GENERAL IMPEDANCE—FUNCTION THEORY*

BY

PAUL I. RICHARDS

Brookhaven National Laboratory

1. Introduction. In the field of lumped-constant circuits, considerable theory is available[1] concerning relations satisfied by all physically realizable impedances. Conditions which are both necessary and sufficient for physical realizability have been derived. It is the purpose of the present paper to extend some of these techniques to more general, distributed-constants circuits.

The basic method of existing theory consists in extending the impedance $Z = R + iX$ to "complex frequencies" $s = \gamma + i\omega$ so that Z becomes a complex function of a complex variable. In the field of lumped-constants, the circuit functions are rational and the extension is effected merely by substituting s for $i\omega$.

2. The concept of impedance. When we pass from the field of lumped-constant to more general circuits, we can no longer say that the impedances are rational functions of ω. This circumstance raises a number of difficulties. First, we must consider singularities other than poles, for it is known that all non-rational functions have at least one singularity which is not a pole. (For example, the input impedance of a shorted line, $\tanh (i\omega l/c)$, has an essential singularity at infinity.) Secondly, we no longer have the relatively simple type of differential equation which occurs in lumped-constant circuits. At our level of generality, there are available only Maxwell's Equations. These considerations make it advisable for us to review anew the general concept of impedance and the means of extending this concept to complex frequencies.

Before embarking on our general discussion, we must stress a point frequently overlooked in the foundations of electro-magnetic theory. When introducing this subject, the remark is often made that specialization to harmonic time dependence is of no great consequence due essentially to the Fourier Integral Theorem. At a later stage, it is then mentioned that the material constants in the resulting equations may depend upon frequency. Often it is *not* mentioned that, in the case of such dispersive media, it is obviously impossible to return to the original Maxwell's Equations for arbitrary time dependence.

Thus, if we wish to be accurate, we should not say merely that the postulates of electro-magnetic theory are contained in Maxwell's Equations. Rather, we must say, at least for dispersive media (and, in actual practice, in general), that the postulates consist of: (a) Maxwell's Equations for harmonic time dependence and (b) Fourier (or Laplace) superposition for arbitrary time dependence.

Our arguments will be based essentially upon (b). The purpose of the discussion was to emphasize that (b) is a basic *postulate* of electro-magnetic theory and is not a mere pseudo-mathematical consequence thereof.

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[1]For an excellent summary of lumped-constant impedance-function theory, see H. W. Bode, *Network analysis and feedback amplifier design*, D. Van Nostrand, New York, 1945.

We must next decide what we shall mean when we speak of an impedance in general. We certainly shall want any impedance Z to satisfy a relation of the type:[2]

$$(\text{response quantity}) = (\text{applied quantity}) \times Z(i\omega). \tag{1}$$

Actually, we shall place no restrictions upon the quantities involved except for those limitations already implied by the form of (1). These implied restrictions, however, merit considerable discussion.

Firstly, note that applied and response quantities cannot always be interchanged. A familiar example in the field of lumped constants would be generator-voltage and short-circuit-output-current for a four terminal network. Under no conditions can the latter be considered "applied". (It is this impossibility of interchange which causes the analytic properties of a transfer function to differ markedly from those of its reciprocal.[1])

Secondly, it is specifically stated in (1) that Z is to depend only upon frequency. This implies that (*i*) the circuit configuration is constant in time and (*ii*) the circuit is linear (i.e. the material constants depend at most on frequency and spatial position). We shall add the further restriction that (*iii*) the circuit is passive; mathematically, this is the statement that the total power absorbed is eventually non-negative.

Finally, (1) implies that the applied and response quantities can be characterized by two single numbers, one of which is determined by the other. Thus the circuit must be so constructed that the entire spatial dependence of all field components is completely determined by frequency, while the absolute magnitude of the field at all points is determined by a single "excitation parameter".

