega_d t_1 - r_0 A_2 \cos \omega_d t_2] \\ &\quad \left. - 2E_2 \cos \theta_2 [A_2 \cos \omega_d t_2 - r_0 A_1 \cos \omega_d t_1] \right\} \Big], \end{aligned} \tag{5.12}$$

where $t_1 = t_0, t_2 = t_0 + t.$ It should be pointed out that W_2 as given by (5.12) is not

purely a probability density, in as much as averages over the phases of the modulation and over the phases of the difference frequency terms involving ω_{d1} , ω_{d2} remain to be taken. Since the modulation and difference terms are not correlated, these averages are independent and, for example, may be determined in the manner of (2.12). Although periodic disturbances are not, strictly speaking, random, they may be treated as such on assuming a random distribution of the time origin with respect to an hypothetical observer. In this way the phase may be considered a random variable uniformly distributed between 0 and 2π , and a statistical average accordingly may be performed for the function containing this random variable. See section 3.10 of reference 2 for a more detailed discussion.

It is evident at once from (5.12) that the presence of a modulated signal introduces considerable mathematical complexity, and comparison with the results of Secs. 2 and 4 would indicate that this form of treatment is perhaps in most cases not so expeditious. Further, it is restricted to narrow-band disturbances, while the approach of Sections 2 to 4 is more general. However, a considerable simplification in (5.12) is possible if we observe that for $\omega' = \omega - \omega_0$, Eq. (5.7) becomes essentially zero, i.e., $\lambda_0(t) \doteq 0$, since the input spectrum is narrow-band and symmetrically distributed about $f = f_c$ and f_c heavily exceeds the effective bandwidth of the noise, as now the contribution from 0 to $-\omega_c$ nearly cancels the integral from 0 to ∞ . It follows that all terms in (5.12) containing $\lambda_0(t)$ may then be safely discarded.

The case of greatest practical interest occurs when the carrier and central noise frequencies coincide; then $\omega_d = 0$, and with the usual condition $\lambda_0(t) = 0$ we have for W_2

$$W_2 = \frac{E_1 E_2}{4\pi^2 \psi^2 (1 - r^2)} \exp - (A_1^2 + A_2^2 - 2r_0 A_1 A_2) / 2\psi (1 - r_0^2) \\ \cdot \exp - [E_1^2 + E_2^2 - 2r_0 E_1 E_2 \cos(\theta_2 - \theta_1) \\ - 2E_1(A_1 - r_0 A_2) \cos \theta_1 - 2E_2 \cos \theta_2 (A_2 - r_0 A_1)] / 2\psi (1 - r_0^2). \quad (5.13)$$

Notice that when $t \rightarrow \infty$ we obtain the square of the first-order density W_1 . Then since $r_0(\infty) = 0$, $\lambda_0(\infty) = 0$, Eq. (5.12) transforms in the more general case where $\omega_d \neq 0$ to

$$\lim_{t \rightarrow \infty} (W_2) = W_1^2 = \left\{ \frac{E}{2\pi\psi} \exp - [A_0(t)^2 + E^2 - 2EA_0(t) \cos \phi] / 2\psi \right\}^2, \quad (5.14)$$

in which $\phi = \theta + \omega_d t$ is a new phase angle. The probability density of the envelope E is found by integrating over all phases θ , or ϕ , between 0 and 2π . The result is

$$W_1(E, A_0(t)) = (E/\psi) \exp - I_0(EA_0(t)/\psi) [A_0(t)^2 + E^2] / 2\psi, \quad (5.15)$$

a generalization of a result derived independently by Goudsmit,²⁷ North,⁷ Rice,² and others, when the carrier is not modulated, i.e., when $A_0(t) = A_0$. The complete first-order probability density requires the average over the phases of the modulation, in the manner explained following (5.12). When there is no signal we have the well-known expression $W_1(E) = E\psi^{-1} \exp(-E^2/2\psi)$. Figure 4b illustrates some typical distributions of W_1 for various values of the ratio $A_0/\psi^{1/2}$, or $\sqrt{2}p^{1/2}$. As we would

²⁷ S. A. Goudsmit, *Comparison between signal and noise*, M.I.T. Rad. Lab. Report 43-21, Jan. 29, 1943.

expect, the presence of the signal shifts the average and the most probable values of the distribution to larger values of the envelope voltage, such that for sufficiently strong signals the most probable value of E coincides with the peak amplitude A_0 .

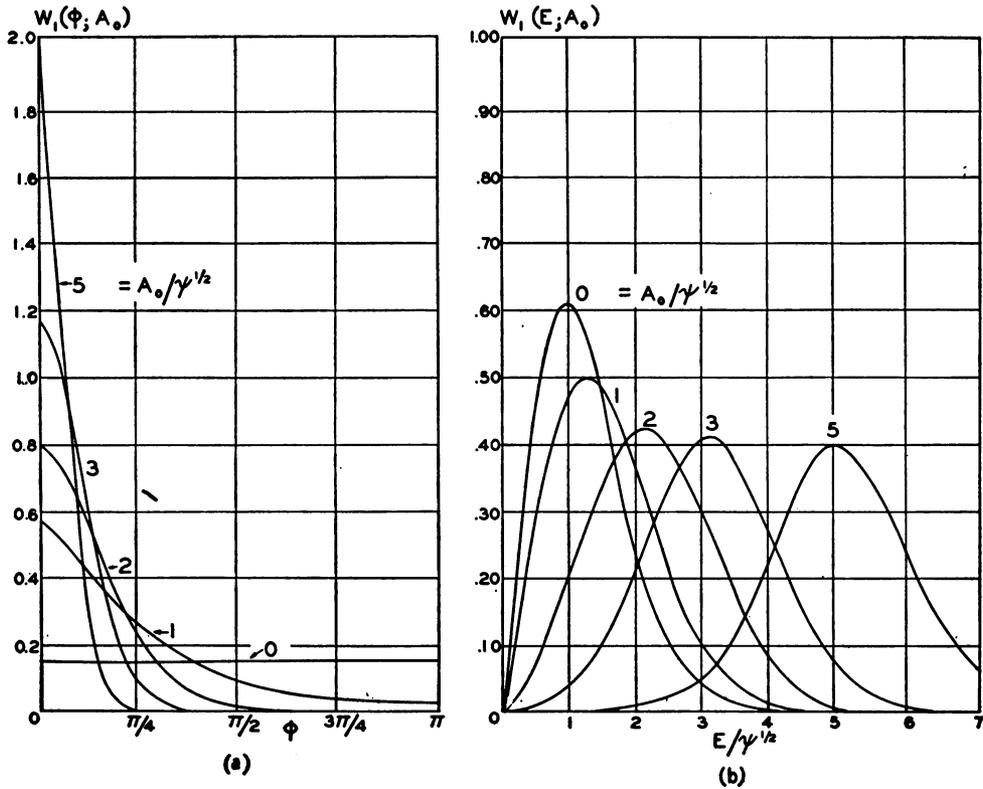


FIG. 4. Curves (a): the probability density of the phase ϕ when the carrier is unmodulated. Curves (b): the probability density of the envelope E of an unmodulated carrier and narrow-band noise, for various ratios of peak signal to r-m-s noise amplitudes, as in the case of (a).

The probability density $W_1(\theta, A_0(t))$ for the phases follows in a similar manner. From (5.14) we have after integration over E with the help of (A3.2) and (A3.7)

$$W_1(\theta, A_0(t)) = \frac{1}{2\pi} e^{-(A_0(t)\sin\phi)^2/2\psi} \left\{ \left(\frac{\pi}{2\pi} \right)^{1/2} A_0(t) \cos\phi + {}_1F_1 \left[-\frac{1}{2}; \frac{1}{2}; -(A_0(t)\cos\phi)^2/2\psi \right] \right\}, \quad (5.16)$$

and again to determine the pure distribution $W_1(\theta)$ we must average over the phases of the modulation $A_0(t)$ as described above. The expression $\phi = \theta + \omega at$ introduced first in (5.14) is a new phase variable, shifted from the origin of θ by an amount that increases linearly with the time; Eq. (5.16) might be considered a periodically fluctuating probability density. However, the statistics of the phase θ or ϕ are not radically affected; the density functions are the same in either case, only the average values of θ and ϕ are different in the interval $0 - 2\pi$. Note that when there is no signal

$W_1(\theta) = 1/2\pi$, as we would expect. Figure 4a illustrates the variation of the phase density for an unmodulated carrier where the ratio $A_0/\psi^{1/2}$ takes the same values as in the example of Fig. 4b for the envelope. For no signal the distribution is uniform; as the signal strength is increased relative to the r-m-s noise voltage the phases are grouped progressively closer about the 0° -phase of the carrier, until for no noise, or what is the same thing here, for an overwhelming signal, the density becomes a delta-function at $\phi = 0$. Strong signals thus dominate the distribution of the phases for the noise, and strong signals tend to obliterate the random phasing. The curves of Fig. 4a are symmetrical about $\phi = 0$, the complete interval being $-\pi < \phi \leq \pi$ or $0 < \phi \leq 2\pi$.

The second-order probability density W_2 for the envelopes E_1 and E_2 , or the phases θ_1 and θ_2 may be determined in the same way that W_1 was. We have finally²⁸

$$\begin{aligned}
 &W_2(E_1E_2; A_1A_2; t) \\
 &= \frac{E_1E_2}{\psi^2(1 - r_0^2)} \exp - [E_1^2 + E_2^2 + A_1^2 + A_2^2 - 2A_1A_2r_0]/2\psi(1 - r_0^2) \\
 &\times \sum_{m=0}^{\infty} \epsilon_m I_m \left(\frac{r_0 E_1 E_2}{\psi(1 - r_0^2)} \right) I_m \left(\frac{A_1 - r_0 A_2}{\psi(1 - r_0^2)} E_1 \right) I_m \left(\frac{A_2 - r_0 A_1}{\psi(1 - r_0^2)} E_2 \right). \tag{5.17}
 \end{aligned}$$

A similar but more complex result obtains from (5.12) if $\omega_d \neq 0$. When there is no modulation (5.17) may be more simply written

$$\begin{aligned}
 W_2(E_1E_2; t) &= \frac{E_1E_2}{\psi^2(1 - r_0^2)} \exp \left[\frac{-A_0^2}{\psi(1 + r_0)} - (E_1^2 + E_2^2)/2\psi(1 - r_0^2) \right] \\
 &\cdot \sum_{m=0}^{\infty} \epsilon_m I_m \left(\frac{r_0 E_1 E_2}{\psi(1 - r_0^2)} \right) I_m \left(\frac{A_0 E_1}{\psi(1 + r_0)} \right) I_m \left(\frac{A_0 E_2}{\psi(1 + r_0)} \right), \tag{5.18}
 \end{aligned}$$

and for no signal at all we have

$$W_2(E_1E_2; t) = \frac{E_1E_2}{\psi^2(1 - r_0^2)} \exp - (E_1^2 + E_2^2)/2\psi(1 - r_0^2) I_0 \left(\frac{r_0 E_1 E_2}{\psi(1 - r_0^2)} \right), \tag{5.19}$$

a result originally derived by Uhlenbeck²⁹ and used by him to determine the correlation function $R(t) = \overline{E_1 E_2}$ for a half-wave linear rectifier when random noise alone is detected.

The probability density for the phases θ_1 and θ_2 , $\omega_d = 0$, follows in the same way from (5.13), the final result taking the form

$$\begin{aligned}
 &W_2(\theta_1\theta_2; A_1A_2; t)_{\omega_d=0} = [4\pi^2(1 - r_0^2)]^{-1} \\
 &\exp [- (A_1^2 + A_2^2 - 2r_0A_1A_2)(1 - r_0^2 - \cos^2 \theta_1 - \cos^2 \theta_2)/2\psi(1 - r_0^2)] \\
 &\times \sum_{m=0}^{\infty} \left[\frac{2r_0 \cos(\theta_2 - \theta_1)}{1 - r_0^2} \right]^m \frac{1}{m!} \left\{ a_1(2\psi)^{1/2} \Gamma \left(\frac{m+3}{2} \right) {}_1F_1 \left(\frac{-m}{2}; \frac{3}{2}; -a_1^2/4 \right) \right\}
 \end{aligned}$$

²⁸ The result (5.17) has been obtained independently by Rice (communication to the author), for the less general case of an unmodulated carrier, $A_1 = A_2 = A_0$.

²⁹ G. E. Uhlenbeck, *Theory of the random process*, M.I.T. Laboratory Report #454, Oct. 15, 1943.

$$\begin{aligned}
& + \Gamma\left(\frac{m}{2} + 1\right) {}_1F_1\left(-\frac{m+1}{2}; \frac{1}{2}; -a_1^2/4\right) \Big\} \\
& \times \left\{ a_2(2\psi)^{1/2} \Gamma\left(\frac{m+3}{2}\right) {}_1F_1\left(-\frac{m}{2}; \frac{3}{2}; -a_2^2/4\right) \right. \\
& \left. + \Gamma\left(\frac{m}{2} + 1\right) {}_1F_1\left(-\frac{m+1}{2}; \frac{1}{2}; -a_2^2/4\right) \right\}, \tag{5.20}
\end{aligned}$$

$a_1 = b_1 \cos \theta_1$, $a_2 = b_2 \cos \theta_2$, (for b_1 , b_2 see Sec. 6).