We shall not attempt an exhaustive discussion of the cases in which this fourth restriction can be met, but a few remarks are in order. The first picture that comes to the reader's mind when this requirement is stated is probably a circuit which—while possibly containing waveguides, antennas, etc.,—has definite small input terminals. But what does "small" mean when the frequency has not been specified? Consider lumped-constant impedance-function theory. Here, definite input terminals are assumed and yet the impedance functions are treated as having physical significance beyond the frequency of visible light and, indeed, at $\omega = \infty$. What is essentially done, of course, is to idealize the physical picture in a self-consistent fashion so as to obtain a theory which is tractable and, at the same time, physically significant within the frequency range of interest. It seems reasonable that the same procedure could be carried out over greater frequency ranges; a case in point is transmission-line theory, which avoids the lumped-constant approximation of treating a line merely as a pair of leads but which ignores the effects of radiation, waveguide modes, and discontinuity reactances.

Following these thoughts one step further, we note that we may dispense with the concept of definite input terminals. We might, for example, choose as the quantities in (1), the transverse dominant-mode components of the forwardly-propagated electric and magnetic fields at some point of a reference plane located within an input waveguide. The characteristics of the waveguide would assure satisfaction of the above condition provided that the higher modes could be excited only through the dominant mode. This could be accomplished by specifying that the guide be fed by an input guide which "tapers to a point generator". In practice this last phrase could mean that

[2]Note that the usual roles of applied and response quantities have been interchanged. This facilitates treatment of transfer impedances.

the taper is to be cut off and attached to an actual generator after the cross-section has become too small to support higher modes; this would involve a set of generators with associated coupling tapers of different lengths for each frequency range (no taper is needed below the cut-off of the main guide), but the impedance as defined above would, nevertheless, be well-defined.

Having reached this level of generality, we are faced with the question of distinguishing between transfer and driving-point impedances. We shall adopt the following definition, the motivation for which is obvious in the light of the wave-guide example.

Definition: If the product of the quantities in (1) has the same algebraic sign as the total power (or energy) absorbed by the circuit, then Z is called a driving-point impedance.

Before closing this section, another consideration should be mentioned. Inasmuch as our arguments will be based upon physical principles, we must require that the circuit can actually be constructed. The significance of this remark will, we hope, become clearer as the argument progresses. For the present, we may remark that a transmission line or waveguide of infinite length must therefore be excluded; though Maxwell's Equations are capable of treating this fiction, the device cannot actually be exhibited in the world of reality. An antenna radiating into empty space (or into a completely absorbing box if we wish to avoid the non-linear ionosphere) can be constructed and used and is therefore *not* to be excluded. Secondly, we should note that truly lossless circuits are also a fiction. We do not wish to exclude these; they are far too useful. However, recognizing their idealized nature, we shall, when convenient, treat them as the limiting case of a lossy circuit. This corresponds to the physical facts: losses are present but, in some cases, effects due specifically to dissipation are small enough to be neglected.

3. Necessary conditions. We now wish to extend our impedance $Z(i\omega)$ to complex frequencies $s = \gamma + i\omega$. Maxwell's Equations certainly have formally valid mathematical solutions for time dependence of the form $e^{\gamma t} \cos(\omega t) = Re(e^{st})$. These solutions can be obtained by mere substitution. With dispersive media, however, there would be considerable doubt as to the correct value of the material constants at such complex frequencies. Moreover, although the substitution process will usually lead to an analytic extension of the original function of a single variable, there exist functions (e.g. jump functions) which cannot be so extended. We shall, therefore, adopt a very different approach, which allows the use of the powerful theory of the Laplace Transformation.

Let us cause the applied quantity, $f(t)$ say, to vary with time in an arbitrary manner, subject only to the condition $f(t) = 0$ for $t < 0$ (without which we could hardly be said to "apply" the quantity). Let $g(t)$ represent the response quantity; since the circuit cannot respond before it is stimulated, $g(t) = 0$ for $t < 0$. If, for the moment, $s = i\omega$, then, by the fundamental superposition *postulate* of electro-magnetic theory ((b) of Section II), we have:

$$Z(s) = \frac{\int_0^\infty e^{-st} g(t)\, dt}{\int_0^\infty e^{-st} f(t)\, dt}. \tag{2}$$

If we now allow s to take on any complex value, the integrals in (2) become the definition of the Laplace Transformations of f and g. Thus, by analytic continuation if necessary, $Z(s)$ has been defined for all complex s. The superposition postulate of electromagnetic theory implies, in particular, that $Z(s)$ as defined by (2) takes the same values on $s = i\omega$ whatever the function $f(t)$ may be. We shall see later that Z is analytic in $\gamma > 0$[3] and, at least, on portions of $\gamma = 0$. By an identity theorem[4] for analytic functions, it then follows that $Z(s)$ is well-defined by (2) despite the arbitrariness of $f(t)$. We shall also see later that Z has the usual physical significance in $\gamma > 0$.

The definition (2) has the great advantage that we may, in particular, choose harmonic time variation of extremely low frequency and thereby utilize established facts.

In accordance with the last remark of Section II, we shall first consider circuits of arbitrarily minute but nevertheless finite loss. Moreover, let $f(t)$ vary harmonically in time at such a frequency $s_0 = i\omega_0$ that the associated wavelength is far greater than the overall ("metallic") dimensions of the circuit. By this statement, we do *not* mean that circuit radiation is to be neglected in the entire analysis but merely that the above wavelength is to be so chosen that radiation at *that* frequency will be extremely small. (For example, a centimeter antenna may be considered as a capacitance at a frequency of one cycle/sec.) Thus, though the circuit may in a sense be infinite, the wavelength is to be chosen very large compared to its material, *physical* dimensions.[5]

Under these conditions ($f(t) = e^{s_0 t}$, $t > 0$, s_0 imag.),[6] the (presumably) established theories and facts of low-frequency electricity show that the response quantity will be given by

$$g(t) = \begin{cases} 0 & t < 0, \\ Ae^{s_0 t} + h(t) & t > 0, \end{cases} \tag{3}$$

where $h(t)$ represents the transient response and is bounded by e^{-at} with small but finite a.[5] (Recall our momentary restriction of finite losses.)

Carrying out the Laplace Transformations in (2), we have:

$$Z(s) = A + (s - s_0)H(s), \tag{4}$$

where $H(s)$ is the Laplace Transform of $h(t)$. Since $h(t)$ is bounded by e^{-at}, it follows[7] that $H(s)$ is analytic in the region $Re(s) \equiv \gamma > -a$. Thus we may finally state the following theorem.

For a circuit with finite loss, however small, there exists an $a > 0$ such that $Z(s)$ is analytic in $\gamma > -a$.

Setting $s = s_0$, (4) immediately gives $Z(s_0) = A$, the correct value according to (3). If we choose another value for s_0, the transient excitation may change somewhat (as it must by (4) since Z is well-defined), but its general character cannot be greatly altered since transient behavior is essentially independent of frequency. (A physicist accustomed to working at high frequencies—and thus with short times—may, however, insist on

[3]Throughout this paper the phrase "in $\gamma > 0$", in accordance with the usual convention, does *not* assert anything about the point $s = \infty$. As mentioned earlier, an essential singularity often occurs there.

[4]C. Carathéodory, *Conformal representation*, Cambridge Univ. Press, 1941 (reprinted), p. 82.

[5]Note that it is precisely at this point that the fictitious infinite transmission line violates the restriction of "physical reality" laid down in Section 2.

[6]For simplicity, we use complex notation. The reader can easily verify explicitly that the same final results are obtained for the true time variations.

[7]G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, Dover Publications, New York, 1943, pp. 43, 194, 401.

preserving a "fine-structure" which the audio worker chooses to neglect. For example, compare the response to a unit step of a short lossless line terminated by a resistor with that computed by assuming the line to be a small inductance—or even a mere pair of leads.) Thus $H(s)$ will remain analytic in $\gamma \geq 0$ and therefore bounded at any point in this region. Consequently $Z(s_0)$ will reduce to the correct value A whatever complex value of s_0 is assumed. The usual physical significance may therefore be attached to $Z(s)$ for complex values of s lying in $\gamma \geq 0$.

We also note that if s is real in (2) then $Z(s)$ is real and thus, by the Schwarz reflection principle,

$$Z(s^*) = Z^*(s). \tag{5}$$

This also follows formally from the reality of the general form of Maxwell's Equations.