6. The correlation function and mean power for the envelope of a signal and noise.

The results of the preceding section may be used in (1.7b) to give us an expression for the correlation function and the mean power associated with the *envelope* of an incoming signal and noise wave. Our primary interest is with the low-frequency output, or detected envelope, but the general treatment will first be outlined in brief fashion below, before returning to a more detailed examination of the former.

For the general non linear device we follow the suggestion of Rice (ref. 2, Sec. 4.3) and use the Fourier transform (1.8) to represent our output current $I = g(V)$ as a function of the input disturbance $V = E \cos(\omega_c t - \theta)$. Expanding the exponential in a series of Bessel functions gives us

$$I = \sum_{l=0}^{\infty} I_l = \sum_{l=0}^{\infty} B_l(E) \cos l(\omega_c t - \theta), \tag{6.1}$$

where $B_l(E)$ represents the envelope of the l th output band, viz:

$$B_l(E) = \frac{i^l \epsilon_l}{2\pi} \int_{\mathbf{c}} f(iz) J_l(Ez) dz, \tag{6.2}$$

which in principle may be observed when all other contributions are eliminated by an appropriate band-pass filter, centered about the l th zone and followed by a linear half-wave rectifier or "envelope tracer," as it is sometimes called. If $W_1(E)$ is the probability density of the input envelope, the density function for the l th region is

$$D_{1l}(E) = B_l(E) W_1(E) = W_1(B_l) B_l(E) dB_l/dE, \tag{6.3}$$

since

$$W_1(B_l) = W_1(E) dE/dB_l, \tag{6.4}$$

where $W_1(B_l)$ is the density function of B_l . This latter quantity, $W_1(B_l)$, may be found from (5.15) and from (6.2) by differentiation. For higher order densities we have

$$W_2(B_{1l}, B_{2l}; t) = W_2(E_1 E_2; t) \left/ \left| \frac{\partial(B_{1l} B_{2l})}{\partial(E_1 E_2)} \right| \right., \text{ etc.} \tag{6.4a}$$

An important consequence of (6.4) and (6.4a) is that these results enable us to determine the probability density of the envelope of a distribution after successive non linear operations, *provided*, of course, that in each operation the concept of the envelope remains, i.e., we have a narrow band wave undergoing rectification and not one whose mean frequency is comparable with the carrier.

The correlation function and the mean power may be written in a similar manner. We have from (2.12), (5.12), and (6.1) for the complete correlation

$$R(t) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} T_0^{-1} \int_0^{T_0'} dt_0' \int_0^{T_0} T_0^{-1} dt_0 \int_0^{\infty} dE_1 \int_0^{\infty} dE_2 \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 B_l(E_1) B_m(E_2) W_2(E_1 E_2; \theta_1 \theta_2; A_1 A_2; t) \cos l(\omega_c t_0 - \theta_1) \cos m[\omega_c(t_0 + t) - \theta_2], \quad (6.5)$$

which simplifies considerably when the noise band is symmetrical, i.e., $\lambda_0 \doteq 0$, the carrier is unmodulated, and $\omega_d = 0$. The spectrum may be found from (1.6). The mean power associated with the l th harmonic region is given when $t=0$ in (6.5), but an alternative and sometimes simpler expression may be obtained with the help of $W_1(E)$ and (2.33a), applied to (6.1), (6.2) after averaging over the phases of the zone central frequencies $l\omega_c$.

Specifically, let us examine the interesting case of the ν th-law half-wave rectifier when a modulated signal and narrow-band noise together enter the apparatus. This is essentially the receiver problem, as we are concerned only with the low-frequency output—namely, the modified envelope. Then it is in keeping with the problem to set $\omega_d = \lambda_0(t) = 0$, the latter on the assumption of a symmetrical spectrum, and we find for the noise contribution to the correlation function as indicated by the subscript N , cf. 2(.11), from (5.17) in (6.5),

$$R(t)_N = \int_0^{\infty} dE_1 \int_0^{\infty} dE_2 B_0(E_1) B_0(E_2) W_2(E_1 E_2; A_1 A_2; t) \\ = \gamma^2 f(A_1 A_2; r_0) \sum_{m=0}^{\infty} \epsilon_m \int_0^{\infty} dE_1 \int_0^{\infty} dE_2 (E_1 E_2)^{\nu+1} I_m \left(\frac{E_1 E_2 r_0}{\psi(1-r_0^2)} \right) I_m(b_1 E_1) I_m(b_2 E_2) \\ \times e^{-(E_1^2 + E_2^2)/2\psi(1-r_0^2)} \quad (6.6)$$

where $f(A_1, A_2; r_0)$ is given by

$$f(A_1, A_2; r_0) = \psi^2(1-r_0^2)^{-1} \exp [(-A_1^2 - A_2^2 + 2r_0 A_1 A_2)/2\psi(1-r_0^2)], \quad \text{and} \\ b_1 = \frac{A_1 - r_0 A_2}{\psi(1-r_0^2)}; \quad b_2 = \frac{A_2 - r_0 A_1}{\psi(1-r_0^2)}; \quad B_0(E_1) = \gamma E_1^{\nu}, \quad \text{etc.} \quad (6.6a)$$

The scale factor γ is related to the tube factor β used throughout this paper by

$$\gamma = \frac{\beta}{2\pi} \int_0^{\pi} \sin^{\nu} \theta d\theta = \frac{\beta}{2\sqrt{\pi}} \frac{\Gamma[(\nu+1)/2]}{\Gamma[(\nu/2+1)]}. \quad (6.7)$$

This follows because in (6.6) we are dealing with the *envelope* E rather than the amplitude V of the wave. Now the output current I consists of the positive halves of the oscillations of βV^{ν} , while the envelope of I is the same as that of βV^{ν} . But the area under the loops of I is to the area under βE^{ν} as the area under a loop of $\sin^{\nu} \theta$ is to an area of unit height and length 2π , so that as far as the low-frequency portion of the output is concerned the loops of I are "smeared" together into a current which varies as γE^{ν} , with γ given by (6.7) above. Compare the appropriate results of the present section with those of the next, where the problem is treated from the point of view of the instantaneous amplitude.

There appear to be three principal ways of evaluating $R(t)_N$, each yielding results different in form and usefulness. The first is achieved with the aid of the transformation

$$E_1 = [\psi(1 - r_0^2)]^{1/2} z^{1/2} e^{\phi/2}, \quad E_2 = [\psi(1 - r_0^2)]^{1/2} z^{1/2} e^{-\phi/2}, \quad (6.8)$$

followed by the successive application of the expression for the product of two Bessel functions (*cf.* (2), p. 148, ref. 21), the integral form of the modified Bessel function of the second kind ((7), p. 182, ref. 21), of argument $n - 2k$, and finally, by ((11), p. 410, ref. 21). The second form follows after expanding $I_m(E_1 E_2 r_0 / \psi(1 - r_0^2))$ and using termwise integration, with the help of (A3.7), and the third employs a contour integral representation of the Bessel functions in (6.6) applied to the addition formula for $I_0[(x^2 + y^2 + 2xy \cos \phi)^{1/2}]$, along with a reversal of the order of integration. We give the final results:

For the first method we have

$$\begin{aligned} R(t)_N &= \gamma^2 2^\nu \psi^\nu (1 - r_0^2)^{\nu+1} e^{-[A_1^2 + A_2^2 - 2r_0 A_1 A_2]^{1/2} \psi(1 - r_0^2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\epsilon_m r_0^m c_1^{m+2n} c_2^m}{2^{m+n} m!} \\ &\times \sum_{k=0}^n \frac{(c_1^{-1} c_2)^{2k} \Gamma(\nu/2 + k + m + 1) \Gamma(\nu/2 - k + m + n + 1)}{k!(k+m)!(n-k)!(m+n-k)!} \\ &\times {}_2F_1\left(\frac{\nu}{2} + k + m + 1; \frac{\nu}{2} - k + m + n + 1; m + 1; r_0^2\right), \quad (6.9) \\ c_1 &= b_1 [\psi(1 - r_0^2)]^{1/2}, \quad c_2 = b_2 [\psi(1 - r_0^2)]^{1/2}. \end{aligned}$$

In the case of modulation we must now apply (2.12) to determine the average over the phases $\omega_A t'_0$. As one can readily see from (6.9) the effort is formidable, even for the simplest modulations. However, a less general but nonetheless important case arises when $A_1 = A_2 = A_0$, a constant. With the assistance of (A3.20b) we obtain for the complete low-frequency correlation function

$$\begin{aligned} R(t) &= \gamma^2 2^\nu \psi^\nu e^{-A_0^2 / \psi(1+r_0)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\epsilon_m r_0^m p^{m+n}}{(1+r_0)^{2m+2n}} \\ &\times \sum_{k=0}^n \frac{\Gamma(\nu/2 + k + m + 1) \Gamma(\nu/2 - k + m + n + 1)}{k!(k+m)!(n-k)!(m+n-k)!} \\ &\times {}_2F_1\left(k - \frac{\nu}{2} - n, -k - \frac{\nu}{2}; m + 1; r_0^2\right), \quad p \equiv A_0^2 / 2\psi. \quad (6.10) \end{aligned}$$

For noise alone the low-frequency correlation is at once found from (6.10), since only the terms for which $m = n = 0$ contribute:

$$\begin{aligned} R(t)_{\text{noise}} &= \gamma^2 2^\nu \psi^\nu \Gamma\left(\frac{\nu}{2} + 1\right)^2 {}_2F_1\left(-\frac{\nu}{2}, -\frac{\nu}{2}; 1; r_0^2\right) \\ &= \frac{\psi^\nu \beta^2 2^{\nu-2}}{\pi} \Gamma\left(\frac{\nu+1}{2}\right)^2 {}_2F_1\left(-\frac{\nu}{2}, -\frac{\nu}{2}; 1; r_0^2\right), \quad (6.11) \end{aligned}$$

the latter from (6.7). For the second method we write

$$\begin{aligned}
 R(t)_N &= \gamma^2 2^\nu (1 - r_0^2)^{\nu+1} \psi^\nu \exp \left\{ - (A_1^2 + A_2^2 - 2r_0 A_1 A_2) / 2\psi (1 - r_0^2) \right\} \\
 &\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\epsilon_m r_0^{2n+m} \Gamma(m+n+\nu/2+1)^2 (c_1 c_2)^m}{(m!)^2 n! (n+m)! 2^m} \\
 &\times {}_1F_1 \left(\frac{\nu}{2} + m + n + 1; m + 1; \frac{c_1^2}{2} \right) {}_1F_1 \left(\frac{\nu}{2} + m + n + 1; m + 1; \frac{c_2^2}{2} \right), \tag{6.12}
 \end{aligned}$$

equivalent to (6.9), while the counterpart of (6.10) is

$$\begin{aligned}
 R(t) &= \gamma^2 2^\nu \psi^\nu (1 - r_0^2)^{\nu+1} e^{-p} \sum_{m=0}^{\infty} \frac{\epsilon_m r_0^m}{(m!)^2} \left(\frac{1 - r_0}{1 + r_0} \right)^m p^m \sum_{n=0}^{\infty} \frac{r_0^{2n} \Gamma(m+n+1+\nu/2)^2}{n! (n+m)!} \\
 &\times {}_1F_1 \left[-n - \frac{\nu}{2}; m + 1; -p \frac{(1 - r_0)^2}{1 + r_0} \right]^2 \tag{6.13}
 \end{aligned}$$

from (A3.2). When there is no signal it is easy to verify that (6.13) reduces to (6.11), again with the help of (A3.20b).

There remains the third approach. This enables us to write finally, for the correlation function

$$\begin{aligned}
 R(t)_N &= \gamma^2 2^\nu \psi^{\nu+2} (1 - r_0^2)^{\nu+2} f(A_1 A_2; r_0) \Gamma(\nu/2 + 1)^2 \sum_{m=0}^{\infty} \frac{(\nu/2 + 1)_m^2}{(m!)^2} \\
 &\times \frac{1}{2\pi i} \int_{\mathbf{D}} \frac{\xi^{\nu+1} e^{\xi} [\xi r_0 + c_1 c_2 / 2]^{2m} d\xi}{[\xi - c_1^2 / 2]^{\nu/2+1} [\xi - c_2^2 / 2]^{\nu/2+1}}. \tag{6.14}
 \end{aligned}$$

The path **D** of integration in the ξ -plane is taken to be sufficiently large so that the series in the integrand converge satisfactorily. The contour also includes any branch points and poles of the integrand of (6.14). Expanding the denominator of (6.14) in series and using the binomial development of $(\xi r_0 + c_1 c_2 / 2)^{2m}$, gives, after termwise integration for the case when there is no modulation, viz., $A_1 = A_2 = A_0$, and $c_1 = c_2 = c$, $c^2 / 2 = p(1 - r_0) / (1 + r_0)$,

$$\begin{aligned}
 R(t) &= 2^\nu \psi^\nu \gamma^2 \Gamma \left(\frac{\nu}{2} + 1 \right)^2 (1 - r_0^2)^{\nu+1} e^{-p} \sum_{m=0}^{\infty} \frac{(\nu/2 + 1)_m^2 (2m)!}{m!^2} \\
 &\times \sum_{n=0}^{2m} \frac{r_0^{2m-n} p^n (1 - r_0)^n}{n!^2 (2m - n)! (1 + r_0)^n} \\
 &\times {}_1F_1 [-2m - \nu - 1 + n; n + 1; -p(1 - r_0) / (1 + r_0)]. \tag{6.15}
 \end{aligned}$$

A somewhat different form of the same expression is found by making the substitution $z = \xi - c^2 / 2$, and again expanding $(\xi r_0 + c^2 / 2)^{2m}$ in a binomial series, followed by termwise integration of the resulting series. The low-frequency correlation is then

$$\begin{aligned}
 R(t) &= 2^\nu \psi^\nu \gamma^2 \Gamma \left(\frac{\nu}{2} + 1 \right) (1 - r_0^2)^{\nu+1} e^{-p} \sum_{m=0}^{\infty} \frac{(\nu/2 + 1)_m^2 (2m)!}{m!^2} \sum_{n=0}^{2m} \frac{(1 - r_0)^n r_0^{2m-n} p^n}{(2m - n)! n!} \\
 &\times \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{p(1 - r_0)}{1 + r_0} \right]^{1+\nu-k} \frac{\Gamma(\nu + 2)}{\Gamma(\nu - k + 2) \Gamma(\nu + n - k + 2)}. \tag{6.16}
 \end{aligned}$$

Notice that when ν is integral, the series in k terminates after $\nu + 2$ terms.

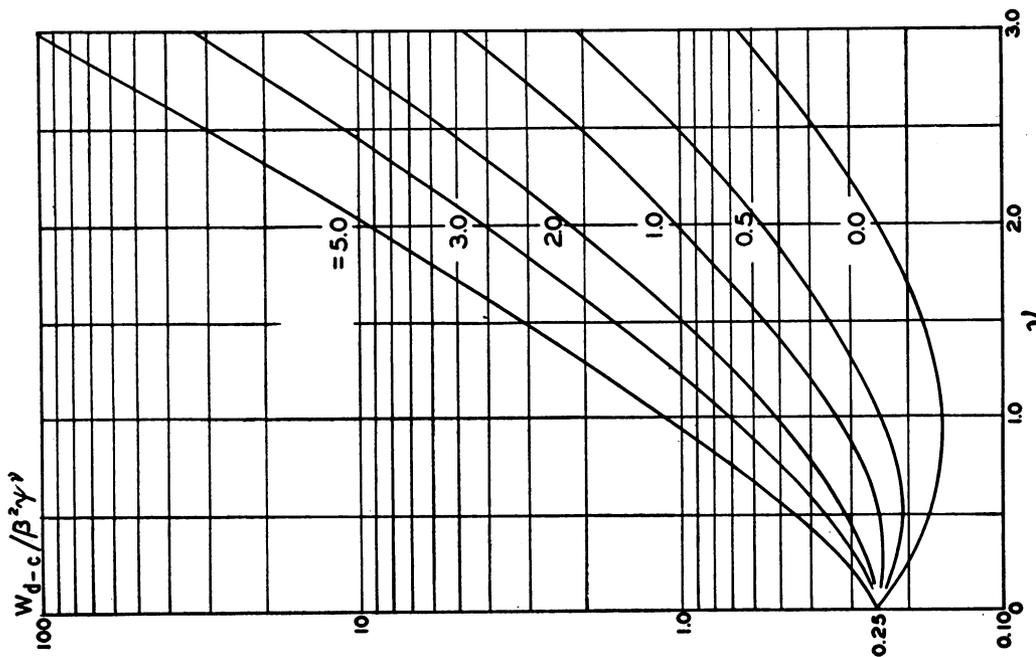


FIG. 6. Mean power output in the d-c component following a half-wave ν -th-law rectifier for various values of the parameter $\phi = A_0^2/2\psi$.

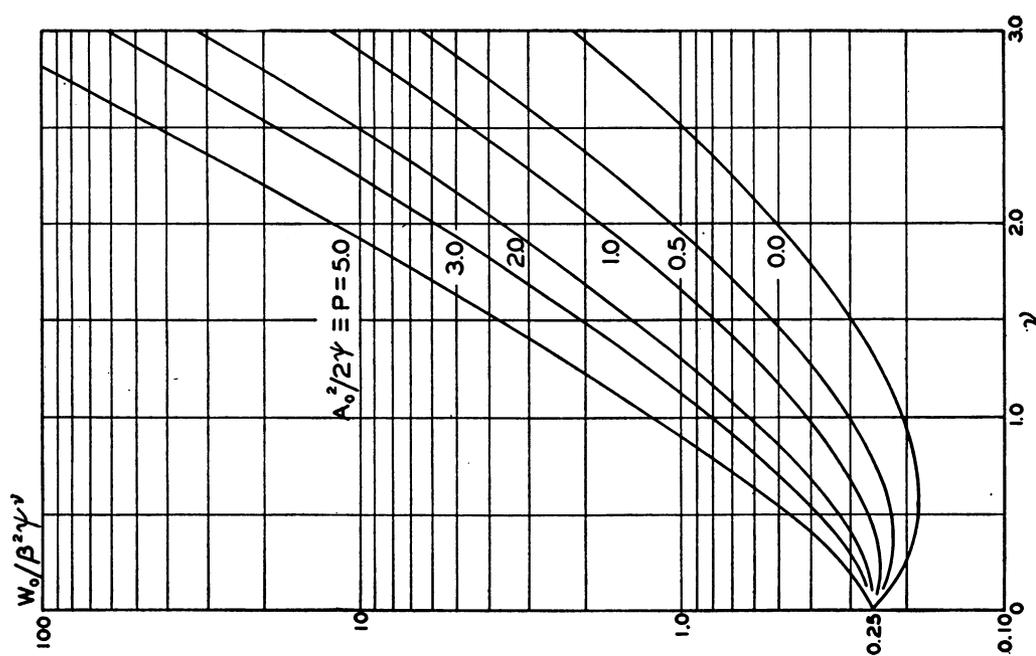


FIG. 5. Mean power output in the low-frequency components following a half-wave ν -th-law rectifier, for a number of values of the parameter $\phi = A_0^2/2\psi$.

The mean total low-frequency power $(W_\tau)_0$ and the power in the $d-c$ component W_{d-c} may be obtained respectively for the various cases considered above on setting $t=0$ and $t = \infty$ in $R(t)$. However, a simpler way of obtaining the same results is to use (6.6) or (6.7) with (5.15) directly. Application of (A3.7) then gives us

$$(W_\tau)_0 = \overline{I_0^2} = \gamma^2 2^\nu \psi^\nu \Gamma(\nu + 1) {}_1F_1(-\nu; 1; -p), \quad A_1 = A_2 = A_0, \quad (6.17)$$

and for the $d-c$

$$W_{d-c} = (\overline{I_0})^2 = \gamma^2 2^\nu \psi^\nu \Gamma\left(\frac{\nu}{2} + 1\right)^2 {}_1F_1\left(-\frac{\nu}{2}; 1; -p\right)^2. \quad (6.18)$$

When the carrier is modulated we have only to average the various $p(t_0^i)^j$ that are present in the above. Curves illustrating $(W_\tau)_0$ and W_{d-c} as a function of the law of the detector are included in Figs. 5 and 6, for a number of values of $p = A_0^2/2\psi$. It is interesting to note that the output powers actually decrease as ν increases in the interval $0 \sim \nu \sim 1.0$ when the signal is weak relative to the noise. Strong signals suppress the noise, and the output is then proportional to p^ν for both total-, low-frequency, and $d-c$ powers, namely,

$$(W_\tau)_0 \cong \gamma^2 2^\nu \psi^\nu p^\nu (1 + \nu^2/p + \dots) \text{ and } W_{d-c} \cong \gamma^2 2^\nu \psi^\nu p^\nu (1 + \nu^2/2p + \dots), \quad (6.19)$$

from (A3.3).

7. General half-wave rectification of random noise. A relatively simple case of considerable interest arises when random noise alone enters a non linear device which passes only the positive amplitudes of the incoming disturbance. The current-voltage characteristic assumes the following general form:

$$\begin{aligned} I &= \beta V^\nu, & V > 0, & \nu > 0 \\ &= 0, & V < 0, & \end{aligned} \quad (7.1)$$

where we restrict ourselves to unsaturated cases, corresponding in practice to (relatively) small-signal rectification. Here V represents the instantaneous input noise voltage, and I is the output current.

The analysis of the present section is more general than that of Sections 5 and 6 in that it is capable of handling broad-band noise, where the concept of the envelope is no longer meaningful; it is less general in that signals are excluded from the discussion. Also, a more detailed study of the higher order spectral regions is included, for the case of narrow-band noise. In fact, Section 7 is in itself a study of a special case of the analysis of Sections 3 and 4, when there is no bias and no carrier.

(a) *Broad-band Noise.* We consider first the case of wide-band noise, whose central frequency, spectrally speaking, is comparable to the bandwidth of the disturbance. See Fig. 7(a) or (b). From (2.9), (2.11), and (7.1) the correlation function of the output in the absence of a signal is

$$\begin{aligned} R(t) &= \frac{\beta^2 \psi^\nu}{2\pi(1-r^2)^{1/2}} \int_0^\infty dX_1 \int_0^\infty dX_2 (X_1 X_2)^\nu \\ &\times \exp \left[- (X_1^2 + X_2^2 - 2rX_1 X_2)/2\psi(1-r^2) \right] \end{aligned} \quad (7.2)$$

where $r(t)$ is the normalized input correlation, cf. Eq. (A2.7). The integration of (7.2) may be effected directly by using polar coördinates: $x = \rho \cos \theta$, $y = \rho \sin \theta$. The latter approach gives us for (7.2)

$$R(t) = \frac{\beta^2 \psi^\nu 2^{-\nu-1}}{[2\pi(1-r^2)^{1/2}]} \int_0^\pi \sin^\nu \phi d\phi \int_0^\infty \rho^{2\nu+1} e^{-a\rho^2} d\rho,$$

$$a = (1 - r \sin \phi)/2(1 - r^2)$$

$$= \frac{\beta^2 \psi^\nu \Gamma(\nu + 1)(1 - r^2)^{\nu+1/2}}{4\pi} \int_0^\pi \frac{\sin^\nu \phi d\phi}{(1 - r \sin \phi)^{\nu+1}}, \tag{7.3}$$

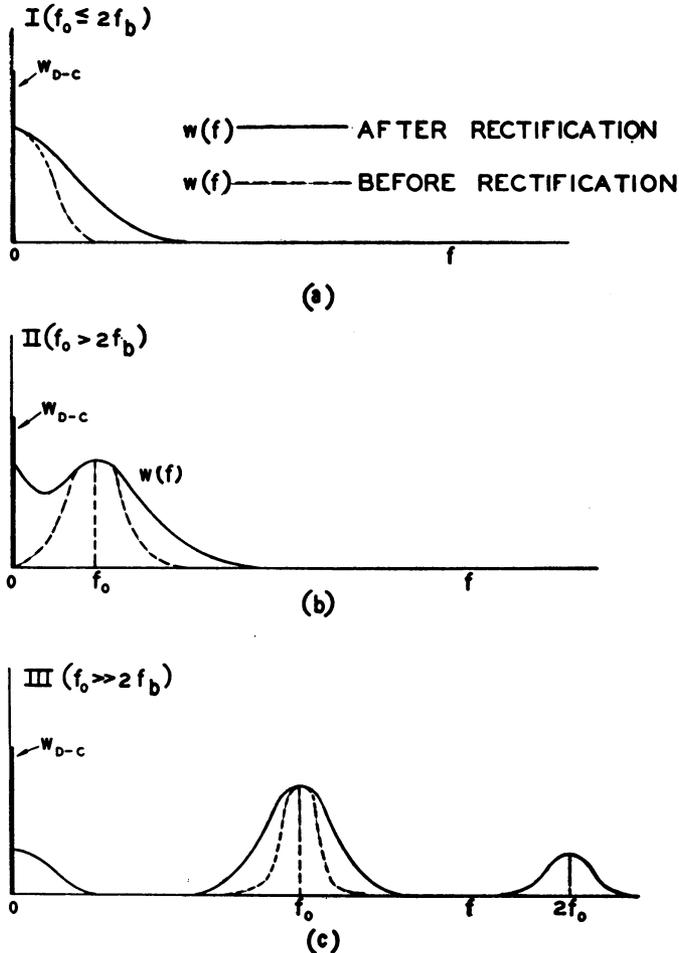


FIG. 7. Examples of narrow-band and broad-band noise before and after rectification. The normalization is arbitrary.