We must, finally, consider circuits which, under appropriate conditions and perhaps only within certain frequency ranges, may be considered "lossless". Inasmuch as such circuits, for reasons given earlier, should really be considered as the limiting case of a lossy circuit, we may suspect that the theorem italicized above will "essentially" hold in $\gamma \geq 0$, and this will prove to be true.

At sufficiently high frequencies, a dissipationless circuit must be a lossless cavity. For lower frequencies we may arrive at the same situation in the following way. It may be that the circuit is not truly a cavity but produces practically zero radiation. Since this very small radiation is neglected anyway, we may surround the circuit by a sufficiently large (but finite) perfectly conducting box; the reflected radiation will then be of the same order of magnitude as that originally neglected.

Now we wish to re-examine the argument leading to (4) and the theorem following therefrom. Certainly we may still choose an s_0 whose wavelength is very much larger than the circuit and its surrounding box. The transient response $h(t)$ in (3) is, again by the established facts of low-frequency systems, bounded but never dies away. The transient will consist of a superposition of the resonant frequencies of the circuit. Now it is shown in the theory of eigen-value problems[8] that, for a lossless cavity, these resonances occur at discrete, isolated frequencies (i.e. they form a sequence with no finite limit point). Thus we may say[6]

$$h(t) = \sum a_n e^{i\omega_n t}. \tag{6}$$

(An added exponentially decaying part may occur if the circuit is not "lossless" at all frequencies; this, it is easily seen, contributes no new information about the region $\gamma \geq 0$.) The Laplace Transform of (6) is

$$H(s) = \sum a_n/(s - i\omega_n). \tag{7}$$

Substituting (7) into (4), we see that $Z(s)$ is analytic in $\gamma > 0$ (as follows merely from the boundedness of $h(t)$),[7] and, since the ω_n have no finite limit point, we may summarize our results as follows. These are valid for both transfer and driving-point impedances.

(*a*) *$Z(s)$ is analytic in $\gamma \geq 0$ except only for isolated poles on $\gamma = 0$;*

(*b*) $Z(s^*) = Z^*(s)$.

Finally, we wish to find an added condition to characterize driving-point impedances

[8] R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Interscience Publishers, New York, 1943, vol. 1, pp. 358, 384.

as defined in Sec. 2. This has been done by Brune[9] who has shown that if the integrated product $f(t)g(t)$ is always positive for all sufficiently large time intervals then:

$$Re(Z) = R \geq 0 \quad \text{in the region} \quad \gamma \geq 0 \tag{8}$$

This argument establishes (8) for lossy circuits, and treatment of a "lossless" circuit as a limit shows that (8) is valid generally.

4. Consequences of necessary conditions. Throughout the remainder of this paper, only driving-point impedances will be considered. For brevity, we shall refer to such merely as "impedances".

Any physical impedance must satisfy any theorems derived from the conditions italicized above and Eq. (8). The author has made a study of the class of functions defined by these conditions; this investigation is purely mathematical and has appeared elsewhere.[10] The results of physical interest will be summarized in this section.

The only impedance which is purely resistive at all (real) frequencies is the constant resistance $Z \equiv R_0$.[11]

The purely reactive functions have many properties analogous to those given for lumped-constant reactances in the well-known Foster's Reactance Theorem. General reactances are analytic except on the real frequency axis, and there the zeros and poles alternate, while the origin is always a zero or pole. The derivative is always positive (never zero), and for all ω,

$$dX/d\omega \geq | X/\omega | \tag{9}$$

Previously, (9) had been obtained only from Lagrange's equations for a lumped-constant network; it is now known to hold for all generalized reactances.

A general reactance is necessarily a complicated function, for, if it has only a finite number of poles on the real frequency axis, it must be a rational function and therefore a lumped-constant reactance. Nevertheless, two expansions used for lumped-constant reactances can be extended to general reactances; the processes involved naturally become infinite. All of the expansions quoted below converge absolutely and uniformly in any bounded region. (Except, of course, for terms involving poles within the region.)

One such expression is the "partial fraction" expansion:

$$Z = As + (K_0/s) + \sum_1^\infty 2K_n s/(s^2 + \omega_n^2) \tag{10}$$

where ω_n are the anti-resonant (angular) frequencies and K_n, the residues at these poles (K_0 may be zero); $A = \lim Z/s$ as s approaches ∞ in $| \arg s | < \pi/2$.