where the substitution $2\theta = \phi$ has been made. Expansion of the denominator and term-wise integration (since $r \sin \phi \leq 1$), with the help of

$$\sin^\nu \phi (1 - r \sin \phi)^{-(\nu+1)} = \sum_{n=0}^{\infty} \frac{r^n (\sin \phi)^{\nu+n} \Gamma(\nu + n + 1)}{n! \Gamma(\nu + 1)} \quad \text{and}$$

$$\int_0^\pi \sin^{\nu+n} \phi d\phi = \frac{\sqrt{\pi} \Gamma[(\nu + n + 1)/2]}{\Gamma[(\nu + n)/2 + 1]} \tag{7.4}$$

yields

$$R(t) = \frac{\beta^2 \psi^\nu (1 - r^2)^{\nu+1/2}}{4\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{r^n \Gamma[(\nu + n + 1)/2] \Gamma(\nu + n + 1)}{n! \Gamma[(\nu + n)/2 + 1]}$$

$$= \frac{\beta^2 2^{\nu-2} \psi^\nu}{\pi} (1 - r^2)^{\nu+1/2} \left\{ \Gamma\left(\frac{\nu + 1}{2}\right)^2 {}_2F_1\left(\frac{\nu + 1}{2}, \frac{\nu + 1}{2}; \frac{1}{2}; r^2\right) \right.$$

$$+ 2r \Gamma\left(\frac{\nu}{2} + 1\right)^2 {}_2F_1\left(\frac{\nu}{2} + 1, \frac{\nu}{2} + 1; \frac{3}{2}; r^2\right) \left. \right\}$$

$$= \frac{\beta^2 2^\nu \psi^\nu}{4\pi} \left\{ \Gamma\left(\frac{\nu + 1}{2}\right)^2 {}_2F_1\left(-\frac{\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}; r^2\right) \right.$$

$$+ 2r \Gamma\left(\frac{\nu}{2} + 1\right)^2 {}_2F_1\left(\frac{1 - \nu}{2}, \frac{1 - \nu}{2}; \frac{3}{2}; r^2\right) \left. \right\}, \tag{7.5}$$

this last with the help of (A3.20b). The series form of (7.5) is more convenient when the spectrum is desired; $R(t)$ may be written accordingly

$$R(t) = \frac{\beta^2 2^\nu \psi^\nu}{4\pi} \sum_{n=0}^{\infty} \left[\frac{(-\nu/2)_n^2 \Gamma[(\nu + 2)/2]^{2n} r^{2n}}{(2n)!} \right.$$

$$\left. + \frac{[(1 - \nu)/2]_n^2 \Gamma[\nu/2 + 1]^{2n} r^{2n+1}}{(2n + 1)!} \right]. \tag{7.6}$$

A number of interesting special cases follow from (7.6). For half-wave linear rectifiers ($\nu = 1$) we obtain

$$R(t)_{\nu=1} = \frac{\beta^2 \psi}{2\pi} \left({}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; r^2\right) + \frac{\pi r}{2} \right)$$

$$= \frac{\beta^2 \psi}{2\pi} \left\{ r \left(\sin^{-1} r + \frac{\pi}{2} \right) + (1 - r^2)^{1/2} \right\}, \tag{7.7}$$

a result derived independently by Van Vleck, North, and Rice (see Sec. 4.7 of reference 2). Half-wave quadratic detectors ($\nu = 2$) give us

$$R(t)_{\nu=2} = \frac{\beta^2 \psi^2}{2\pi} \left(\frac{\pi}{2} (1 + 2r^2) + 4r {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; r^2\right) \right)$$

$$= \frac{\beta^2 \psi^2}{2\pi} \left\{ \left(\frac{\pi}{2} + \sin^{-1} r \right) (1 + 2r^2) + 3r(1 - r^2)^{1/2} \right\}. \tag{7.8}$$

We may continue in this fashion when ν is integral and write $R(t) = f(r, (1 - r^2)^{1/2})$,

$\sin^{-1}\nu$) with the help of the recurrence relations for the hypergeometric function.

The total mean output power W_r , the d - c power W_{d-c} , and the mean total a - c power W_{a-c} all follow at once from (2.26d), (2.33), (7.5), and (A3.20a). We have finally

$$W_r = \frac{\beta^2 \psi^\nu 2^{\nu-1}}{\sqrt{\pi}} \Gamma(\nu + \frac{1}{2}); \quad W_{d-c} = \frac{\beta^2 \psi^\nu 2^{\nu-2}}{\pi} \Gamma\left(\frac{\nu+1}{2}\right)^2; \quad W_{a-c} = W_r - W_{d-c} \quad (7.9)$$

Curves of W_r , W_{d-c} , and W_{a-c} are shown in Fig. 8. It is evident that characteristics for which ν is large (>2) exhibit outputs which are chiefly a - c . The spectra associated with the output may be found from (1.6) and (7.6), where integrals of the type

$$c_{0,n}(f) = 4 \int_0^\infty r(t)^n \cos \omega t dt, \quad \omega = 2\pi f,$$

must be considered. Observe that when ν is an even integer, the first series in (7.6) terminates after the term for which $n = \nu/2$, and when ν is odd, after $n = (\nu-1)/2$. In particular, the mean output spectrum for the linear rectifier is

$$W(f)_{\nu=1} = \frac{\beta^2 \psi}{2\pi} \left\{ 2\delta(f-0) + \frac{\pi}{2} c_{0,1}(f) + \sum_{n=1}^\infty \frac{(2n)! c_{0,2n}(f)}{n! 2^{2n} (2n-1)^2} \right\}, \quad (7.10)$$

and for a quadratic detector

$$W(f)_{\nu=2} = \frac{\beta^2 \psi^2}{2\pi} \left\{ \pi \delta(f-0) + 4c_{0,1}(f) + \pi c_{0,2}(f) + 4 \sum_{n=1}^\infty \frac{(2n)! c_{0,2n+1}(f)}{[n! 2^{2n} (2n-1)]^2 (2n+1)} \right\}, \quad (7.11)$$

where $\delta(f-0)$ is the familiar delta-function. Curves illustrating (7.10) and (7.11) when the input spectrum is Gaussian, Eq. (A2.9), $\omega_c = 0$, are given in reference 9.

(b) *Narrow-band Noise.* The output now consists, as explained above in more detail in section 3, of bands of noise located about harmonics of the central frequency f_c . The spectra associated with these harmonic zones are "smeared out," or distorted from their original shape, as Fig. 7(c) indicates. Our result (7.6) is expressed in a form convenient for obtaining the correlation functions associated with the various harmonic regions lf_c , $l=0, 1, 2, 3, \dots$, and hence the spectral distribution of the mean power and the mean total power itself pertaining to these regions.

Here the input correlation function is given by (A2.8). When this is substituted into (7.6) we obtain series involving $(\cos \omega_c t)^{2n}$ and $(\cos \omega_c t)^{2n+1}$. The expansion in multiples of $\omega_c t$ is then used, and the correlation function of the l th zone follows from (7.6) on taking those values of n such that $j = (2n-l)/2 (\geq 0)$ is integral for even l , and $j = (2n-l+1)/2 (\geq 0)$ for odd values of $l (>0)$. We may write finally

$$R_l(t) = \pi^{-1} \beta^2 \psi^\nu 2^{\nu-2} \Gamma\left(\frac{\nu+1}{2}\right)^2 \epsilon_l \cos l\omega_c t \sum_{n=l/2}^\infty \frac{(-\nu/2)_n r_0(t)^{2n}}{(n-l/2)!(n+l/2)!}, \quad (7.12a)$$

$$l = 0, 2, 4, \dots,$$

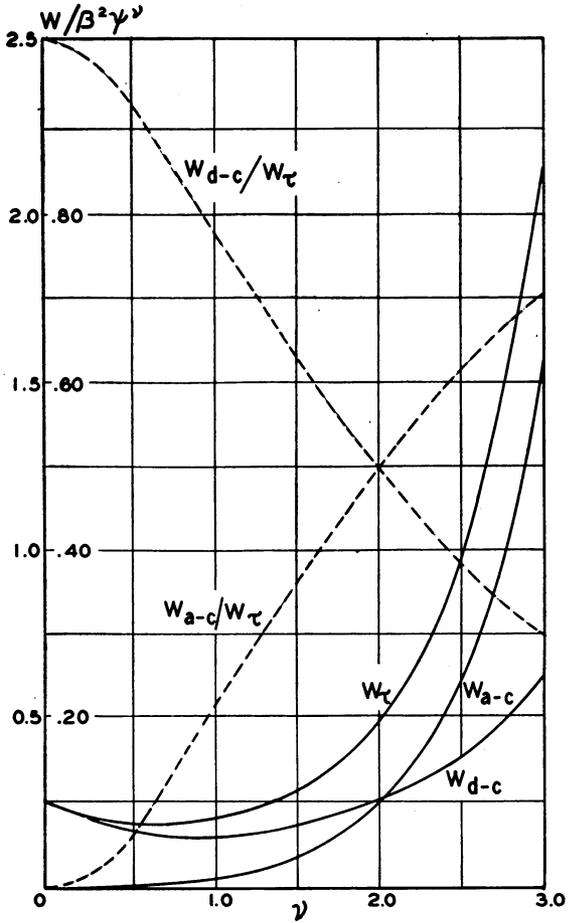
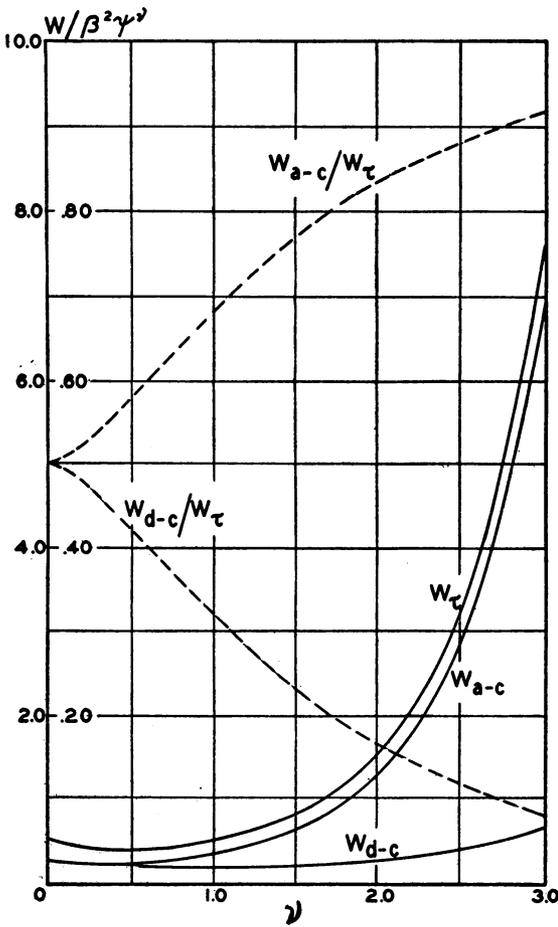


FIG. 8. The mean total, a-c, and d-c power outputs of a half-wave ν th-law detector when the incoming disturbance is broadband noise, as a function of ν . The right-hand scale applies for the ratios.

FIG. 9. The mean total and a-c powers associated with the low-frequency output when the incoming random noise is narrow-band, as a function of ν . The d-c power output is the same as that in Fig. 8, and the right-hand scale applies in the case of the ratios.

and

$$R_l(t) = \pi^{-1}\beta^2\psi^\nu 2^{\nu-1}\Gamma\left(\frac{\nu}{2} + 1\right)^2 \cos l\omega_c t \sum_{n=(l-1)/2}^{\infty} \frac{[(1-\nu)/2]_n^2 r_0(t)^{2n+1}}{[(2n-l+1)/2]! [(2n+l+1)/2]!},$$

$$l = 1, 3, 5, \dots \quad (7.12b)$$

We observe that contributions to the spectrum about the frequency lf_c arise from terms for which $n \geq l/2, (l-1)/2$; values of n less than this do not appear. Now (7.12a) and (7.12b) may be summed; for even values of l we set $n-l/2 = m, m=0, 1, 2, \dots$, and find that

$$\begin{aligned} \sum_{n=l/2}^{\infty} \frac{[-\nu/2]_n^2 r_0^{2n}}{(n-l/2)!(n+l/2)!} &= r_0^l \sum_{m=0}^{\infty} \frac{[-\nu/2]_{m+l/2}^2 r_0^{2m}}{m!(m+l)!} \\ &= \frac{r_0^l (-\nu/2)_{l/2}^2}{l!} \sum_{m=0}^{\infty} \frac{[(l-\nu)/2]_m^2 r_0^{2m}}{m!(l+1)_m} \\ &= \frac{r_0^l [-\nu/2]_{l/2}^2}{l!} {}_2F_1\left(\frac{l-\nu}{2}, \frac{l-\nu}{2}; l+1; r_0^2\right), \end{aligned} \tag{7.13}$$

and the correlation function appropriate to the even harmonics is then

$$\begin{aligned} R(t) &= \frac{\beta^2 \psi^{\nu-2}}{\pi} \epsilon_1 \left(-\frac{\nu}{2}\right)_{l/2}^2 \frac{\Gamma[(\nu+1)/2]^2}{l!} r_0^l \cos \omega_c t \\ &\quad \times {}_2F_1\left(\frac{l-\nu}{2}, \frac{l-\nu}{2}; l+1; r_0^2\right), \quad l = 0, 2, 4, \dots \end{aligned} \tag{7.14}$$

A similar procedure for odd values of l , letting $n-(l-1)/2 = m$, $m = 0, 1, 2 \dots$ in (7.12b) yields

$$\begin{aligned} R_l(t) &= \frac{\beta^2 \psi^{\nu-1}}{\pi} \left(\frac{1-\nu}{2}\right)_{(l-1)/2}^2 \frac{\Gamma[\nu/2+1]^2}{l!} r_0^l \cos \omega_c t \\ &\quad \times {}_2F_1\left(\frac{l-\nu}{2}, \frac{l-\nu}{2}; l+1; r_0^2\right), \quad l = 1, 3, 5, \dots \end{aligned} \tag{7.15}$$

Notice that when the detector characteristic is an even power of the incoming wave, only those zones for which l is even and equal to or less than ν (even) are produced, while all *odd* harmonic zones ($l = 1, 3, 5, 7 \dots$) appear in the output. A like situation is encountered in odd-powered characteristics: only the zones for which $0 < l \leq \nu$ (l and ν odd) are generated, whereas all even regions exist. The above is analogous to the half-wave rectification of a sine wave if we identify the narrow-band input noise with a sinusoid having the frequency $f_0 = f_c$. The amplitudes, and hence the powers, of the zones about lf_c corresponding to the sinusoidal components $lf_0 = lf_c$ are *not* equal, as a Fourier analysis readily shows, but their number and location in the spectrum are.