Schelkunoff has suggested[12] that there might exist such expansions and, pending their proof, has shown in considerable detail how they may be used to calculate the input reactance of complicated distributed-constant structures. The anti-resonant

[9] O. Brune, *Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency*, J. Math. Phys. **10**, 191-236 (1931).

[10] P. Richards, *A special class of functions with positive real part in a half-plane*, Duke Math. Journal, **14**, 777–786 (1947).

[11] Proved for lumped-constant circuits by O. Zobel, *Distortion correction in electrical circuits with constant resistance recurrent networks*, Bell Syst. Tech. J. **7**, 438-534 (1928).

[12] S. A. Schelkunoff, *Representation of impedance functions in terms of resonant frequencies*, Proc. I. R. E., vol. 32, pp. 83-90 (Feb. 1944).

frequencies are, of course, relatively easily evaluated. The K_n (i.e. equivalent L and C) can be found from energy storage. The first term is less easily calculated; it can be shown to be $(d/ds)[Z - (K_0/s)]$ at $s = 0$ minus the series $\sum_1^{\infty} 2K_n/\omega_n^2$. For this reason and also because the series (10) converges rather slowly, it is preferable to employ the following. Use Z or $Y = 1/Z$, whichever is zero at zero frequency. Then, setting $Z'(0)$—or $Y'(0)$—equal to L (the low-frequency inductance—or capacitance), we have

$$Z = Ls - 2s^3 \sum_1^{\infty} K_n/[\omega_n^2(s^2 + \omega_n^2)] \tag{11}$$

This series converges much faster than (10) and the terms are more easily evaluated. For examples of such calculations, the reader is referred to Schelkunoff, op. cit.

The validity of (10) justifies the common practice of approximating a resonant cavity with an equivalent lumped circuit in the vicinity of resonance.

Also, if, in any (or all) terms of (10) or (11), s is replaced by $s + \delta(\delta > 0)$, the series converges as before but now represents a circuit with small losses; K_n and ω_n may, to a first approximation, be evaluated as above, and the value of δ for each term can then be obtained from the "Q" of the circuit at that resonant frequency.

Another useful expansion is analogous to Foster's cannonical form for lumped-constant reactances. Here it becomes an infinite product. If $Z(0) = 0$,

$$Z = Ls \prod_1^{\infty} \frac{1 + (s^2/\Omega_n^2)}{1 + s^2/\omega_n^2} \tag{12}$$

while if $Z(0) = \infty$,

$$Z = \frac{1}{Cs} \prod_1^{\infty} \frac{1 + (s^2/\Omega_n^2)}{1 + (s^2/\omega_n^2)} \tag{13}$$

where L or C is the low-frequency inductance or capacitance, Ω_n are the series-resonant (angular) frequencies, and ω_n the anti-resonant frequencies (both numbered by increasing magnitude).[13]

The various terms in (12, 13) are much more easily evaluated than those in (11). This is somewhat compensated by slower convergence. As a simple example of the use of these products in calculation of reactances, consider a shorted line. Suppose we had found (say, merely by considering traveling-wave reflection) that the line is series-resonant at $\Omega_n = n\pi c/l$ and anti-resonant at $\omega_n = (2n + 1)\pi c/2l$. The low-frequency inductance is then easily calculated, and substitution in (12) immediately gives the reactance at all frequencies. In this simple case, the product can be evaluated for general s because it is known to be the infinite product expansion of $L \tanh (ls/c)$. These same principles can obviously be extended to any circuit and will yield impedance formulas not easily obtained in any other fashion.

We turn now to general, non-reactive impedances. The resistance will, in general, assume an *absolute* minimum at some point because $R \geq 0$. At such a point, the impedance behaves locally very much like a reactance. The derivative $dZ/ds = dX/d\omega$ is real and positive and even satisfies (9). The lower bound R_0 of R represents essentially

[13]After developing (12, 13), the author discovered the recently available article, K. Franz, *Das Reactanztheorem für beliebige Hohlräume*, Elek. Nach. Tech., Bd 21, Heft 1/2, S 8, (Jan/Feb 1944).