Of particular interest in practice is the low-frequency output $l = 0$. The complete low-frequency correlation function is seen from (7.14) to be just (6.11), which in the instance $\nu = 1$ may also be expressed in terms of the complete elliptic integrals E and K of modulus r_0 :

$$R_0(t)_{\nu=1} = \frac{\beta^2 \psi}{\pi^2} \left\{ E(r_0) - \frac{1}{2}(1-r_0^2)K(r_0) \right\}. \tag{7.16}$$

For the quadratic rectifier the form of $R_0(t)_{\nu=2}$ is quite simple:

$$R_0(t)_{\nu=2} = \beta^2 \psi^2 (1+r_0^2)/4. \tag{7.17}$$

When ν is even we obtain a polynomial in r_0 , viz:

$$\begin{aligned}
 R_0(t)_{\nu=2n} &= \beta^2 \psi^{2n} \left[\frac{(2n)!}{2^{n+1}n!} \right]^2 {}_2F_1(-n, -n; 1; r_0^2) \\
 &= \left[\frac{\beta \psi^n (2n)!}{2^{n+1}n!} \right]^2 \sum_{j=0}^n {}_n C_j^2 r_0^{2(2n-j)}.
 \end{aligned}
 \tag{7.18}$$

The *d-c* component may be found when *j* is set equal to *n*.

Again we may determine the various powers by setting *t* = 0 or *t* = ∞ in (7.14) and (7.15) and using (A3.20a). The mean total power in the low-frequency region and in the *d-c* are respectively

$$\begin{aligned}
 (W_\tau)_0 = R_0(0) &= \frac{\beta^2 \psi^\nu 2^{\nu-2}}{\pi} \frac{\Gamma(\nu + 1)\Gamma[(\nu + 1)/2]^2}{\Gamma(\nu/2 + 1)^2}; \\
 W_{d-c} = R_0(\infty) &= \frac{\beta^2 \psi^\nu 2^{\nu-2}}{\pi} \Gamma\left(\frac{\nu + 1}{2}\right)^2,
 \end{aligned}
 \tag{7.19}$$

the latter agreeing with our previous result (7.9), as we would expect in virtue of the general theorem stated at the end of Appendix II. The higher (*l* > 0) spectral regions are pure *a-c*. The mean *a-c* power as in general

$$(W_{a-c})_l = R_l(0) = \frac{\beta^2 \psi^\nu 2^{\nu-2}}{\pi} \epsilon_l \left(-\frac{\nu}{2}\right)_{l/2}^2 \frac{\Gamma[(\nu + 1)/2]^2 \Gamma(\nu + 1)}{\Gamma[(\nu + l)/2 + 1]^2},$$

l = 2, 4, . . . (7.20a)

and

$$(W_{a-c})_l = \frac{\beta^2 \psi^\nu 2^{\nu-1}}{\pi} \left(\frac{1 - \nu}{2}\right)_{(l-1)/2}^2 \frac{\Gamma(\nu/2 + 1)^2 \Gamma(\nu + 1)}{\Gamma[(\nu + l)/2 + 1]^2}, \quad l = 1, 3, 5, \dots \tag{7.20b}$$

The contributions from all harmonic zones to the total output power $\sum_l R_l(0)$ is simply the first equation in (7.9). Figures 9 and 10 show *W_l* as a function of *ν* for several different values of *l*. It is interesting to note from Fig. 9 that when *ν* < 2 the *d-c* power exceeds the low-frequency *a-c*, while for *ν* > 2 the reverse is true. Contrast this with the behavior of broad-band noise, Fig. 8. We see also in Fig. 10 that over the usual range of *ν*, i.e., 1 < *ν* < 2, the power associated with the higher zones (*l* ≥ 3) is quite negligible for most purposes compared with that in the zones 0, 1, and 2. In the instance of narrow-band noise the fundamental (*l* = 1) spectral region appears to have the greatest relative mean power. Examples of spectra for regions 0 and 1 are to be found in reference 9.

8. Small-signal detection and full-wave rectification. The method of section 7 is particularly well suited to the study of the important practical case of small-signal detection. This process is characterized by input amplitudes sufficiently small so that cut-off, corresponding to large grid-voltage swings in the negative direction, and saturation, arising from excessively large positive swings, are both avoided. For periodic disturbances or in general for waves limited in amplitude, it is possible to achieve this condition. For noise we must modify the "small-signal" concept by specifying that the instantaneous voltage amplitude of the input does not enter the regions of saturation and cut-off (*vide* Fig. 1) more than a given percentage of the time. In

our analysis we assume that this percentage is small enough to permit us to replace the physically bounded dynamic characteristic by one unaffected by cut-off or saturation.

We may represent the small-signal dynamic path, or dynamic transfer characteristic, as it is sometimes called,³⁰ by the following

$$I = \alpha_0 + \alpha_1 V + \alpha_2 V^2 + \dots + \alpha_k V^k + \dots = \sum_{k=0}^{\infty} \alpha_k V^k, \tag{8.1}$$

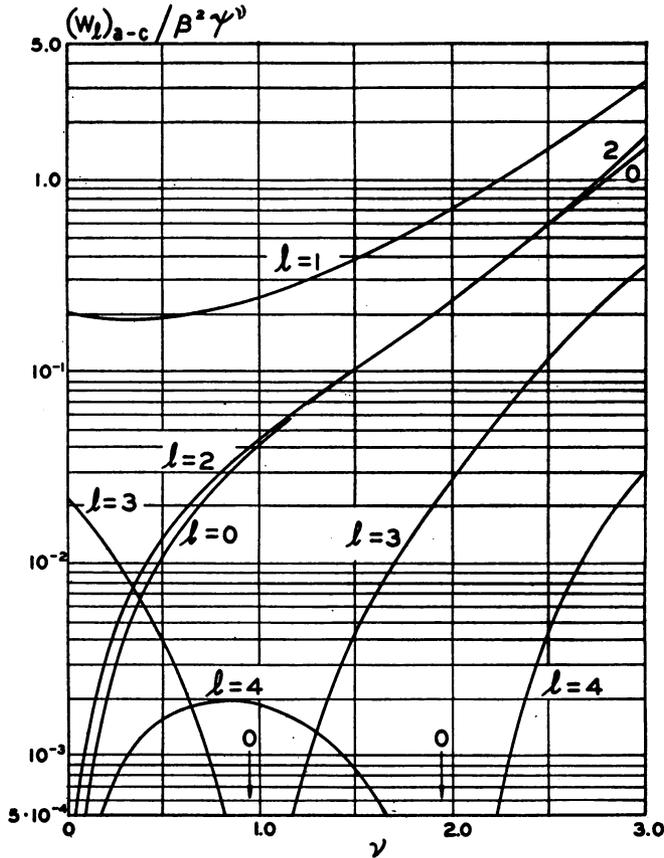


FIG. 10. The mean a-c power outputs associated with the various output harmonic regions ($l=0-4$), as a function of ν .

where V is again identified as the disturbance on the grid and I is the instantaneous current output; the series (8.1) is assumed to converge. The quantities α_k are dimensional constants, for resistive tube loads, and may be described in terms of the tube parameters $\mu, r_b, r_p, \partial r_p / \partial e_b$ etc. (See Eqs. 3.41, 3.42, reference 30.) For reactive loads, however, the α_k are complicated functions of the frequency and hence of the input and output waves. As before, our study is restricted to cases where the plate loads are primarily resistive.

³⁰ H. J. Reich, *Theory and application of electron tubes*, McGraw-Hill (1944) pp. 74-77.

(a) *Noise Alone.* When the applied voltage $V(t)$ is random noise, the correlation function of the output is from (2.9) and 1.7b) in normalized form

$$R(t)_N = [2\pi(1 - r^2)^{1/2}]^{-1} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy I(x\psi^{1/2}) I(y\psi^{1/2}) e^{-(x^2+y^2-2xyr)/2(1-r^2)}. \tag{8.2}$$

The substitution of (8.1) in (8.2) gives

$$R(t)_N = [2\pi(1 - r^2)^{1/2}]^{-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha_j \alpha_k \psi^{(j+k)/2} \int_{-\infty}^{\infty} x^j dx \int_{-\infty}^{\infty} y^k dy e^{-(x^2+y^2-2xyr)/2(1-r^2)}. \tag{8.3}$$

The integral in (8.3) may be evaluated with the help of polar coördinates, the method of section 7, or by expanding $\exp. (xyr/(1-r^2))$ and applying (A3.20b). The result in either case is

$$\begin{aligned} Q_{j,k}(r) &= [2\pi(1 - r^2)^{1/2}]^{-1} \int_{-\infty}^{\infty} e^{-x^2/2(1-r^2)} x^j dx \int_{-\infty}^{\infty} y^k e^{-y^2/2(1-r^2)} e^{xyr/(1-r^2)} dy \\ &= 0, \quad j + k \text{ odd,} \\ &= 2r \cdot \frac{2^{(j+k)/2}}{\pi} \Gamma\left(\frac{j}{2} + 1\right) \Gamma\left(\frac{k}{2} + 1\right) \\ &\quad \times {}_2F_1\left(\frac{1-j}{2}, \frac{1-k}{2}; \frac{3}{2}; r^2\right), \quad j, k \text{ both odd,} \\ &= \frac{2^{(j+k)/2}}{\pi} \Gamma\left(\frac{j+1}{2}\right) \Gamma\left(\frac{k+1}{2}\right) \\ &\quad \times {}_2F_1\left(-\frac{j}{2}, -\frac{k}{2}; \frac{1}{2}; r^2\right), \quad j, k \text{ both even.} \end{aligned} \tag{8.4}$$

Observe that the hypergeometric functions terminate. Equation (8.2) may be written finally

$$R(t)_N = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha_j \alpha_k \psi^{(j+k)/2} Q_{j,k}(r), \quad j + k \text{ even.} \tag{8.5}$$

Only when $j+k$ is even is there a non vanishing output. This quite naturally follows from the basic assumption for our random noise that it contains no $d-c$ and consequently has the average value zero, i.e., $\bar{x} = \bar{y} = 0$, and so $\overline{x^j}$ and $\overline{y^k}$ also vanish (j, k odd) for the infinite time average. Then it is clear that $\overline{x^j y^k}$ on the average must be zero provided $j+k$ is odd, as borne out by (8.4). This also follows at once from (A1.4) et seq. when $\nu_k = 0$.

The output spectrum is obtained from (8.5) and (1.6), and the total output power is found as before on setting $t=0$ in $R(t)_N$. With the aid of (A3.20a) we have

$$W_{\tau} = R(0) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha_j \alpha_k \left(\frac{\psi}{2}\right)^{(j+k)/2} (j+k)! / \left(\frac{j+k}{2}\right)!, \quad j + k \text{ even.} \tag{8.6}$$

In practice only the first few terms are generally significant, because α_j and α_k ap-

proach zero rapidly for larger values of j and k . The d - c output is easily derived on setting $t = \infty$ in (8.5). The expression is

$$W_{d-c} = R(\infty) = \left\{ \sum_{j=0}^{\infty} \alpha_j \left(\frac{\psi}{2}\right)^{j/2} j! / \left(\frac{j}{2}\right)! \right\}^2, \quad j = 0, 2, 4, \dots \tag{8.7}$$

Equation (8.7) shows that only the even terms in the dynamic characteristic contribute to the d - c . The mean power associated with the continuum is now to be obtained from (8.6)–(8.7) and the fundamental relation $W_c = W_r - W_{d-c}$.

When the input noise is broadband there is no separation of the output into spectral bands centered about harmonics of the central frequency as explained in section 3. The spectrum must be calculated directly from (8.5). On the other hand, when the noise is narrow-band, a resolution into harmonics is possible, for which the various correlation functions and hence the powers and spectra may be determined. We note again that the input correlation is given by (A2.8).

Let us consider first the low-frequency output, as it is perhaps of greatest practical interest. Now we observe from (8.4) that only when j and k are both even does $r(t)$ exist in even powers and therefore contributes to the d - c and low-frequency continuum. Our procedure for selecting the terms that contribute is similar to that outlined in section 7. Application of the expansion of $(\cos \omega_c t)^{2n}$ to the hypergeometric function in (8.4) yields

$$\begin{aligned} & {}_2F_1\left(-\frac{j}{2}, -\frac{k}{2}; \frac{1}{2}; r^2\right) \\ &= \sum_{n=0}^{i/2, k/2} \frac{(-j/2)_n (-k/2)_n r_0^{2n}}{\left(\frac{1}{2}\right)_n n! 2^{2n}} \sum_{q=0}^n \epsilon_{n-q} {}_{2n}C_q \cos 2(n-q)\omega_c t, \quad j, k \text{ even}, \end{aligned} \tag{8.8}$$

where either limit applies on the first summation, according to whichever is the lesser, $j/2$ or $k/2$. Only those terms for which $q = n$ contribute to the low frequency and d - c . We have then

$$\begin{aligned} & {}_2F_1\left(-\frac{j}{2}, -\frac{k}{2}; \frac{1}{2}; r^2\right)_0 \\ &= \sum_{n=0}^{i/2, k/2} \frac{(-j/2)_n (-k/2)_n r_0^{2n} {}_{2n}C_n}{\left(\frac{1}{2}\right)_n n! 2^{2n}} = {}_2F_1\left(-\frac{j}{2}, -\frac{k}{2}; 1; r_0^2\right) \quad j, k \text{ even}, \end{aligned} \tag{8.9}$$

since $(1/2)_n = (2n)!/n!2^{2n}$. Equation (8.5) becomes finally

$$\begin{aligned} R_0(t)_N &= \frac{1}{\pi} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha_j \alpha_k (2\psi)^{(j+k)/2} \Gamma\left(\frac{j+1}{2}\right) \Gamma\left(\frac{k+1}{2}\right) \\ &\quad \times {}_2F_1\left(-\frac{j}{2}, -\frac{k}{2}; 1; r_0^2\right), \quad j, k \text{ even}. \end{aligned} \tag{8.10}$$

The low-frequency continuum is calculated from (8.10) when all the constant terms in $\sum {}_2F_1$ corresponding to the d - c are removed. The mean total power for the harmonic zone $l = 0$ follows from (8.10) and (A3.20a), when $t = 0$. The d - c power output is

seen to be precisely (8.7), as we would expect, and the continuum power $(W_c)_0$ follows from $(W_r)_0 - W_{a-c} = (W_c)_0$.

The above procedure may be applied to the higher spectral zones $l \geq 1$ where, following the analysis of Sec. 7(b), we may derive the interesting partial sum results

$$\begin{aligned}
 {}_2F_1\left(-\frac{j}{2}, -\frac{k}{2}; \frac{1}{2}; r_0^2 \cos^2 l\omega_c t\right) &= \sum_{l=0,2,\dots}^{\infty} \epsilon_{l/2} \cos l\omega_c t \frac{r_0^l}{l!} (-j/2)_{l/2} (-k/2)_{l/2} \\
 &\times {}_2F_1\left(\frac{l-j}{2}, \frac{l-k}{2}; l+1; r_0^2\right), \quad i, k \text{ even}, \tag{8.11}
 \end{aligned}$$

and

$$\begin{aligned}
 r_0 \cos \omega_c t {}_2F_1\left(\frac{1-j}{2}, \frac{1-k}{2}; \frac{3}{2}; r_0^2 \cos^2 \omega_c t\right) &= \sum_{l=1,3,\dots}^{\infty} \cos l\omega_c t \frac{r_0^l}{l!} \left(\frac{1-j}{2}\right)_{(l-1)/2} \left(\frac{1-k}{2}\right)_{(l-1)/2} \\
 &\times {}_2F_1\left(\frac{l-j}{2}, \frac{l-k}{2}; l+1; r_0^2\right), \quad j, k \text{ odd.} \tag{8.12}
 \end{aligned}$$

From (8.11) and (8.12) substituted into (8.5) one may write at once for the correlation functions of the l respective output bands

$$\begin{aligned}
 R_l(t)_N &= \epsilon_{l/2} \frac{r_0^l \cos l\omega_c t}{l!} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha_j \alpha_k (2\psi)^{(j+k)/2} (-j/2)_{l/2} (-k/2)_{l/2} \Gamma\left(\frac{j+1}{2}\right) \Gamma\left(\frac{k+1}{2}\right) \\
 &\times {}_2F_1\left(\frac{l-j}{2}, \frac{l-k}{2}; l+1; r_0^2\right), \quad j, k \text{ even} \tag{8.13}
 \end{aligned}$$

for even values of l , and when l is odd we obtain

$$\begin{aligned}
 R_l(t)_N &= \frac{2r_0^l \cos l\omega_c t}{l!} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha_j \alpha_k (2\psi)^{(j+k)/2} \left(\frac{1-j}{2}\right)_{(l-1)/2} \left(\frac{1-k}{2}\right)_{(l-1)/2} \\
 &\times \Gamma\left(\frac{j}{2} + 1\right) \Gamma\left(\frac{k}{2} + 1\right) {}_2F_1\left(\frac{l-j}{2}, \frac{l-k}{2}; l+1; r_0^2\right), \quad j, k \text{ odd.} \tag{8.14}
 \end{aligned}$$

From these relations it is observed that for given values of j and k it is not possible to have contributions to zones for which l exceeds j or k , whichever is the lesser. Equation (8.13) and (8.14) also enable us to determine the various mean $a-c$ powers in the continuum of the output, on setting $t=0$ as before, and with the aid of (A3.20a).

(b) *Signal and Noise.* Instead of the input voltage being merely random noise, it now consists of a mixture of noise and a signal S , where the latter may or may not be modulated. Since the noise and signal are independent, there can be no correlation between them. Further, if the signal is modulated, correlation may or may not exist between the modulation and the carrier, but in any case there is still no correlation

with the noise; (see the first part of section 2). Accordingly, in determining the correlation function that part, R_N , is obtained by a suitable average over the random noise components independently of the contribution arising from the signal. The complete correlation follows from (2.12).

Let S_1 and S_2 be the input signal voltages at times separated by an interval t . Then the total input voltage may be represented in normalized form by

$$\begin{aligned} v_1 &= V_1/\psi^{1/2} = (S_1 + X)/\psi^{1/2} = s_1 + x; \\ v_2 &= V_2/\psi^{1/2} = (S_2 + Y)/\psi^{1/2} = s_2 + y. \end{aligned} \tag{8.15}$$

The contribution to the correlation, attributable to the noise, follows from (8.2) where now the input is given by (8.15). We find then that

$$R_N = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha_j \alpha_k \psi^{(j+k)/2} \int_{-\infty}^{\infty} (x + s_1)^j dx \int_{-\infty}^{\infty} (y + s_2)^k W_2(x, y; t) dy, \tag{8.16}$$

with $W_2(x, y; t)$ given by (2.9) after suitable normalization. We note several methods of evaluating these integrals. Unfortunately, none of these methods yields very simple results, although they appear to be the best available. However, in practice the dynamic characteristic (8.1) may be expressed with reasonable accuracy when only the first few terms are considered; the higher coefficients $\alpha_j, \alpha_k, j, k \geq 3$ are in many cases negligible, and the complexity of the results is consequently much reduced. One approach is that used in evaluating (8.3). Applying it to (8.16) one obtains for the coefficient of $\alpha_j \alpha_k \psi^{(j+k)/2}$:

$$\begin{aligned} I_{j,k} &= \int \int_{-\infty}^{\infty} (x + s_1)^j (y + s_2)^k W_2(x, y; t) dx dy \\ &= j!k!2^{-(j+k)/2} \sum_{p=0}^j \sum_{q=0}^k \frac{2^{(p+q)/2} s_1^p s_2^q}{p!q![(j-p)/2]![(k-q)/2]!} \\ &\times {}_2F_1\left(\frac{p-j}{2}, \frac{q-k}{2}; \frac{1}{2}; r^2\right), \quad j-p, k-q \text{ even,} \\ &= 2rj!k!2^{-(j+k)/2} \sum_{p=0}^j \sum_{q=0}^k \frac{2^{(p+q)/2} s_1^p s_2^q}{p!q![(j-p-1)/2]![(k-q-1)/2]!} \\ &\times {}_2F_1\left(\frac{1-j+p}{2}, \frac{1-k+q}{2}; \frac{3}{2}; r^2\right), \quad j-p, k-q \text{ odd,} \end{aligned} \tag{8.17}$$

and when $j+k-(p+q)(\geq 0)$ is odd the integrals vanish. We may also derive (8.17) with the aid of the transformation to polar coördinates employed in section 7.

Now let us assume that the modulation A_0 and the carrier $\cos \omega_0 t$ are uncorrelated. Then the input signal S may be written

$$S_1(t_0, t'_0) = A_c(t'_0) \cos \omega_0 t_0, \quad S_2(t_0 + t, t'_0 + t) = A_0(t'_0 + t) \cos \omega_0(t_0 + t). \tag{8.18}$$

The complete correlation may be found from (2.12) on averaging over the phases t_0 and t'_0 , and from (2.12) the mean power spectrum follows at once, the mean power on setting $t=0$. As an example, consider the general small-signal quadratic rectifier $\alpha_j, \alpha_k \neq 0, j \leq 2, k \leq 2$. With the help of (8.17)³¹ we find for R_N

³¹ See Part II of reference 8, section II b.

$$\begin{aligned}
R_N = & \alpha_0^2 + \psi^{1/2}\alpha_0\alpha_1(s_1 + s_2) + \psi\alpha_0\alpha_2(2 + s_1^2 + s_2^2) + \psi\alpha_1^2(r + s_1s_2) \\
& + \psi^{3/2}\alpha_1\alpha_2(s_1 + s_2)(s_1s_2 + 2r + 1) \\
& + \psi^2\alpha_2^2(1 + 2r^2 + s_1^2 + s_2^2 + 4rs_1s_2 + s_1^2s_2^2). \tag{8.19}
\end{aligned}$$

If the noise is narrow-band, $r = r_0 \cos \omega_c t$ and further, if the carrier is tuned to the center of the noise band, as is the case in receivers or transmitters, then $\omega_c = \omega_0$. Thus, when (8.19) is substituted into the expression for the complete correlation function (2.12) and when (8.18) is used, we observe that the terms involving $s_1, s_2, s_1^2, s_2^2, s_1^2s_2, s_1s_2^2, s_1s_2, s_1r, s_2r$, but *not* s_1s_2r , contribute only to the d - c and to frequencies in the neighborhood of f_c and $2f_c$. The low-frequency output, exclusive of direct current, is from (8.19) in conjunction with (2.12).

$$\begin{aligned}
R(t)_{L.F.} = & \psi^2\alpha_2^2 \left\{ r_0(t)^2 + r_0(t)T_0'^{-1} \int_0^{T'} (A_{01}A_{02})_{L.F.} dt' \right. \\
& \left. + (4T_0')^{-1} \int_0^{T'} (A_{01}^2A_{02}^2)_{L.F.} dt' \right\}, \tag{8.20}
\end{aligned}$$

where we have written A_{01} (for $A_0(t'_0)$) and A_{02} for $A_0(t'_0 + t)$, and the subscript $L-F$ indicates that d - c components are to be removed. Observe that the low-frequency output is composed of three contributions: the first in (8.20) is that arising from the input noise alone, the second represents the cross-modulation of the noise and signal components, and the third is the detected signal envelope, squared, of course, since in this instance the dynamic characteristic is quadratic. We remark that only the term in α_2^2 , cf. (8.1), is capable of rectifying; hence the low-frequency output alone results from it. Special cases of (8.19) and (8.20) have been given by Rice, ref. 2, Eqs. (4.10.1) and (4.10.3), when the incoming wave is noise alone, or consists of noise and an unmodulated carrier.

As before, *vide* (8.6) and (8.7), it is possible to deduce (from 8.16) and (8.17) certain general relations for the output, total continuum, and d - c power.

(c) *Full-Wave Rectification.* By full-wave rectification it is meant that the dynamic characteristic is such that $I_{\text{out}} = \beta |V_{\text{in}}|^r$. When the incoming wave is noise alone, we have merely to multiply the results obtained in section 7 by 2^r . However, if the input contains a signal as well, Eq. (7.2) multiplied by 2^r still applies, provided we write for the lower limits of integration $-s_1$ and $-s_2$ respectively, in place of $x=y=0$. Analytically the problem is now best handled by the methods of section 2 and 3.

We wish to thank Mr. Rice and Prof. J. H. Van Vleck with whom we have discussed these problems from time to time, and also Prof. L. Brillouin, for their helpful criticism of this paper.