Franz proves (12, 13) for cavities of arbitrary shape. For a comparison of his work and that of the present author, see Appendix B.

a constant series resistance because the function $Z - R_0$ again satisfies (8) and the italicized conditions immediately above.

Except for an added purê reactance, the resistive part of a generalized impedance specifies its reactance completely. If $R(\omega)$ is the resistive part (on $\gamma = 0$), the formula is:

$$X(c) = \frac{2c}{\pi} \int_0^\infty \frac{R(t) - R(c)}{t^2 - c^2} \, dt \tag{14}$$

This formula was proved for the case of lumped-constant circuits by Bode.[1]

The various other formulas of Bode have not yet been extended for the completely general case. The original proofs were based on Cauchy's Integral Formula and require that various integrals over a large semi-circle in the right half-plane approach zero with increasing radius. For rational functions, this proof offers no great difficulty, but a general impedance will have an essential singularity at infinity. This means that the behavior of Z, as s approaches infinity, may vary radically with the direction of approach.

If, however, Z is bounded on the real frequency axis, then it may be shown that it is bounded in the entire right half-plane. This fact immediately makes many[14] of Bode's formulas available for this class of functions.

However, the phase-area law:

$$\int_{f=0}^{f=\infty} X \, d(\log f) = \frac{\pi}{2} (R(\infty) - R(0))$$

is certainly not valid unless $R(\infty)$ has meaning. Even if Z is bounded, $R(\infty)$ may not exist; for example, the impedance of a shorted section of slightly lossy line, $\tanh (b + s)$ $(b > 0)$, is bounded on the real frequency axis, but oscillates indefinitely as $\omega \to \infty$. If Z is bounded *and* $R(\infty)$ exists, then Z will approach $R(\infty)$ uniformly in the entire right half-plane. The phase-area law is then easily proved by Bode's method[15] for this class of functions.

Appendix A

It is well-known that the characteristic impedances of transmission lines and waveguides may have branch points on the real-frequency axis. We have indicated in Sec. 2 our reasons for excluding infinite lines or guides, but it might be well to point out explicitly why the input impedance of a finite line or guide cannot have such a branch point if the terminating impedance has none.

Recall that the propagation function θ of a line or guide has exactly the same branch points as the characteristic impedance Z_0. In both cases, the point is such that varying the complex frequency once around the branch point merely changes the sign of Z_0 and θ.

Now, in the expression for the input impedance, Z_0 and θ appear only in the combinations $Z_0 \tanh \theta L$ and $(\tanh \theta L)/Z_0$. Since the hyperbolic tangent is an odd function of its complex variable, these expressions do not change at all when the point in question is circled. Therefore, by definition, the input impedance does not have a branch point.

[14]Specifically (Bode, op. cit.):
p 301, No's I(b), I(d), III(b), IV(b), V(c), VI(b)
p 335, No's I, III, IV, V

[15]Bode, op. cit., p 286. Use $(Z - R(\infty))/s$ as the integrand instead of Z/s. Formulas II, p 335, are also valid for this class of functions.

Appendix B

It was mentioned in Sec. 4 that Franz has proved (12), (13). His method employs the theory[8] of the distribution of the eigen-values of the wave equation. When the developments of this paper were originally evolved, the author was unaware of these theories and so did not use the following condition, which is an immediate corollary of the present argument of Sec. 3.

Lim$(n(f)/f)$ *as* f *approaches* ∞ *exists, where* $n(f)$ *is the number of resonances below* f. Franz uses this condition in his proof. Since it may be shown that (8) and the italicized conditions immediately above (8) do not imply this condition, the author's theorem is, from a purely mathematical point of view, more general than that of Franz.

Similarly, the addition of this extra condition does not invalidate any of the present results since it merely means working with a subclass of those functions satisfying (8) and the italicized conditions above it. It is quite likely, however, that restrictions such as that just given could be used to obtain further results not implied by our earlier conditions alone. Consideration of slightly lossy cavities immediately suggests that a more general restriction may exist on the asymptotic behavior of the number of maxima of R on the real frequency axis.