Appendix I. In this appendix are briefly summarized the analytical results obtained when the solution to the generalized, s -dimensional problem of random flights is used, with the aid of the characteristic function and central limit theorem,^{10,32} to derive the multivariate Gaussian distribution law, so fundamental in all problems of the type considered in this paper.

The generalized random-walk problem,⁴ resolved by a method due originally to

³² J. V. Uspensky, *Introduction to mathematical probability*, McGraw-Hill (1937).

Markoff,³³ leads to an expression for the characteristic function $F_N(\xi)$, whose associated probability density $W_N(\mathbf{R})$ it is our task to find: viz.

$$F_N(\xi) = \exp \left\{ i \sum_{k=1}^s \nu_k \xi_k - \frac{1}{2} \sum_{k=1}^s \sum_{l=1}^s \mu_{kl} \xi_k \xi_l \right\}, \quad N \gg 1, \quad (A1.1)$$

where the quantities ν_k and μ_{kl} are given by

$$\nu_k = \sum_{j=1}^N \bar{x}_{jk}; \quad \mu_{kl} = \sum_{j=1}^N \overline{x_{jk} x_{jl}}, \quad N \gg 1. \quad (A1.2)$$

Here \mathbf{r}_j is the j th displacement vector, with components $r_{jk} = x_{jk}$ ($k=1, 2, \dots, s$), and the range of values of x_j extends from $-\infty$ to $+\infty$; N is the total number of displacements, here indefinitely large, and \mathbf{R} is the resultant displacement. Since F_N is the Fourier transform of W_N , we have finally, by the usual methods,³⁴ the familiar result that³⁵

$$\begin{aligned} W_N(\mathbf{R}) &= \frac{\exp -\frac{1}{2}(\tilde{\mathbf{X}} - \tilde{\mathbf{v}})\mathbf{M}(\mathbf{X} - \mathbf{v})}{(2\pi)^{s/2} |\boldsymbol{\mu}|^{1/2}} \\ &= [(2\pi)^s |\boldsymbol{\mu}|]^{-1/2} \exp \left\{ -\frac{1}{2} \sum_{k=1}^s \sum_{l=1}^s \frac{\mu^{kl}}{|\boldsymbol{\mu}|} (X_k - \nu_k)(X_l - \nu_l) \right\}, \end{aligned} \quad (A1.3)$$

where X_k is the k th component of \mathbf{R} and $\bar{X}_k = \nu_k$; here \mathbf{M} is an $(s \times s)$ matrix reciprocal to the matrix $\boldsymbol{\mu}$; $|\boldsymbol{\mu}|$ is the determinant of $\boldsymbol{\mu}$, and μ^{kl} is the cofactor of μ_{kl} . Since the average of $\exp(i \sum_k \xi_k X_k)$ is, by definition, the characteristic function, it is a simple matter to show that

$$\bar{X}_k = \nu_k = \sum_{j=1}^N \bar{x}_{jk}; \quad \overline{X_{k_1} X_{k_2}} = \mu_{k_1 k_2} + \nu_{k_1} \nu_{k_2} = \sum_{j=1}^N (\overline{x_{jk_1} x_{jk_2}} + \overline{x_{jk_1}} \cdot \overline{x_{jk_2}}) \quad (A1.4)$$

by expanding this and (A1.1) and equating coefficients of $\xi_k, \xi_{k_1} \xi_{k_2}$, etc. In the same way higher moments may be found, it being noted that if $\nu_k = 0$, all odd-order moments vanish.

Appendix II: Some Remarks on Correlation Functions, Spectra, and Power.

The fundamental relationship between the mean power—or mean square amplitude spectrum—and the correlation function, defined below, is well known. We mention it briefly here along with a short discussion of some of the more significant and useful properties of the relation in our work, some of which do not seem to have been treated previously. There appear to be several approaches, yielding similar results, one, for example, through the use of Fourier series,¹ the other with the help of Plancherel's theorem.³⁶

Now let us consider $g(t)$ to represent a suitable function of the time, and let us

³³ A. Markoff, *Wahrscheinlichkeitsrechnung* (Leipzig, 1912).

³⁴ See ref. 3; also sec. 10:16 of Margenau and Murphy, *The mathematics of physics and chemistry* (D. Van Nostrand, 1943) and ref. 2, sec. 3.5.

³⁵ \mathbf{X} and \mathbf{v} are column matrices, and \sim indicates the transposed matrix.

³⁶ See E. C. Kemble, *Fundamental principles of quantum mechanics* (McGraw-Hill, p. 36, 1937); also M. Plancherel, *Rend. di Palermo* 30, 289 (1910). The theorem is also known by Parseval's name.

require that $g(t)$ be in general different from zero in a long time interval T , but be zero outside this interval. With this in mind we may consider the *correlation function* for $g(t)$, which is defined as

$$R(t) \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} \overline{g(t_0)g(t_0+t)} dt_0 = \overline{[g(t_0)g(t_0+t)]_{av.}}, \quad (\text{A2.1})$$

where now $g(t_0)$ vanishes outside the interval $0 < t < T$. The average in (A2.1) is to be performed over all phases of the disturbance, and the bar indicates the average computed over any random variables in the wave: for it often happens that $g(t)$ is a stochastic or random function of the time t , or at least that some component of $g(t)$ is randomly distributed. Analytically such randomness is introduced by treating the function as involving a certain number of parameters, and then taking these parameters to be random variables, distributed according to a certain law. Now for any given set of parameters there will be a definite correlation function $R(t)$, and as we shall see, a definite mean-square amplitude spectrum also. However, it is the *average* of this set of correlation functions and spectra that is important for our work. These averages may be obtained by averaging over the ranges of the parameters, with the help of their distributions, as indicated by the superscript bar on (A2.1) and elsewhere. We remark further that should $g(t)$ comprise two or more incommensurable periodic disturbances, the definition (A2.1) is easily extended to include the separate averages over the respective phases of the additional components, inasmuch as the correlation function of such a wave is simply the product of the correlation functions of the separate parts. An important example of this type of function is the modulated carrier, *vide* Sec. 2, where no correlation between carrier and modulation exists.

Letting

$$g_1 = g(t_0), \quad (0 < t_0 < T); \quad g_2^* = g(-t_0 + t), \quad (t_0 - T < t < t_0), \quad (\text{A2.2})$$

and with the help of the Fourier transforms of g_1 and g_2^* and Plancherel's theorem, we obtain finally for (A2.1) the correlation function

$$R(t) = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{\overline{|S(f)|^2}}{T} e^{-i\omega t} df = \int_0^{\infty} W(f) \cos \omega t df, \quad t > 0, \quad (\text{A2.3})$$

where $W(f)$ is the mean spectral density defined as

$$W(f) = \lim_{T \rightarrow \infty} \frac{2\overline{|S(f)|^2}}{T}, \quad f > 0, \quad (\text{A2.4})$$

and $S(f)$ is the amplitude spectrum of $g(t)$. The inversion of (A2.3) gives us at once the other relation

$$W(f) = 4 \int_0^{\infty} R(t) \cos \omega t dt. \quad (\text{A2.5})$$

The limit in (A2.4) is assumed exist. When it does not, the physical significance of (A2.4) is this: it represents the power at some discrete frequency f_0 rather than a spectral distribution over a continuous range of frequencies. Then analytically in the

limit $W(f)$ is observed to become infinite at $f=f_0$ and be zero elsewhere, such that $\int_0^\infty W(f)df$ converges. The spectral density $W(f)$ accordingly exhibits the properties of a delta-function.

On setting $t=0$ we find from (A2.1) and (A2.3) that

$$R(0) = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{\overline{g(t_0)^2}}{T} dt_0 = [\overline{g(t_0)^2}]_{\text{av.}} = \int_0^\infty W(f)df, \quad (\text{A2.6})$$

showing that the mean total power may be obtained from the correlation function by putting t equal to zero in the latter; this is in agreement with our definition (A2.4) of the spectral density. Furthermore, *the mean total power is observed from (A2.6) to be independent of the shape of the spectrum*, depending only on the integral $\int_0^\infty W(f)df$. It should also be mentioned that the spectrum (A2.4) can never give us the time-variation of the wave, since information about the phases is always wanting. Hence an infinite number of different functions $g(t)$ may be combined to give the same value of $W(f)$.

It is interesting to observe what happens after very long times. For a purely stochastic disturbance, which from (A2.1) is seen to be independent of the average over T , $R(\infty)$ becomes zero: there ceases to be any correlation at all between an event at time t_0 and one at t_0+t , $t > \infty$ later. But for periodic components $R(\infty)$ approaches no definite limit, since this part of the disturbance is indefinitely repeated and can never be said ultimately to die down to zero in time. By considering the oscillatory or constant parts of the expression for the correlation function in the limit $t \rightarrow \infty$, we can determine the contribution to the total power arising from the periodic part of the wave, for with the help of (A2.5) and the delta-function the result is seen to be (the sum of) the mean powers in the respective components. These quantities may also be identified as the coefficients of the constant or of the trigonometric parts of the correlation function $R(t)$. The constant part corresponds to *d-c*, the others, to the various discrete frequencies. Thus, in turn, the power in the continuum is $R(0) - R(\infty)$, and may be obtained from that portion of the correlation function which vanishes at $t = \infty$. Examples are given in Sections 2 and 7.

It is convenient to use the *normalized* correlation function $r(t)$, where

$$r(t) \equiv \psi(t)/\psi(0) = \int_0^\infty w(f) \cos \omega t df / \int_0^\infty w(f) df; \quad \rho(t) \equiv R(t)/R(0). \quad (\text{A2.7})$$

Here $\psi(t)$ is chosen to represent the correlation of a wave entering some non linear device and $w(f)$ is its mean power spectrum; $\rho(t)$, correspondingly, is the normalized correlation of the output, when $R(t)$ is so distinguished. It follows from (A2.7) that $\rho(0) = r(0) = 1$, which is the maximum value of $\rho(t)$ and $r(t)$.

The important special case in which the incoming noise is confined to a symmetrical band of frequencies narrow compared with the central frequency f_c , i.e., $w = w(f - f_c)$ (see Fig. 7c), leads to

$$\begin{aligned} r(t) &= \psi(0)^{-1} \int_0^\infty w(f - f_c) \cos \omega t df = \left\{ \psi(0)^{-1} \int_{-\infty}^\infty w(f') \cos \omega' t df' \right\} \cos \omega_c t \\ &\equiv r_0(t) \cos \omega_c t, \end{aligned} \quad (\text{A2.8})$$

on the change of variable $f - f_c = f'$, where we have ignored the spectral "tail" at $f = 0$, inasmuch as $w(-f_c)$ is assumed to be very much smaller than $w(0)$. As an example, consider the Gaussian spectrum $w(f) = W_0 \exp [-(f - f_c)^2 / f_b^2]$, $f_c / f_b \gg 1$; for this $\psi(t)$ and $\psi(0)$ are readily shown to be

$$\psi(t) = \psi(0)r_0(t) \cos \omega_c t = \left(\frac{W_0 \omega_b}{2\pi^{1/2}} \right) e^{-\omega^2 t_b^2 / 4} \cos \omega_c t; \quad \psi(0) = \frac{W_0 \omega_b}{2\pi^{1/2}}; \quad (A2.9)$$

W_0 is the maximum spectral density. In a similar fashion correlation functions for other input spectral distributions are easily determined with the help of (A2.7) or (A2.8).

One may expand the output correlation function $R(t)$ in a power series in $\psi(t)$ (or $r(t)$), the input correlation, and hence as a function of $\cos \omega_c t$ ($l = 0, 1, 2, \dots$), for the case of narrow-band noise, by virtue of (A2.8) and the resolution of $(\cos \omega_c t)^n$ into harmonics.⁹ The series then becomes

$$R(t) = \sum_{l=0}^{\infty} G_l(t) \cos \omega_c t = \sum_{l=0}^{\infty} R_l(t), \quad (A2.10)$$

where now $R_l(t)$ is the correlation of the l th harmonic zone generated in the output. The spectrum of these bands follows at once from (A2.5), showing also that the various "resolved" spectra $W_l(f)$ are distributed about the harmonics lf_c ($l = 0, 1, 2, \dots$). Examination of one such region $W_l(f)$ shows that here, too as in (A2.6), the mean power in any given band is independent of the spectral distribution of the band. Further, since the input spectrum enters only through $\psi(t)$ and $\psi(0)$ (= the mean input power) *the mean power in the l -th band is also independent of the original spectral shape $w(f)$ of the incoming wave.* This is, strictly speaking, only practically true, not rigorously so, as distortion of the input always spreads its spectrum and thus spectral "tails" from one region overlap those of another. This overlapping is quite insignificant most of the time, as long as the band is narrow, the criterion of narrowness depending on what is considered a negligible spectral ordinate.

Appendix III: Special Functions and Integrals. A quantity that appears often in our analysis, cf. Secs. 4 or 5, is the confluent hypergeometric function

$${}_1F_1(\alpha; \beta; x) = 1 + \frac{\alpha x}{\beta 1!} + \frac{\alpha(\alpha + 1)}{\beta(\beta + 1)} \frac{x^2}{2!} + \dots + \frac{(\alpha)_n x^n}{(\beta)_n n!} + \dots, \quad (A3.1)$$

where $(\alpha)_n = \alpha(\alpha + 1) \dots (\alpha + n - 1)$, and $(\alpha)_0 = 1$, as usual. This function has the important property, known as Kummer's transformation,²¹ that

$${}_1F_1(\alpha; \beta; x) = e^x {}_1F_1(\beta - \alpha; \beta; -x), \quad (A3.2)$$

and it may be shown that the asymptotic development of ${}_1F_1$ takes the form,²⁷

$${}_1F_1(\alpha; \beta; -x) \simeq \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} x^{-\alpha} \left\{ 1 + \frac{\alpha(\alpha - \beta + 1)}{x 1!} + \frac{\alpha(\alpha + 1)(\alpha - \beta + 1)(\alpha - \beta + 2)}{x^2 2!} + \dots \right\}, \quad R(x) > 0. \quad (A3.3)$$

²⁷ See, for example, Whittaker and Watson, *Modern analysis* (Cambridge Univ. Press, 1940), Chapter XVI.

The expression (A3.3) is useful in determining the limiting forms of spectra and power distribution when the root-mean-square noise voltage $\psi^{1/2}$ is much less than either the cut-off voltage b_0 or the amplitude of the carrier; see Sec. 4. The confluent hypergeometric functions of both negative and positive argument may also be expressed in terms of the modified Bessel functions of the first kind, for certain combinations of values of α and β . Examples are given in Appendix II of Bennett's paper⁶ and also in ref. 2.

Another function of considerable interest is

$$\phi^{(j)}(b) = \left. \frac{d^{(j)}}{dx^j} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \right|_{x=b} \equiv (-1)^j H_j(b) e^{-b^2/2} / (2\pi)^{1/2}, \quad j = 0, 1, 2, \dots \quad (A3.4)$$

Here the $H_j(b)$ are Hermitian polynomials³⁸ of order j . It is not difficult to show, by successive differentiations with respect to b_0 when $n=0$, that the ϕ -functions may be given as the following infinite integrals

$$\begin{aligned} \int_0^\infty z^{2n} \cos b_0 z e^{-\psi z^2/2} dz &= \left(\frac{\pi}{2}\right)^{1/2} (-1)^n \psi^{-n-1/2} e^{-b_0^2/2\psi} H_{2n}(b_0/\psi^{1/2}) \\ &= \pi (-1)^n \psi^{-n-1/2} \phi^{(2n)}(b_0/\psi^{1/2}) \\ \int_0^\infty z^{2n-1} \sin b_0 z e^{-\psi z^2/2} dz &= \left(\frac{\pi}{2}\right)^{1/2} (-1)^{n+1} \psi^{-n} e^{-b_0^2/2\psi} H_{2n-1}(b_0/\psi^{1/2}) \\ &= \pi (-1)^n \psi^{-n} \phi^{(2n-1)}(b_0/\psi^{1/2}), \quad n = 0, 1, 2, \dots, \end{aligned} \quad (A3.5)$$

and in this connection

$$\phi^{(-1)}(b_0/\psi^{1/2}) = \frac{1}{\sqrt{2\pi}} e^{-b_0^2/2\psi} H_{-1}(b_0/\psi^{1/2}) = \frac{1}{2} \Theta(b_0/\sqrt{2\psi}), \text{ where } \Theta(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv$$

is the familiar error function, tabulated, for example, in Jahnke and Emde and in Pierce's Tables. Tables of $\phi^{(n)}$ for $n=0, \dots, 6$ are given in T. C. Fry's *Probability and its engineering uses* (D. Van Nostrand, 1928, pp. 456, 457). Additional values may be obtained from the recurrence relation

$$\phi^{(n+1)}(b) = - \{ b\phi^{(n)}(b) + n\phi^{(n-1)}(b) \}, \quad n = 0, 1, 2, \dots \quad (A3.6)$$

We may also express $\phi^{(n)}$ in terms of the confluent hypergeometric function ${}_1F_1(\alpha; \beta; -x)$. To do this we need Hankel's exponential integral³⁹

$$\begin{aligned} \int_0^\infty J_\nu(az) z^{\mu-1} e^{-q^2 z^2} dz &= \frac{\Gamma[(\nu + \mu)/2]}{2q^\mu \Gamma(\nu + 1)} \left(\frac{a}{2q}\right)^\nu {}_1F_1\left(\frac{\nu + \mu}{2}; \nu + 1; -\frac{a^2}{4q^2}\right), \\ R(\mu + \nu) &> 0, \quad | \arg q | < \pi/4, \end{aligned} \quad (A3.7)$$

which is readily established by expanding the Bessel function and integrating term-wise with help of the Γ -function. Since

$$e^{\pm ib_0 z} = (\pi b_0 z / 2)^{1/2} \{ J_{-1/2}(b_0 z) \pm i J_{1/2}(b_0 z) \}, \quad (A3.8)$$

³⁸ W. Kapteyn, Proc. Royal Acad. Amster. **16**, 1191 (1914), and G. Szegő, Amer. Math. Soc. Colloq. Pub. **23**, 101-104 (1939).

³⁹ Reference 21, **13.3**.

it follows at once for (A3.6) when this is substituted into (A3.7) that

$$\left. \begin{aligned} \phi^{(2n)}(b_0/\psi^{1/2}) &= \frac{(-1)^n(2n)!}{2^n n!(2\pi)^{1/2}} {}_1F_1\left(\frac{2n+1}{2}; \frac{1}{2}; -\frac{b_0^2}{2\psi}\right), \\ \text{and} \\ \phi^{(2n-1)}(b_0/\psi^{1/2}) &= \frac{(-1)^n(2n)!}{2^n n!(2\pi)^{1/2}} \frac{b_0}{\psi^{1/2}} {}_1F_1\left(\frac{2n+1}{2}; \frac{3}{2}; -\frac{b_0^2}{2\psi}\right), \\ & \qquad \qquad \qquad n = 0, 1, 2, \dots \end{aligned} \right\} \quad (\text{A3.9})$$

We notice also from (A3.1), (A3.4), and (A3.9) that the Hermitian polynomials may be written

$$\begin{aligned} H_{2n}(b) &= \frac{(-1)^n(2n)!}{2^n n!} {}_1F_1\left(-n; \frac{1}{2}; \frac{b^2}{2}\right); \\ H_{2n-1}(b) &= \frac{(-1)^{n+1}(2n)!}{2^n n!} b {}_1F_1\left(1-n; \frac{3}{2}; \frac{b^2}{2}\right), \quad n = 0, 1, 2, \dots \end{aligned} \quad (\text{A3.10})$$

From these results it is evident that

$$\begin{aligned} \phi^{(2n)}(0) &= \frac{(-1)^n(2n)!}{2^n n!(2\pi)^{1/2}} = H_{2n}(0)/(2\pi)^{1/2}; \\ (2\pi)^{1/2}\phi^{(2n-1)}(0) &= -H_{2n-1}(0) = 0. \end{aligned} \quad (\text{A3.11})$$

Now in the theory of the ν th-law non linear device, when $\nu (>0)$ is not necessarily an integer, the integrand of the fundamentals integrals (2.20) contains a branch point at the origin, rather than a simple pole when ν is integral. Accordingly, results like (A3.5) and (A3.7) must be extended to include these more general cases. The first integral to be established is

$$I_1 = \int_{\mathbf{C}} e^{-c^2 z^2} z^{2\mu-1} dz = \frac{i\pi e^{-\mu\pi i}}{c^{2\mu}\Gamma(1-\mu)} = ic^{-2\mu}\Gamma(\mu)e^{-\pi\mu i} \sin \pi\mu, \quad |\arg c| < \pi/4, \quad (\text{A3.12})$$

where the contour \mathbf{C} is the usual one of Eq. (1.8), extending from $-\infty$ to $+\infty$ along the real axis and is indented downward in an infinitesimal semi-circle about $z=0$. Here the argument of z is zero on the positive portion of \mathbf{C} and is $-\pi$ on the negative part. In the neighborhood of the origin the contribution over the semicircle vanishes in the limit, provided that for the moment we require $R(\mu) > 0$. Then we may set $z = e^{-\pi i}t, 0 \leq t \leq \infty, z < 0; z = t, z > 0$, so that finally

$$\begin{aligned} I_1 &= (1 - e^{-2\pi\mu i}) \int_0^\infty t^{2\mu-1} e^{-c^2 t^2} dt = (1 - e^{-2\pi\mu i})\Gamma(\mu)/2c^{2\mu} = \pi i e^{-\pi\mu i}/c^{2\mu}\Gamma(1-\mu), \\ & \qquad \qquad \qquad R(\mu) > 0, \quad |\arg c| < \pi/4. \end{aligned} \quad (\text{A3.13})$$

Now $I_1(\mu)$ is certainly analytic for all values of μ , and therefore by analytic continuation we may extend the domain of I_1 to include all values of μ for which $I_1(\mu)$ is finite. This removes the restriction $R(\mu) > 0$ and gives us (A3.12).

The generalization of (A3.7) follows from (A3.12). We have only to expand the Bessel function to obtain

$$I_2 = \int_C z^{\mu-1} J_\nu(az) e^{-q^2 z^2} dz = \left(\frac{a}{2}\right)^\nu \sum_{n=0}^\infty \frac{(a/2)^{2n} (-1)^n}{n! \Gamma(\nu + n + 1)} \int_C z^{\nu+2n-\mu-1} e^{-q^2 z^2} dz, \tag{A3.14}$$

termwise integration being allowed because of the absolute convergence of $J_\nu(az)$. Application of (A3.12) with the aid of

$$\Gamma\left(1 - n - \frac{\nu}{2} - \frac{\mu}{2}\right) = \Gamma\left(1 - \frac{\nu + \mu}{2}\right) / (-1)^n \left(\frac{\nu + \mu}{2}\right)_n$$

gives us finally

$$I_2 = \frac{\pi i^{1-\nu-\mu} (a/2q)^\nu}{q^\mu \Gamma(\nu + 1) \Gamma[1 - (\nu + \mu)/2]} {}_1F_1\left(\frac{\mu + \nu}{2}; 1 + \nu; -\frac{a^2}{4q^2}\right), \tag{A3.15}$$

| arg q | < π/4.

Equation (A3.15) enables us to evaluate the generalized version of (A3.5), namely,

$$I_3 = \int_C z^\mu e^{+izb_0 - c^2 z^2} dz, \quad | \arg c | < \pi/4. \tag{A3.16}$$

The substitution of (A3.8) for $e^{\pm izb_0}$ and application of I_2 gives

$$I_3 = \frac{\pi i^{-\mu}}{c^{\mu+1}} \left\{ {}_1F_1\left(\frac{\mu + 1}{2}; \frac{1}{2}; -x^2\right) / \Gamma\left(\frac{1 - \mu}{2}\right) \right. \\ \left. \pm 2x {}_1F_1\left(\frac{\mu + 2}{2}; -\frac{3}{2}; -x^2\right) / \Gamma\left(-\frac{\mu}{2}\right) \right\}, \tag{A3.17}$$

where $x = b_0/2c$ and $| \arg. c | < \pi/4$, for convergence at infinity. Note that only when μ is an integer can I_3 be expressed as a ϕ -function, cf., Eq. (A3.9).

In the limit of vanishing noise voltage, $\psi \rightarrow 0$, or infinite signal or bias and finite noise power, it is necessary to extend the Weber-Schafheitlin integral and evaluate

$$I_4 = \int_C J_\alpha(az) J_\beta(bz) dz / z^\gamma. \tag{A3.18}$$

Again, as in the case of the generalized Γ -function, Eq. (A3.12), analytic continuation can be used in precisely similar fashion to give us finally⁴⁰

$$I_4 = \frac{\pi e^{-\pi i(\alpha+\beta-\gamma)/2} b^\beta {}_2F_1\left[\frac{1}{2}(\alpha+\beta-\gamma+1), \frac{1}{2}(\beta-\gamma-\alpha+1); \beta+1; b^2/a^2\right]}{2\gamma^{-1} a^{\beta-\gamma+1} \Gamma(\beta+1) \Gamma[(1+\gamma-\alpha-\beta)/2] \Gamma[(1+\gamma+\alpha-\beta)/2]}, \quad 0 \leq b \leq a, \tag{A3.19}$$

subject only to the restrictions that $R(\gamma) > 0$. The function ${}_2F_1$ has two useful properties, needed in the present work, which we list below:

⁴⁰ Reference 21, Sec. 13.4 gives the value of $\int_0^\infty J_\alpha(az) J_\beta(bz) dz / z^\gamma$.

$${}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}, \quad \begin{array}{l} R(\gamma) \neq 0, -1, -2, \dots \\ R(\gamma - \alpha - \beta) \neq 0, -1, -2, \dots \end{array}, \quad (\text{A3.20a})$$

and

$${}_2F_1(\alpha, \beta; \gamma; x) = (1 - x)^{\gamma - \alpha - \beta} {}_2F_1(\gamma - \beta, \gamma - \alpha; \gamma; x). \quad (\text{A3.20b})$$

(See Chapter XIV of reference 37 for a general treatment.)